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Coisotropic quasi-periodic motions for a constrained system of rigid bodies

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Abstract

We consider a constrained system of four rigid bodies located in axisymmetric potential and gyroscopic force fields and interacting by means of angular velocities. We describe an integrable case (not in Liouville sence!) when 12-dimensional phase space of the above system is fibered by the coisotropic invariant tori, the majority of which carry quasi-periodic motions with 7 independent frequences.

1 Introduction

Let (M, ω^2) be a smooth symplectic $2n$-dimentional manifold. A Hamiltonian system on M is usually called (completely) integrable if it possesses n independent smooth integrals $F_i,\ i = 1, \ldots, n$, with pairwise vanishing Poisson brackets. The regular compact common level manifolds M_c of F_i are tori on which the motion is quasi-periodic with $r = r(c) \leq n$ independent frequences (see [1–2]). As far as we know untill recent time there where no natural examples of Hamiltonian systems possessing quasi-periodic motions with $r > n$ independent frequences. Some results in that direction were obtained by the author [3–4].

In the present paper we show that the quasi-periodic motions with $r = 7$ independent frequences arises in 12-dimensional Hamiltonian system which governs the evolution of 4 constrained rigid bodies in presense of potential and gyroscopic forces of special type.

In section 2 we give a general construction of mechanical system with phase space fibered by coisotropic invariant tori. Recall that a submanifold L of M is coisotropic if the orthogonal complement (relative to ω^2) of T_xL is contained in T_xL for each $x \in L$. The motion on M will be called coisotropic if the closure of its orbit forms a coisotropic manifold. The above construction is based on a reduction procedure for Hamiltonian systems with locally Hamiltonian symmetries (see [5–6] for more information). This procedure modifies the reduction methods developed in [1–2, 7–11].

In section 3 we consider a system consisting of 4 constrained rigid bodies located in axisymmetric potential and gyroscopic force fields and interacting by means of angular velocities. We do not discuss here whether the accepted hypotheses about the character of gyroscopic forces are correct from the physical point of view but refer to [12–13]. The description of our system may appear to be rather cumbersome. However, as will be shown in section 6, the majority of conditions formulated in sections 3,4 holds true for the system of Lagrange tops located in coaxial homogeneous gravitational and "magnetic" fields.

In section 4 we show that the system under consideration can be reduced to the direct product of mechanical system on TS^2 with nontrivial gyroscopic force form and linear Hamiltonian system with one degree of freedom.

The simplest case of integrable reduced system is considered in section 5. Here we show, as well, how one can reconstruct the solution of the initial system using that of the reduced one. As a result we obtain the explicit formula (modulo the solution of the reduced system) for coisotropic quasi-periodic motions. It may be noted that in the case considered in section 6 one can express the solution of the reduced system in terms of elliptic functions.

The perturbation problem for the coisotropic quasi-periodic motions is discussed in section 7.

2 Hamiltonian systems on twisted cotangent bundles of manifolds admitting free torus action

Let $\mathcal{M}$ be an n-dimensional smooth Riemannian manifold with metric $\ll \cdot, \cdot \gg$. A mechanical system with configuration space $\mathcal{M}$ is a Lagrangian system on $T\mathcal{M}$ with Lagrangian function of the form $L = \frac{1}{2} \ll \dot{q}, \dot{q} \gg -U(q)$. Here $\dot{q} \in T_q\mathcal{M},\ U : \mathcal{M} \to \mathbf{R}$ is a smooth potential. This system can be transformed into a Hamiltonian one on the symplectic manifold $M = (T^*\mathcal{M}, d\Lambda)$, ($\Lambda$ denote the Liouville 1-form pdq) in a standard way. Denote by $(\cdot|\cdot)$ the pairing between spaces of covectors and vectors and by $\mathcal{A} : T\mathcal{M} \to M$ the nondegenerate symmetric bundle map defined by $\ll \cdot, \cdot \gg = (\mathcal{A} \cdot |\cdot)$. Then the Hamiltonian of the transformed system is

$$H = T + U \circ pr = \frac{1}{2}(p|\mathcal{A}^{-1}(q)p) + U(q), \qquad (1)$$

where $pr : M \to \mathcal{M}$ is the natural projection of cotangent bundle, $p \in M,\ q = pr(p)$.

The gyroscopic force field is determined by a skew symmetric bundle map $G : T\mathcal{M} \to M$, which gives rise to a 2-form on $\mathcal{M}$

$$\Gamma(\xi, \eta) = (G(q)\eta|\xi), \quad \xi, \eta \in T_q\mathcal{M}.$$

We shall call Γ the gyroscopic force form [14]. The case of closed form Γ is especially interesting from the physical point of view [15]. Only that case will be considered in what follows.

The equations of motion for the mechanical system located into gyroscopic force field are of Hamiltonian form relative to the new symplectic structure

$$\omega^2 = d\Lambda + pr^*\Gamma \tag{2}$$

but with the former Hamiltonian function (1). The symplectic manifold $(T^*\mathcal{M}, d\Lambda + pr^*\Gamma)$ is called the twisted cotangent bundle [16]. Henceforth the case of non-trivial second cohomology group $H^2(\mathcal{M}; \mathbf{R})$ will be considered. We shall assume that Γ represents a nontrivial cohomology class $[\Gamma]$.

Let us now describe the reduction procedure on (M, ω^2) for a mechanical system possessing an abelian group of explicit symmetries. Let $\mathcal{M}$ admit the smooth isometric action of k-dimensional torus T^k. Suppose that the potential U and 2-form Γ are torus-invariant. The natural torus action on M preserves Hamiltonian (1) as well as both symplectic structures $d\Lambda$ and ω^2. Thus H together with Poisson structures generated by $d\Lambda$ and ω^2 drops on the quotient.

The nature of the reduced space and the reduced Poisson structures have been well studied in case of torus action admitting the momentum map (see [2, 14]). If $[\Gamma] \neq 0$ then the momentum map, generally speaking, need not exist. In this situation the so-called 2-cocycle of torus action plays an important role. We need some notations to define this cocycle.

Let $\mathcal{T}^k$ denotes the Lie algebra of T^k, Y_a be the infinitesimal generator of $a \in \mathcal{T}^k$ on $\mathcal{M}$, X_a be the same on M, $\boldsymbol{m} : M \to (\mathcal{T}^k)^*$ be the momentum mapping of T^k-action on $(M, d\Lambda)$, $I_0 : M \to T\mathcal{M}$ and $I : M \to T\mathcal{M}$ be the bundle maps (Hamiltonian operators) for the symplectic structures $d\Lambda$ and ω^2 respectively, so that, for example, $\iota(Ip)\omega^2 := \omega^2(Ip, \cdot) = -(p|\cdot)$. Obviously that $pr_* X_a = Y_a$ and $X_a = I_0 d(\boldsymbol{m}|a)$.

