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On Remarkable Reductions of the Nonlinear Dirac Equation

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Abstract

The three ansatzes are constructed for the nonlinear Dirac equation.

In a number of papers [1–3] we constructed exact solutions of the nonlinear Dirac equation

$$i\sum_{\mu=0}^{3}\gamma_\mu\psi_{x_\mu} - \lambda(\overline{\psi}\psi)^k\psi = 0, \qquad \lambda, k \subset R^1 \tag{1}$$

by reducing it to systems of ordinary differential equations.

In (1) $\gamma_\mu, \mu = \overline{0,3}$ are (4 x 4) Dirac matrices (see, e.g. [4]); $\psi = \psi(x_0, x_1, x_2, x_3)$ is a four-component complex function; $\overline{\psi} = \psi^+\gamma_0$; $k, \lambda \in R^1$.

But we found in [2] that it is quite useful to study reductions of the Dirac equation (1) to two-dimensional systems of partial differential equations (PDE). For this, one has to utilise symmetry properties of equation (1) and to construct ansatzes invariant under inequivalent two-parameter subgroups of the symmetry group of system (1) [4].

In the present paper we will show that three ansatzes obtained in the above mentioned manner reduce the nonlinear Dirac equation to integrable two-dimensional systems of PDE. Omitting intermediate computations we adduce ansatzes for the spinor field $\psi(x)$ and corresponding reduced equations:

1) $\psi = \varphi(\xi, w), \quad \xi = x_0 + x_3, \quad w = x_2,$

$$i(\gamma_0 + \gamma_3)\varphi_\xi + i\gamma_2\varphi_w = \lambda(\overline{\varphi}\varphi)^k\varphi, \tag{2}$$

2) $\psi = w^{-1/2}\exp\left(-\frac{1}{2}\arctan\frac{x_1}{x_2}\right)\varphi(\xi,w), \quad w = x_1^2 + x_2^2$,

$$i(\gamma_0+\gamma_3)\varphi_\xi + i\gamma_2\varphi_w = \lambda w^{-k}(\overline{\varphi}\varphi)^k\varphi; \tag{3}$$

3) $\psi = \xi^{-1/2}\exp\left(-\frac{1}{2}\gamma_1(\gamma_0+\gamma_3)x_1/(x_0+x_3)\right)\varphi(\xi,w), \quad w = x_2,$

$$i(\gamma_0+\gamma_3)\varphi_\xi + i\gamma_2\varphi_w = \lambda \xi^{-k}(\overline{\varphi}\varphi)^k\varphi. \tag{4}$$

Surprisingly enough, equations (2)–(4) proved to be integrable in closed form. Let us consider the system of PDE (2). We rewrite it in the following equivalent form:

$$\varphi_w = i\gamma_2 f_w\varphi + \gamma_2(\gamma_0+\gamma_3)F, \tag{5a}$$

$$\varphi_\xi = F, \qquad \lambda(\overline{\varphi}\varphi)^k = f_w. \tag{5b}$$

System (5a) can be considered as a system of ordinary differential equations with respect to the variable w. Its general solution is obtained by the method of variation of arbitrary constants

$$\varphi(\xi,w) = e^{if\gamma_2}[\chi(\xi) + \int_0^w e^{-if\gamma_2}\gamma_2(\gamma_0+\gamma_3)F dw], \tag{6}$$

where $\chi(\xi)$ is an arbitrary four-component complex function.

Since $(\gamma_0+\gamma_3)\varphi_\xi = (\gamma_0+\gamma_3)F$, we get from (6) the following relation:

$$(\gamma_0+\gamma_3)F = (\gamma_0+\gamma_3)e^{if\gamma_2}(\dot\chi + if_\xi\gamma_2\chi).$$

Substituting the obtained result into (6) yields

$$\varphi(\xi,w) = e^{if\gamma_2}[\chi + \gamma_2(\gamma_0+\gamma_3)\int_0^w e^{2if\gamma_2}(\dot\chi + if_\xi\gamma_2\chi)dw]. \tag{7}$$

The only thing left is to substitute (7) into the second equation from (5b). As a result, we get an integrodifferential equation for $f = f(\xi,w)$

$$f_w = \lambda(A + B\int_0^w ch2f dw + C\int_0^w sh2f dw)^k, \tag{8}$$

where

$$A = \overline{\chi}\chi, \qquad B = \overline{\chi}\gamma_1(\gamma_0+\gamma_3)\dot\chi - \dot{\overline{\chi}}\gamma_1(\gamma_0+\gamma_3)\chi,$$
$$C = i(\overline{\chi}(\gamma_0+\gamma_3)\dot\chi - \dot{\overline{\chi}}(\gamma_0+\gamma_3)\chi). \tag{9}$$

In the same way one establishes that general solutions of systems of PDE (3), (4) is given by the formula (7), where $f = f(\xi,w)$ is a solution of integrodifferential equations

$$w^k f_w = \lambda(A + B\int_0^w ch2f dw + C\int_0^w sh2f dw)^k,$$

$$\xi^k f_w = \lambda(A + B\int_0^w ch2f dw + C\int_0^w sh2f dw)^k, \tag{10}$$

correspondingly.