The bilinear skew-symmetric 2-form $\mathcal{C}$ on $\mathcal{T}^k$ defined correctly by

$$\mathcal{C}(a, b) := \Gamma(Y_a, Y_b) = \omega^2(X_a, X_b)$$

is called the 2-cocycle of T^k-action on (M, ω^2) The nontrivial case $\mathcal{C} \neq 0$ will be considered henceforth.

Let $\mathsf{p}{:}\mathcal{M} \to \mathcal{N} := \mathcal{M}/T^k$ be the projection of principal bundle associated with T^k-action and $\boldsymbol{\omega}$ be the connection form on $(\mathcal{M}, \mathcal{N}, \mathsf{p})$ generated by the metric.

Definition The metric and 2-form Γ are called concordant if there exist a subspace $L \subset \mathcal{T}^k$ satisfying the following conditions: 1) $\mathcal{T}^k = K \oplus L$, $K := Ker\mathcal{C}$; 2) $\Gamma(Y, Y_a) = 0$ for each horisontal vector field Y and each $a \in L$.

By $\mathcal{P}_K$ and $\mathcal{P}_L$ we denote the projection operators onto K and L respectively. Define the 1-form $\boldsymbol{\beta}$ on $\mathcal{M}$ with values in K by

$$(\boldsymbol{\beta}(\cdot)|a) = -(\iota(Y_a)\Gamma)(\cdot), \quad a \in K.$$

It drops to 1-form $\tilde{\boldsymbol{\beta}}$ on $\mathcal{N}$. Having accepted the following two hypotheses we shall simplify the reduction procedure:

$\boldsymbol{H}_1$: the metric and Γ are concordant;

$\boldsymbol{H}_2$: $\boldsymbol{\beta}$ is exact: $\boldsymbol{\beta} = d\boldsymbol{\mu}$, $\boldsymbol{\mu}{:}\mathcal{M} \to K$ being a smooth T^k-invariant map.

One can define $\tilde{\boldsymbol{\mu}} : \mathcal{N} \to K$ such that $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}} \circ \mathsf{p}$.

Define the map $\boldsymbol{J} : M \to K^*$ by

$$(\boldsymbol{J}|a) = (\boldsymbol{m} + \boldsymbol{\mu}|a), \quad a \in K.$$

Since $\iota(X_a)\omega^2 = -d(\boldsymbol{J}|a)\ \forall a \in K$, $\boldsymbol{J}$ is called the momentum map of K-action on (M, ω^2).

To perform the reduction, identify $\boldsymbol{m}^{-1}(0)/T^k$ with $T^*\mathcal{N}$, denote by $\pi_0 : \boldsymbol{m}(0) \to T^*\mathcal{N}$ the projection of principal T^k-bundle, define the maps

$$P_0 : M \to \boldsymbol{m}^{-1}(0) : p \to p - (\boldsymbol{m}(p)|\boldsymbol{\omega}_q),$$

where $\boldsymbol{\omega}_q = \boldsymbol{\omega}|T_q\mathcal{M},\ q = pr(p)$, and

$$\pi := \pi_0 \circ P_0 \times (\boldsymbol{m} + \mathcal{P}^* \boldsymbol{\mu} \circ pr) : M \to T^*\mathcal{N} \times (\mathcal{T}^k)^* := N.$$

Then π is the projection of principal T^k-bundle over N. Observe that the projection $\pi_0 \circ P_0 \times \boldsymbol{m}$ is used to reduce the canonical Poisson structure on M. The reduced Poisson structure (r.P.s) may be described in a following way (see [5–6]).

The map $\boldsymbol{J}$ drops to $\tilde{\boldsymbol{J}}$ on N. For every $c \in K^*$ the manifold $\tilde{\boldsymbol{J}}^{-1}(c)$ is a symplectic leaf of r.P.s. As a symplectic manifold $\tilde{\boldsymbol{J}}^{-1}(c)$ splits into a direct product of two symplectic manifolds:

1) the twisted cotangent bundle of $\mathcal{N}$ with gyroscopic force form $\tilde{\Gamma}_c = \tilde{\alpha} + (c - \tilde{\boldsymbol{\mu}}|\tilde{\boldsymbol{\Xi}})$, where $\tilde{\alpha}$ is the horisontal part of Γ and $\tilde{\boldsymbol{\Xi}}$ is the curvature form of $\boldsymbol{\omega}$ (both being regarded as 2-forms on $\mathcal{N}$);

2) the affine symplectic $2l$-dimensional space $\{w \in (\mathcal{T}^k)^* : (w|a) = (c|a)\ \forall a \in K\}$, $2l = dimL$, which itself is the symplectic leaf of the Poisson brackets $\{\cdot, \cdot\}$ on $(\mathcal{T}^k)^*$ defined by

$$\{(w|a), (w|b)\} = \mathcal{C}(a, b).$$

We shall denote by $I_N : T^*N \to TN$ the Hamiltonian operator of r.P.s. To obtain the reduced Hamiltonian, i.e the function $\tilde{H} : N \to \mathbf{R}$ for which $H = \tilde{H} \circ \pi$ holds, define the smooth family of symmetric operators $\{\mathcal{B}(\tilde{q}) : \mathcal{T}^k \to (\mathcal{T}^k)^*\}_{\tilde{q} \in \mathcal{N}}$ by

$$(\mathcal{A}Y_a|Y_b)_q = (\mathcal{B}(\tilde{q})a|b),\ \tilde{q} = \mathsf{p}(q),$$

denote by $(\tilde{\mathcal{A}} \cdot |\cdot)$ the quotient metric on $\mathcal{N}$ and drop the potential to $\tilde{U} : \mathcal{N} \to \mathbf{R}$. Then

$$\tilde{H} = \frac{1}{2}\Big((\tilde{p}|\tilde{\mathcal{A}}^{-1}(\tilde{q})\tilde{p}) + (w - \mathcal{P}_K{}^* \tilde{\boldsymbol{\mu}}(\tilde{q})|\mathcal{B}^{-1}(\tilde{q})(w - \mathcal{P}_K{}^* \tilde{\boldsymbol{\mu}}(\tilde{q}))\Big) + \tilde{U}(\tilde{q}),$$

where $\tilde{p} \in T^*\mathcal{N},\ \tilde{q} = pr(\tilde{p}),\ w \in (\mathcal{T}^k)^*$.

The special case is noteworthy when $\mathcal{B}(\tilde{q}) = \mathcal{B}$ does not depend on $\tilde{q}$ and the equality $(\mathcal{B}a|b) = 0$ holds for every $a \in K$ and $b \in L$. In this case the above affine symplectic space may be identified with the subspace $K^{\perp}$ orthogonal to K, and the reduced system splits into direct product of mechanical system on $T\mathcal{N}$ with gyroscopic force form Γ_c, kinetic energy generated by quotient metric and potential

$$\frac{1}{2}\Big(c - \tilde{\boldsymbol{\mu}}(\tilde{q})|\mathcal{P}_K \mathcal{B}^{-1} \mathcal{P}_K{}^*(c - \tilde{\boldsymbol{\mu}}(\tilde{q}))\Big) + \tilde{U}(\tilde{q}).$$

Suppose now that the reduced system is completely integrable. Let $N' \subset N$ be an open subset on which the action-angle variables can be constructed (see [2–17]). Otherwise, there exist a map $\tilde{\boldsymbol{F}} : N' \to \mathbf{R}^m : x \to (\tilde{F}_1(x), \ldots, \tilde{F}_m(x)), \quad m = (dimN - dimK)/2$, with properties:

1) $\tilde{F}_i$ has zero Poisson bracket with $\tilde{H}$ as well as with every $\tilde{F}_j$, $i, j = 1, \ldots, m$;

2) each vector field $I_N d\tilde{F}_i$ generates a circle action on N';

3) vector fields $I_N d\tilde{F}_i$ $(i = 1, \ldots, m)$ are independent at every $x \in N'$.

Then for each $c^* \in K^*$, $\tilde{c} \in \tilde{\boldsymbol{F}}(N')$ the common level manifold $M_c = \boldsymbol{J}^{-1}(c^*) \bigcap \boldsymbol{F}^{-1}(\tilde{c})$, $\boldsymbol{F} := \tilde{\boldsymbol{F}} \circ \pi$, is the coisotropic invariant torus of the flow generated by IdH, $dimM_c = r := n + (k - dimK)/2$. Generally speaking, M_c need not be the minimal set. The necessary condition for a motion on M_c to be quasi-periodic and coisotropic consists in non-resonant property of ω^2 on M_c [18]. The symplectic structure ω^2 is called nonresonant with respect to coisotropic torus M_c if at least one (and then each) of the leaves of the integrable foliation generated by $Ker(\omega^2|M_c)$ is dense in M_c. It turns out that if the orbit of K-action on T^k is dense then the ω^2 is nonresonant on each M_c.

Next, let $y = (y_1, \ldots, y_s)$, $s = (dimN + dimK)/2$, be the coordinates in the open domain $\mathcal{D} \subset K^* \times \tilde{\boldsymbol{F}}(N')$, $mes\mathcal{D} < \infty$. Then $\tilde{H}$ depends only on y, so that $\tilde{H} = H^*(y)$, and in the case, where the map $\mathcal{D} \to \mathbf{R}P^s : y \to \frac{\partial H^*}{\partial y_1} : \cdots : \frac{\partial H^*}{\partial y_s}$ is nondegenerate there exist a subset $\mathcal{D}' \in \mathcal{D}$ such that $mes\mathcal{D}' = mes\mathcal{D}$ and the flow of IdH on the torus corresponding to $y \in \mathcal{D}'$ is quasi-periodic with r independent frequencies. This means that the majority of coisotropic tori are nonresonant.

3 Description of the mechanical system

Denote by $(\cdot, \cdot)$ the scalar product and by $[\cdot, \cdot]$ the vector product in $\mathbf{R}^3$. Consider a mechanical system $\mathcal{S}$ consisting of four rigid bodies anchored at fixed points O^j, $j = 1, \ldots, 4$. Let $\{\boldsymbol{e}_1^0, \boldsymbol{e}_2^0, \boldsymbol{e}_3^0\} = \{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ be a resting right-handed orthonormal basis in $\mathbf{R}^3$ and $\{\boldsymbol{e}_1^j, \boldsymbol{e}_2^j, \boldsymbol{e}_3^j\}$ be the moving right-handed orthonormal basis associated with the principal axes of inertia of j-th body. Suppose that the system $\mathcal{S}$ is constrained in a following way

$$\boldsymbol{h}_1 : \qquad \boldsymbol{e}_3^1 = \cdots = \boldsymbol{e}_3^4.$$

The constrained system will be denoted by $\mathcal{S}'$.

Let $Q^j \in SO(3)$ be the operator that determines the position of the j-th body, i.e., $\boldsymbol{e}_i^j = Q^j \boldsymbol{e}_i^0$, $i = 1, 2, 3$. Denote by $\mathcal{L}$ the configuration space of $\mathcal{S}$, i.e., the direct product of 4 copies of $SO(3)$. The configuration space of $\mathcal{S}'$ is the submanifold

$$\mathcal{M} = \{(Q^1, \ldots, Q^4) \in \mathcal{L} : Q^1 \boldsymbol{e}_z = \cdots Q^4 \boldsymbol{e}_z\} \sim SO(3) \times T^3$$

of $\mathcal{L}$. Denote by g_z^t the one-parametrical rotation group around the z-axis. $\mathcal{M}$ naturally inherits free action of torus $T^4 = \mathbf{R}^4/2\pi\mathbf{Z}^4 = \{(\psi_1, \ldots, \psi_4)|mod2\pi\}$ defined on $\mathcal{L}$ by

$$(Q^1, \ldots, Q^4) \to (Q^1 g_z^{\psi_1}, \ldots, Q^4 g_z^{\psi_4}).$$

The quotient $\mathcal{M}/T^4$ may be identified with the unit sphere $S^2 : x^2 + y^2 + z^2 = 1$ in $\mathbf{R}^3$. To do this assign to any orbit of torus action the point on S^2 indicated by $\boldsymbol{e}_3^1$. Thus $\mathcal{M}$ carries the structure of principal T^4-fiber bundle $(\mathcal{M}, S^2, \mathsf{p})$. Suppose now that

$\boldsymbol{h}_2$: the ellipsoid of inertia of j-th rigid body possesses $\boldsymbol{e}_3^j$-axial symmetry, $j = 1, \ldots, 4$.

Then the kinetic energy of the constrained system is T^4-invariant. Now we are going to describe this kinetic energy. Denote by $\mathbf{\Omega}_s^j$ the angular velocity (in space) of the j-th rigid body. In what follows the well-known fact will often be used: the angular velocity may be concidered as $\mathbf{R}^3$- valued 1-form on $SO(3)$ which is equivariant under left- and invariant under right $SO(3)$-action.