We have succeeded in integrating equations (8), (10). Since the procedure of integration demands rather tedious computations, we adduce only the final solution of equation (8). Its explicit form depends on relations between B and C, so one has to consider four inequivalent cases.

Case 1 $B = \pm C, \quad B \neq 0$

a) $k \neq -1, \quad f = \pm\frac{1}{2}ln(\varepsilon + \frac{2\lambda}{(k+1)B}(A + Bg)^{k+1}),$

$$\int_0^{g(\xi,w)} \left[\varepsilon \pm \frac{2\lambda}{(k+1)B}(A + B\tau)^{k+1}\right]^{-1} d\tau = w;$$

b) $k = -1, \quad f = \pm\frac{1}{2}[ln(\varepsilon + \frac{2\lambda}{B}ln(A + Bg)],$

$$\int_0^{g(\xi,w)} \left[\varepsilon \pm \frac{2\lambda}{B}ln(A + B\tau)\right]^{-1} d\tau = w;$$

Case 2 $B^2 > C^2 <=> B = \alpha(\xi)ch2\beta(\xi), \quad C = \alpha(\xi)sh2\beta(\xi)$

a) $k \neq -1, \quad f = -\beta + \frac{1}{2}arch[1 + (\varepsilon + \frac{2\lambda}{\alpha(k+1)}(A + \alpha g)^{k+1})^2]^{1/2},$

$$\int_0^{g(\xi,w)} \left[1 + (\varepsilon + \frac{2\lambda}{\alpha(k+1)}(A + \alpha\tau)^{k+1})^2\right]^{-1/2} d\tau = w;$$

b) $k = -1, \quad f = -\beta + \frac{1}{2}arch[1 + (\varepsilon + \frac{2\lambda}{\alpha}ln(A + \alpha g))^2]^{1/2},$

$$\int_0^{g(\xi,w)} \left[1 + (\varepsilon + \frac{2\lambda}{\alpha}ln(A + \alpha\tau))^2\right]^{-1/2} d\tau = w;$$

Case 3 $B^2 < C^2 <=> B = \alpha(\xi)sh2\beta(\xi), \quad C = \alpha(\xi)ch2\beta(\xi)$

a) $k \neq -1, \quad f = -\beta + \frac{1}{2}arsh[-1 + (\varepsilon + \frac{2\lambda}{\alpha(k+1)}(A + \alpha g)^{k+1})^2]^{1/2},$

$$\int_0^{g(\xi,w)} \left[-1 + (\varepsilon + \frac{2\lambda}{\alpha(k+1)}(A + \alpha\tau)^{k+1})^2\right]^{-1/2} d\tau = w;$$

b) $k = -1, \quad f = -\beta + \frac{1}{2}arsh[-1 + (\varepsilon + \frac{2\lambda}{\alpha}ln(A + \alpha g))^2]^{1/2},$

$$\int_0^{g(\xi,w)} \left[-1 + (\varepsilon + \frac{2\lambda}{\alpha}ln(A + \alpha\tau))^2\right]^{-1/2} d\tau = w;$$

Case 4 $B = C = 0, \quad f = \lambda A^k w$.

In the above formulae $\varepsilon = 0, \pm 1$. Substituting the results obtained into (7) we obtain the general solution of system (2).

Equation (10) can also be integrated but we omit corresponding formulae.

Substitution of (7) into one of the ansatzes for the spinor field $\psi(x)$ from (2)–(4) yields exact solutions of the nonlinear Dirac equation (1) which contain four arbitrary complex functions on $\xi = x_0 + x_3$. Let us remind that solutions of system of PDE (1) obtained by symmetry reduction to systems of ordinary differential equations do not contain arbitrary functions [3, 4].

References

[1] Fushchych W.I., Zhdanov R.Z., On some exact solutions of a system of nonlinear differential equations for spinor and vector fields, *J.Phys. A: Math. Gen.*, 1987, V.20, N 13, 4173–4190.

[2] Fushchych W.I., Zhdanov R.Z., Symmetry and exact solutions of the nonlinear Dirac equation, *Fizika Elem. Chastits i Atomn. Yadra,* 1988, V.19, N 5, 1154–1196.

[3] Fushchych W.I., Zhdanov R.Z., Symmetry and exact solutions of non-linear spinor equations, *Phys. Reports.*, 1989, V.172, N 4, 123–174.

[4] Fushchych W.I., Zhdanov R.Z. Nonlinear Spinor Equations: Symmetry and Exact Solutions, Kiev, Naukova Dumka, 1992.