Putting $\Omega_i^j = (\mathbf{\Omega}_s^j, \boldsymbol{e}_i^j)$ and using formula $\dot{Q}^j \boldsymbol{e}_z = [\mathbf{\Omega}_s^j, \boldsymbol{e}_3^j]$ one can easily check that

$$\Omega_2^1 \boldsymbol{e}_1^1 - \Omega_1^1 \boldsymbol{e}_2^1 = \cdots = \Omega_2^4 \boldsymbol{e}_1^4 - \Omega_1^4 \boldsymbol{e}_2^4$$

on $\mathcal{M}$ holds. As a consequence, the following identities are valid on $\mathcal{M}$:

$$(\Omega_1^1)^2 + (\Omega_2^1)^2 = \cdots = (\Omega_1^4)^2 + (\Omega_2^4)^2,$$

$$\Omega_1^1 \wedge \Omega_2^1 = \cdots = \Omega_1^4 \wedge \Omega_2^4.$$

Define now scalar and $\mathbf{R}^4$-valued 1-forms on $\mathcal{M}$ by

$$\Omega_i = \Omega_i^1 | \mathcal{M}, \quad \boldsymbol{\omega} = (\Omega_3^1, \ldots, \Omega_3^4) | \mathcal{M} = (\omega_1, \ldots, \omega_4).$$

We identify the Lie algebra $\mathcal{T}^4$ of torus T^4 with coordinate space $\mathbf{R}^4$. Now it turns out that $\boldsymbol{\omega}$ is a connection form on $(\mathcal{M}, S^2, \mathsf{p})$.

Let I_i^j, $i = 1, 2, 3$, be the principal moments of inertia of the j-th body. From $\boldsymbol{h}_2$ it follows that $I_1^j = I_2^j$. Thus the kinetic energy of $\mathcal{S}'$ is

$$T = \frac{1}{2}\delta_0 \left((\Omega_1)^2 + (\Omega_2)^2 \right) + \frac{1}{2}(D\boldsymbol{\omega}, \boldsymbol{\omega}),$$

where $\delta_0 = 2\sum_{j=1}^4 I_1^j$, $D = diag(\delta_1, \ldots, \delta_4) := diag(I_3^1, \ldots, I_3^4)$.

Next we suppose that

$\boldsymbol{h}_3$: the total potential of the constrained system is a T^4-invariant function $U = \tilde{U} \circ \mathsf{p}$, $\tilde{U} : S^2 \to \mathbf{R}$.

Now we describe the gyroscopic forces acting upon the system. Suppose that the "internal" ones satisfy

$\boldsymbol{h}_4$: the total moment of force by which j-th body acts upon i-th one is equal to $c_{ij}\Omega_3^j \boldsymbol{e}_3^i$.

In order that the "internal" forces be gyroscopic the following conditions must be imposed on the coefficients

$$\boldsymbol{h}_5 : c_{ij} = -c_{ji}, \qquad i, j = 1, \ldots, 4.$$

Thus the 2-form of the "internal" gyroscopic forces is

$$\mathcal{C}(\boldsymbol{\omega}, \boldsymbol{\omega}) = \sum_{1 \le i < j \le 4} c_{ij} \omega_i \wedge \omega_j.$$

In order that this form be closed we accept

$$\boldsymbol{h}_6 : \quad \sum_{i=1}^{4} c_{ij} = 0, \qquad j = 1, \ldots, 4.$$

Indeed, taking into account the Maurer-Cartan equations $d\Omega_3^j = -\Omega_1^j \wedge \Omega_2^j$ and denoting $-\Omega_1 \wedge \Omega_2$ by Ξ we obtain the curvature form (on $\mathcal{M}$) for $\boldsymbol{\omega}$

$$d\boldsymbol{\omega} = (1,1,1,1)\Xi.$$

Thus, $(1,1,1,1) \in K$ implies that the 2-form of "internal" gyroscopic forces is closed.

The external gyroscopic forcesis assumed to be of Lorenz type, i.e.

$\boldsymbol{h}_7$: the force acting at the moving "point-charge" is $\boldsymbol{F}_B = q[\boldsymbol{r}, \boldsymbol{B}(\boldsymbol{r})]$, where q denotes the algebraic value of "point-charge", $\boldsymbol{r}$ denotes its position vector in resting space, $\boldsymbol{B} = rot\boldsymbol{A}$ is a solenoidal field ("magnetic induction").

Omitting, for the time, index j and denote by $q(\boldsymbol{R})$ the "charge" value at a point whose position vector relative to the body is $\boldsymbol{R}$. The position vector of the same point in the resting space is $\boldsymbol{r} = Q\boldsymbol{R} + \boldsymbol{r}_0$, where $\boldsymbol{r}_0$ is the position vector of the point O.

The total moment of $\boldsymbol{F}_B$ (in body) is

$$\boldsymbol{M}_B(\boldsymbol{\Omega}_c) = [\boldsymbol{\Omega}_c, \int q(\boldsymbol{R})(\boldsymbol{R}, Q^{-1}\boldsymbol{B}(Q\boldsymbol{R} + \boldsymbol{r}_0))\boldsymbol{R}\, dV],$$

where $\boldsymbol{\Omega}_c = Q^{-1}\boldsymbol{\Omega}_s$ is the angular velocity relative to the body. Identify $\mathbf{R}^3$ with $(\mathbf{R}^3)^*$ by means of $(\cdot,\cdot)$, and consider the gyroscopic force form determined by $\boldsymbol{M}_B$:

$$\Gamma_B(\xi,\eta) = (\boldsymbol{\Omega}_c(\xi), \boldsymbol{M}_B(\boldsymbol{\Omega}_c(\eta))), \qquad \xi,\eta \in T_QSO(3).$$

We shall show that Γ_B is exact.

Observe first that

$$\frac{1}{2} d\boldsymbol{r} \wedge [d\boldsymbol{r}, \boldsymbol{B}(\boldsymbol{r})] = d(\boldsymbol{A}(\boldsymbol{r}), d\boldsymbol{r}). \tag{3}$$

Introducing a map $SO(3) \to \mathbf{R}^3 : Q \to Q\boldsymbol{R} + \boldsymbol{r}_0$, with $\boldsymbol{R}$ being fixed, one obtains that the pull-back of 1-form $d\boldsymbol{r}$ to $SO(3)$ is 1-form $Q[\boldsymbol{\Omega}_c(\cdot), \boldsymbol{R}]$. It makes possible to calculate the pull-back of 2-forms in both sides of (3) on vectors $\xi, \eta \in T_QSO(3)$. As a result, one can obtain

$$(\boldsymbol{\Omega}_c(\xi), [\boldsymbol{\Omega}_c(\eta), (\boldsymbol{R}, Q^{-1}\boldsymbol{B}(Q\boldsymbol{R} + \boldsymbol{r}_0))\boldsymbol{R}] =$$
$$\iota(\eta)\iota(\xi) d(Q^{-1}\boldsymbol{A}(Q\boldsymbol{R} + \boldsymbol{r}_0), [\boldsymbol{\Omega}_c, \boldsymbol{R}]).$$

This implies

$$\Gamma_B = d(\boldsymbol{a}(Q), \boldsymbol{\Omega}_c),$$

where $\boldsymbol{a}(q) = \int q(\boldsymbol{R})[\boldsymbol{R}, Q^{-1}\boldsymbol{A}(Q\boldsymbol{R} + \boldsymbol{r}_0)]\, dV$. Denote by $\boldsymbol{a}^j(Q)$ the map $\boldsymbol{a}(Q)$ constructed for j-th rigid body.

The following assumption is necessary to guarantee torus-invariance of gyroscopic force form:

$$\boldsymbol{h}_8 : \quad g_z^t \boldsymbol{a}^j(Qg_z^t) = \boldsymbol{a}^j(Q), \qquad t \in \mathbf{R}, \quad j = 1, \ldots, 4.$$

This is always true if j-th body and the function $q^j(\boldsymbol{R})$ possess $\boldsymbol{e}_3^j$-axial symmetry.

Observe that 1-form

$$(\boldsymbol{a}^j(Q), \boldsymbol{\Omega}_c^j) = [\sum_{i=1}^{2} (\boldsymbol{a}^j(Q), \boldsymbol{e}_i^0)\Omega_i^j] + (\boldsymbol{a}^j(Q), \boldsymbol{e}_z)\Omega_3^j \tag{4}$$

is invariant under right g_z^t-action. The same is true for the function

$$f_j(Q) = (\boldsymbol{a}^j(Q), \boldsymbol{e}_z),$$

1-form Ω_3^j and, as a consequence, for the 1-form σ_j enclosed in brackets in (4). Moreover, σ_j vanishes on the infinitesimal generator of right g_z^t-action, thus correctly defining 1-form $\tilde{\sigma_j}$ on S^2. Function f_j also drops to $\tilde{f}_j$ on S^2. Finally, we conclude that the total gyroscopic force form of the constrained system is

$$\Gamma = d(\mathsf{p}^*\tilde{\sigma}) + d(\tilde{\boldsymbol{f}} \circ \mathsf{p}, \boldsymbol{\omega}) + \mathcal{C}(\boldsymbol{\omega}, \boldsymbol{\omega}),$$

where $\tilde{\sigma} = \sum \tilde{\sigma_j}$, $\tilde{\boldsymbol{f}} = (\tilde{f}_1, \ldots, \tilde{f}_4)$. To ensure the concordance of Γ with metric defined by the kinetic energy T we accept

$$\boldsymbol{h}_9 : \quad c_{12} \neq 0, \qquad \sum_{j=1}^{4} \delta_j^{-1} c_{ij} \tilde{f}_j(s) = \text{const}, \qquad s \in S^2, \quad i = 1, 2.$$

We recall that $\mathcal{T}^4$ have been identified with the coordinate space $\mathbf{R}^4$. This allows as to identify $\mathcal{T}^4$ with $(\mathcal{T}^4)^*$ by means of standard scalar product $(\cdot, \cdot)$ in $\mathbf{R}^4$. Introduce the D-scalar product by $(D\cdot, \cdot)$.. Obviously that the space spanned by vectors $\boldsymbol{a}_j = (\delta_1^{-1} c_{j1}, \ldots, \delta_4^{-1} c_{j4})$, $\quad j = 1, 2$, is D-orthogonal to $K = Ker\mathcal{C}$. Thus $\boldsymbol{h}_9$ guarantees the decomposition $\mathcal{T}^4 = K \oplus L$ and the property $\iota(Y_{a_j})\Gamma = \mathcal{C}(\boldsymbol{a}_j, \boldsymbol{\omega})$, $\quad j = 1, 2$, which yelds the required concordance.

4 Description of the reduced system

Let $\boldsymbol{\varepsilon}_1 = (trD)^{-1/2}(1, 1, 1, 1)$, $\boldsymbol{\varepsilon}_2$ be D-orthonormal basis in K, and $\boldsymbol{\varepsilon}_3$, $\boldsymbol{\varepsilon}_4$ be D-orthogonal basis in L normed by $\mathcal{C}(\boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4) = 1$, $\quad (D\boldsymbol{\varepsilon}_i, \boldsymbol{\varepsilon}_i) := 1/\nu$, $\quad i = 3, 4$. Introduse the dual basis $\boldsymbol{\varepsilon}_i^* = D\boldsymbol{\varepsilon}_i$, $\quad i = 1, 2$; $\qquad \boldsymbol{\varepsilon}_i^* = \nu D\boldsymbol{\varepsilon}_i$, $\quad i = 3, 4,$. We may identify K^* with the subspace spanned by $\boldsymbol{\varepsilon}_1^*, \boldsymbol{\varepsilon}_2^*$ and $K^\perp$ with that spanned by $\boldsymbol{\varepsilon}_3^*, \boldsymbol{\varepsilon}_4^*$. Put

$$\tilde{\boldsymbol{\mu}} = \sum_{i=1}^{2} (\tilde{\boldsymbol{f}}, \boldsymbol{\varepsilon}_i)\boldsymbol{\varepsilon}_i^* := \sum_{i=1}^{2} \tilde{\mu}_i \boldsymbol{\varepsilon}_i^*.$$

Then $\boldsymbol{H}_2$ of sect.1 holds.

The momentum map of T^4-action on $T^*\mathcal{M}$ is $\boldsymbol{m} = (M_3^1, \ldots, M_3^4)$ where M_3^j is a projection of j-th body angular momentum on e_3^j-axis. Now we have $\boldsymbol{J} = \sum_{i=1}^{2} \left((\boldsymbol{m}, \boldsymbol{\varepsilon}_i) + \mu_i \right) \boldsymbol{\varepsilon}_i^*$, and $\tilde{\boldsymbol{J}} = \sum_{i=1}^{2} (\boldsymbol{w}, \boldsymbol{\varepsilon}_i)\boldsymbol{\varepsilon}_i^*$, $\quad \boldsymbol{w} = (w_1, \ldots, w_4) \in (\mathcal{T}^4)^*$. The functions $q = (\boldsymbol{w}, \boldsymbol{\varepsilon}_3)$, $\quad p = (\boldsymbol{w}, \boldsymbol{\varepsilon}_4)$, whose Poisson bracket is 1, determine the coordinates on $K^\perp$, thus transforming it into standard symplectic space $\mathbf{R}^2_{(p,q)}$.

To obtain the potential of the reduced system notice that $\mathcal{P}_K^*(\cdot) = \sum_{i=1}^{2} (\cdot, \boldsymbol{\varepsilon}_i)\boldsymbol{\varepsilon}_{i1}^*$. Finally, observe that the horisontal part of Γ drops to $\tilde{\alpha} = d\tilde{\sigma} + \sqrt{trD}\tilde{\mu}_1\tilde{\Omega}$. Thus, the reduced system on common level manifold $\tilde{J}_i := (\boldsymbol{w}, \boldsymbol{\varepsilon}_i) = c_i$, $\quad i = 1, 2$, is equivalent to a direct product of

I. Mechanical system on TS^2 with kinetic energy $\tilde{T}$ correctly determined by $\frac{1}{2}\delta_0((\Omega_1)^2 + (\Omega_2)^2)$, potential $V = \tilde{U}(s) + \frac{1}{2}\sum_{i=1}^{2}(c_i - \tilde{\mu}_i)^2$, $s \in S^2$, and gyroscopic force form $d\tilde{\sigma} + c'_1\tilde{\Omega}$, $c'_1 = c_1\sqrt{trD}$.

II. Hamiltonian system on $\mathbf{R}^2_{(q,p)}$ with Hamiltonian function $\nu(q^2 + p^2)/2$.

5 Integrability

Now we focus on the simplest integrable case of a system with z-axial symmetry. Observe that left g_z^t-action on $\mathcal{L}$ defined by $(Q^1, \ldots, Q^4) \to (g_z^t Q^1, \ldots, g_z^t Q^4)$ gives rise to S^1-action on $\mathcal{M}$. This action commutes with T^4-action and preserves T, $\boldsymbol{\omega}$. Thus, if we suppose that

$$\boldsymbol{h}_{10}: \quad \boldsymbol{a}^j \circ g_z^t = \boldsymbol{a}^j, \quad j = 1, \ldots, 4; \qquad U \circ g_z^t = U \quad \forall t \in \mathbf{R},$$

then the system I possesses z-axial symmetry, which, as a consequence of $H^1(S^2) = 0$, gives rise to a single-valued extra integral. Observe that if $\boldsymbol{A}(g_z^t \boldsymbol{r}) = g_z^t \boldsymbol{A}(\boldsymbol{r})$ and j-th body is anchored at the z-axis then $\boldsymbol{a}^j$ satisfies $\boldsymbol{h}_{10}$.

The infinitesimal generator for the lifting on M of the above S^1-action is globally Hamiltonian vector field with Hamiltonian function

$$\begin{aligned} F \;=\; & \sum_{j=1}^{4}(\boldsymbol{M}_s^j, \boldsymbol{\Omega}_s^j(\frac{d}{dt}|_{t=0} g_z^t Q^j)) + (\boldsymbol{a}^j(Q^j),\ \boldsymbol{\Omega}_c^j(\frac{d}{dt}|_{t=0} g_z^t Q^j))|M = \\ & \sum_{j=1}^{4}(\boldsymbol{M}_s^j + Q^j \boldsymbol{a}^j(Q^j), \boldsymbol{e}_z)|M, \end{aligned}$$

where $\boldsymbol{M}_s^j$ is the angular momentum (in resting space) of j-th body. We may think of $Q^j \boldsymbol{a}^j(Q^J)$ as the moment of vector potential $\boldsymbol{A}$ for j-th body. Now we formulate our main results.

Theorem 1 *Let the system $\mathcal{S}$ satisfies $\boldsymbol{h}_{1-10}$. Then the corresponding Hamiltonian system $\dot{x} = IdH$ on twisted cotangent bundle of $\mathcal{M}$ possesses 5 first integrals H, J_1, J_2, $\sum_{i=3}^{4} (\boldsymbol{m}, \varepsilon_i)^2$, F each generic common level of which is 7-dimensional coisotropic torus. The symplectic structure ω^2 is nonresonant on this torus iff*

$$\boldsymbol{h}_{11}: \qquad |c_{12}k_1 + c_{13}k_2 + c_{23}k_3| \neq 0 \qquad \forall (k_1, k_2, k_3) \in \mathbf{Z}^3 \setminus \{0\}.$$

P r o o f It remains only to explain the connection between $\boldsymbol{h}_{11}$ and the property of K-action orbit on T^4 to be dense. To do this, observe that if $\boldsymbol{h}_5$ and $\boldsymbol{h}_6$ hold then K is spanned by the vectors $(1, 1, 1, 1)$ and $(c_{23}, -c_{13}, c_{12}, 0)$. But the K-action orbit is dense on T^4 iff there does not exist a nontrivial integer vector which is orthogonal to K. The last condition is equivalent to $\boldsymbol{h}_{11}$.□

Now let us construct the action variables. First, we examine the mechanical system I. By virtue of $\boldsymbol{h}_8$ there exist a map $\boldsymbol{b}^j : \mathbf{R}^3 \to \mathbf{R}^3$ for which $Q\boldsymbol{a}^j(Q) = \boldsymbol{b}^j(Q\boldsymbol{e}_z)$ holds. Put $\boldsymbol{b}(\boldsymbol{r}) = \sum_{j=1}^{4} \boldsymbol{b}^j(\boldsymbol{r})$. It turns out that the Lagrangian system with kinetic energy $\delta_0\|\dot{\boldsymbol{r}}\|^2/2$, potential $V(\boldsymbol{r}/\|\boldsymbol{r}\|)$, gyroscopic force form

$$\hat{\Gamma} = d([\boldsymbol{b}(\boldsymbol{r}), \boldsymbol{r}], d\boldsymbol{r}) + \frac{1}{2}{c'}_1[d\boldsymbol{r}, \boldsymbol{r}] \wedge d\boldsymbol{r}$$

and the constraint $\|\boldsymbol{r}\| = 1$ governs the motion of vector $\boldsymbol{e}_3^1$, thus defining the system which is equivalent to I.

The vector field $-y\boldsymbol{e}_x + x\boldsymbol{e}_y$ generates the action of g_z^t on $\mathbf{R}^3$. In view of $\boldsymbol{h}_{10}$ there exist such functions $\Phi : [-1, 1] \to \mathbf{R}$ and $\tilde{V} : [-1, 1] \to \mathbf{R}$ that

$$([\boldsymbol{b}(\boldsymbol{r}), \boldsymbol{r}], -y\boldsymbol{e}_x + x\boldsymbol{e}_y)|S^2 = \Phi(z), \qquad V|S^2 = \tilde{V}(z, c_1, c_2),$$

where $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$. Note that $\Phi(-1) = \Phi(1) = 0$. Introducing the parametric representation of S^2

$$x = \sin(\theta)\cos(\phi), \qquad y = \sin(\theta)\sin(\phi), \qquad z = \cos(\theta), \qquad \theta \in [0, \pi],\ \phi|\text{mod } 2\pi$$

we obtain

$$\tilde{T} = \frac{\delta_0}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta), \quad \tilde{\Gamma} = \hat{\Gamma}|S^2 = d(\Phi(\cos\theta)d\phi) - {c'}_1 \sin\theta d\theta \wedge d\phi.$$

Put

$$p_\theta = \delta_0\dot{\theta}, \qquad p_\phi = \delta_0\dot{\phi}\sin^2\theta \tag{5}$$

to find the Hamiltonian

$$\tilde{H}_I = (p_\theta^2 + p_\phi^2/\sin^2\theta)/2\delta_0 + \tilde{V}(\cos\theta, c_1, c_2)$$

and the symplectic structure $\tilde{\omega}_I^2 = d(p_\theta d\theta + p_\phi d\phi) + \tilde{\Gamma}$ for the Hamiltonian system corresponding to I. Elimination of cyclic coordinate ϕ and p_ϕ, by means of Noetherian first integral

$$\bar{F}_1 := p_\phi + \Phi(\cos\theta) + {c'}_1\cos\theta = c_3, \tag{6}$$

leads to the Hamiltonian system on (θ, p_θ)-plane with Hamiltonian of the form $\bar{H}_I = p_\theta^2/2\delta_0 + W(\theta, c_1, c_2, c_3)$. We restrict ourselves to the domain of $c_1, \ldots, c_4$ for which the level curve $\varsigma : \bar{H}_I = c_4$ is closed. Define the action variable by

$$\bar{F}_2 = \frac{1}{2\pi}\oint_\varsigma p_\theta d\theta = \bar{F}_2(c_1, \ldots, c_4)$$

Now the required action variables on $T^*S^2 \times (\mathcal{T}^4)^*$ are

$$\tilde{F}_1 = \bar{F}_1|_{c_1=\tilde{J}_1}, \quad \tilde{F}_2 = \bar{F}_2(c_1, c_2, \bar{F}_1, \tilde{H}_I)|_{c_1=\tilde{J}_1,\ c_2=\tilde{J}_2}, \quad \tilde{F}_3 = \frac{1}{2}\sum_{i=3}^{4}(\boldsymbol{w}, \boldsymbol{\varepsilon}_i)^2.$$

Put $y_i = \tilde{J}_i, \quad i = 1, 2; \qquad y_{i+2} = \tilde{F}_i, \quad i = 1, 2, 3$. The reduced Hamiltonian depends only on y_i: $\tilde{H} = H_I^*(y_1, \ldots, y_4) + \nu y_5$. One can varify that the map $(y_1, \ldots, y_4) \to (\lambda_1, \ldots, \lambda_4)$, where $\lambda_i = \partial H_I^*/\partial y_i$, is non-degenerate except for some set of measure 0.

Conclusion The majority of coisotropic tori from Theorem 1 are the carriers of quasi-periodic motions with 7 independent frequencies.

To describe the evolution of the system $\mathcal{S}$ satisfying $\boldsymbol{h}_{1-10}$ we find θ from the equation

$$\dot{\theta}^2 = 2(c_4 - W)/\delta_0 \tag{7}$$

and define $\phi(t)$ using (5), (6), thus obtaining the solution of system I. The solution of system II is $q(t) = c_5\sqrt{2}\sin(\nu t + \tau), \quad p(t) = c_5\sqrt{2}\cos(\nu t + \tau), \quad \tau \in \mathbf{R}$.

The orientation of j-th rigid body is determined by the Eulerian angels θ, ϕ, ψ^j [19]. We use the representation $\Omega_3^j = \dot{\psi}^j + \dot{\phi}\cos\theta$ to find the function ψ_0^j from $\Omega_3^j = 0$. By means of $\theta(t), \phi(t), \psi_0^j(t)$ we construct the "horisontal part" $Q_0^j(t) : \mathbf{R} \to SO(3)$ of j-th body evolution. Finally, observing that the vertical part of vector field IdH is $\sum_{i=1}^2 \lambda_i X_{\varepsilon_i} + \nu(qX_{\varepsilon_3} + pX_{\varepsilon_4})$ we obtain the evolution of j-th body in the form

$$Q^j(t) = Q_0^j(t) g_z^{\psi_*^j(t)}$$

where $\quad \psi_*^j(t) = \sum_{i=1}^2 {\varepsilon_i}^j \lambda_i(t) + \nu({\varepsilon_3}^j \int q(t)dt + {\varepsilon_4}^j \int p(t)dt), \quad$ and ${\varepsilon_i}^j$ is j-th component of $\boldsymbol{\varepsilon}_i$.

6 Constrained Lagrange tops in homogeneous fields

In this case $\boldsymbol{h}_2$ holds and $\tilde{U} = \rho z$ for some $\rho > 0$. Suppose that $\boldsymbol{B} = 2\chi \boldsymbol{e}_z, \quad \chi \in \mathbf{R}$. Then $\boldsymbol{A}(\boldsymbol{r}) = \chi[\boldsymbol{e}_z, \boldsymbol{r}] + const$. We may set

$$\boldsymbol{a}(Q) = \chi \int q(\boldsymbol{R})[\boldsymbol{R}, [Q^{-1}\boldsymbol{e}_z, \boldsymbol{R}]]\, dV = SQ^{-1}\boldsymbol{e}_z,$$

where the symmetric operator $S : \mathbf{R}^3 \to \mathbf{R}^3$ is constructed similarly to the inertia operator after replacing the function of mass distribution by $\chi q(\boldsymbol{R})$. Obviously $\boldsymbol{h}_{10}$ holds, and to satisfy $\boldsymbol{h}_8$ we need to require that the matrix of the operator S^j, corresponding to the j-th body, takes the form $diag(\beta_j, \beta_j, \gamma_j)$ in the basis $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$.

Now we find by a direct calculation:

$$\begin{aligned} \tilde{f}_j &= \gamma_j z; \qquad \boldsymbol{b}^j(\boldsymbol{r}) = (\gamma_j - \beta_j)(zx\boldsymbol{e}_x + zy\boldsymbol{e}_y) + (\beta_j + (\gamma_j - \beta_j)z^2)\boldsymbol{e}_z; \\ \Phi(z) &= \sum_{j=1}^4 \beta_j(1 - z^2); \qquad \tilde{\mu}_i = z\sum_{j=1}^4 \gamma_j \varepsilon_i^j, \quad i = 1, 2. \end{aligned}$$

Substituting $z = \cos\theta$ into (7) we obtain the equation $\quad (\dot{z})^2 = P_4(z, c_1, c_2, c_3)$, where P_4 is a polynomial of 4-th order with respect to z. Thus the dependence of z on t is given by an elliptic function.

Finally, $\boldsymbol{h}_9$ takes the form

$$c_{12} \neq 0, \qquad \sum_{j=1}^4 c_{ij}\gamma_j/\delta_j = 0, \quad i = 1, 2. \tag{8}$$

Consider now the case when

$\boldsymbol{h}_{12}$: the mass distriburion in bodies, is proportional to the "charge" distribution.

This implies that S^j is proportional to the inertia operator of j-th rigid body. Hence, γ_j/δ_j does not depend on j and (8) becomes a consequence of $\boldsymbol{h}_6$ if $c_{12} \neq 0$. We have proved

Theorem 2 *Consider the mechanical system consisting of four heavy Lagrange tops constrained in accordance with* $\boldsymbol{h}_1$ *and satisfying* $\boldsymbol{h}_{4-7}$, $\boldsymbol{h}_{11}$, $\boldsymbol{h}_{12}$. *Then each set* $\mathcal{D}$, $mes\mathcal{D} < \infty$, *in the phase space of the system containes a subset* $\mathcal{D}'$ *such that* $\mathcal{D}' = mes\mathcal{D}$ *and each point of* $\mathcal{D}'$ *gives rise to coisotropic quasi-periodic motion.*

7 Perturbation problem

The question arise whether the above quasi-periodic motions persist under small perturbations of the system. The answer can be given in a framework of KAM-theory after applying the results of [20]. We consider the case when perturbations may break the symmetry properties of rigid bodies, of scalar and vector potentials, but the total gyroscopic force form remains to be close. Moreover, at first we suppose that the restrictions of perturbed and unperturbed gyroscopic force forms to at least one (and then to each) orbit of T^4-action represent the same cohomology class. In this case, replacing $\boldsymbol{h}_{11}$ by somewhat stronger requirement

$$|c_{12}k_1 + c_{13}k_2 + c_{23}k_3| > \gamma \left(\sum |k_i|\right)^{-3} \qquad \forall (k_1, k_2, k_3) \in \mathbf{Z}^3 \setminus \{0\}, \tag{9}$$

with γ being a positive number, one can apply the results of [20] to prove KAM-like theorem on persistance of coisotropic quasi-periodic motions. It should be observed here that in our case not only the Hamiltonian function but the symplectic structure, as well, is perturbed. This difficulty can be remedied if we note that the projections of the quasi-periodic motions obtained above does not pass through the poles of S^2. Thus $\mathcal{M}$ may be replaced by $\mathcal{M}_1 = \mathsf{p}^{-1}(S^2 \setminus \{\textit{neighbourhoods of the poles}\})$. The perturbed gyroscopic force form is represented on $\mathcal{M}_1$ as $\Gamma + d\vartheta$, a 1-form ϑ being small. Now one can kill such a perturbation by means of appropriate additional perturbation of the Hamiltonian.

Next, if the above hypothesis on the cohomology classes fails then we shall think of c_{12}, c_{13}, c_{23} as parameters which range in some bounded domain $\mathcal{G} \in \mathbf{R}^3$. These parameters uniquely determine a skew-symmetric matrix of 4-th order satisfying $\boldsymbol{h}_{5-6}$. Let $\mathcal{G}' \in \mathcal{G}$ be a subset for which (9) holds. Then $mes(\mathcal{G} \setminus \mathcal{G}')$ vanishes together with γ. Now we average the perturbed gyroscopic force form by T^4-action and take the vertical part of the obtained 2-form to get 2-form $\mathcal{C}_\epsilon(\boldsymbol{\omega}, \boldsymbol{\omega}) = \sum_{1 \le i < j \le 4} c_{\epsilon ij} \omega_i \wedge \omega_j$. It turns out that $(1,1,1,1) \in Ker\mathcal{C}_\epsilon$. Indeed, there exist a connection form $\boldsymbol{\omega}_\epsilon$ on $\mathcal{M}$ with curvature form $\boldsymbol{\Xi}_\epsilon = d\boldsymbol{\omega}_\epsilon$ taking values in $Ker\mathcal{C}_\epsilon$ (see Proposition 1 in [6]). Since $\mathcal{C}_\epsilon$ is close to $\mathcal{C}$ we may choose $\boldsymbol{\omega}_\epsilon$ close to $\boldsymbol{\omega}$. The 2-form $\boldsymbol{\Xi}_\epsilon$ drops to $\tilde{\boldsymbol{\Xi}}_\epsilon$ on S^2 which is an integer one as well as $\tilde{\boldsymbol{\Xi}}$. This yields $\int_{S^2} \tilde{\boldsymbol{\Xi}} = \int_{S^2} \tilde{\boldsymbol{\Xi}}_\epsilon = 4\pi(1,1,1,1) \in Ker\mathcal{C}_\epsilon$. Now we conclude that $\mathcal{C}_\epsilon$ is uniquelly determined by the vector $(c_{\epsilon 12}, c_{\epsilon 13}, c_{\epsilon 23})$. One may expect with probability close to 1 that the above vector falls into $\mathcal{G}'$. If this is the case we choose $\tilde{\boldsymbol{f}}_\epsilon(z)$ close to $\tilde{\boldsymbol{f}}(z)$ in such a way that

$$\boldsymbol{h}'_9: \qquad \sum_{j=1}^{4} \delta_j^{-1} c_{\epsilon ij} \tilde{f}_{\epsilon j}(z) = \text{const}, \qquad i = 1, 2.$$

Finally, when regarding the system with kinetic energy $\delta_0((\Omega_1)^2+(\Omega_2)^2)/2+(D\boldsymbol{\omega}_\epsilon,\boldsymbol{\omega}_\epsilon)/2$, potential U and gyroscopic force form $d(\mathsf{p}^*\tilde{\sigma})+d(\tilde{\boldsymbol{f}}_\epsilon\circ\mathsf{p},\boldsymbol{\omega}_\epsilon)+\mathcal{C}_\epsilon(\boldsymbol{\omega}_\epsilon,\boldsymbol{\omega}_\epsilon)$ as an unperturbed one we reduce the case under consideration to the above case of unperturbed cocycle.

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