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Differential Equations for the Components of the Canonical Chua's Circuit and its General Properties

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Abstract

Three third-order differential equations are obtained for the canonical system. These equations are equivalent to the initial system and characterize its components. The first equation is Lipschitz continuous, the second one has discontinuous right hand side and the third one is an equation with impulse action. The equation in variations which corresponds to solutions of the first equation is investigated; analysis of its solutions is given; conditions for presence of periodic solutions for Chua's system are discussed. The canonical form for the general system of Chua's type is obtained.

We will consider equations for the canonical Chua's circuit [1]

$$\begin{aligned} \frac{dv_1}{dt} &= \frac{1}{C_1}\left[\frac{v_2-v_1}{R} - f(v_1)\right], \qquad \frac{dv_2}{dt} = \frac{1}{C_2}\left[\frac{v_1-v_2}{R} + i_3\right], \\ \frac{di_3}{dt} &= \frac{1}{L}[-v_2 - R_0 i_3], \end{aligned} \tag{1}$$

where $f(v_1)$ is a piesewice-linear function of the form

$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)[|v_1 + B_p| - |v_1 - B_p|], \tag{2}$$

$C_1, C_2, L, R, R_0, G_a, G_b, B_p$ are the constants of the circuit.

In terms of dimensionless variables equations (1) take the form

$$\frac{dx}{d\tau} = \alpha[y - x - f(x)], \qquad \frac{dy}{d\tau} = x - y + z, \qquad \frac{dz}{d\tau} = -\beta y - \gamma z, \tag{3}$$

where

$$f(x) = \begin{cases} bx + a - b & \text{for} \quad x \geq 1, \\ ax & \text{for} \quad |x| \leq 1, \\ bx - a + b & \text{for} \quad x \leq -1, \end{cases} \tag{4}$$

α, β, γ and a, b are the parameters of the system, τ is the dimensionless time. All these parameters (including the time) are naturally connected with that of Chua's circuit.

Let us rewrite $f(x)$ in the form

$$f(x) = ax + (b - a)f_1(x),$$

where

$$f_1(x) = \begin{cases} x - 1 & \text{for} \quad x \geq 1, \\ 0 & \text{for} \quad |x| \leq 1, \\ x + 1 & \text{for} \quad x \leq -1, \end{cases} \tag{5}$$

so system (3) will take the form

$$\frac{dx}{d\tau} = \alpha[y - \delta x + m f_1(x)], \quad \frac{dy}{d\tau} = x - y + z, \quad \frac{dz}{d\tau} = -\beta y - \gamma z, \tag{6}$$

where

$$\delta = 1 + a, \quad m = a - b. \tag{7}$$

Variables v_1, v_2, i_3 (respectively x, y, z) in equations (1) (respectively (6)) will be called the components of the canonical Chua's circuit. For the case $\gamma = 0$ system (6) is thoroughly studied (see [2–8]). For the case $\beta = 0$ the equation for z is split up and system (6) is easily integrated on the invariant manifold $z = 0$. For $\alpha m = 0$ system (6) is linear so we will suppose that

$$\alpha\beta m \neq 0. \tag{8}$$

Under this assumption we will consider separately the equations for the components of the canonical Chua's circuit.

1. Differential equations

One can see from the form of system (6), that the solution $x(t) = (x(\tau), y(\tau), z(\tau))$ has components of different order of smoothness: $x(\tau) \in$

$C^1(\mathbb{R}), y(\tau) \in C^2(\mathbb{R}), z(\tau) \in C^3(\mathbb{R})$, where $C^r(\mathbb{R})$ stands for the space of r times continuously differentiated on $\mathbb{R}$ functions. This implies specific character of differential equations for the components of the circuit. Let us obtain these equations.

Equation for $z = -\beta z_1$

From (6) we have

$$y = \dot{z}_1 + \gamma z_1 \stackrel{\Delta}{=} l_1(z_1), \quad x = \dot{y} + y + \beta z_1 = \ddot{z}_1 + (\gamma+1)\dot{z}_1 + (\beta+\gamma)z_1 =$$

$$\ddot{z}_1 + (\gamma+1)\dot{z}_1 + \beta_1 z_1 \stackrel{\Delta}{=} l(z_1), \tag{9}$$

where $\beta_1 = \beta + \gamma$. This leads to the equation for z_1

$$L[z_1] \stackrel{\Delta}{=} l(\dot{z}_1) + \alpha[\delta l(z_1) - l_1(z_1)] = \alpha m f_1(l(z)). \tag{10}$$

Equation for y

From (6) we have

$$z = \dot{y}+y-x, \quad \ddot{y}+\dot{y}-\dot{x}+\beta y+\gamma(\dot{y}+y-x) = 0, \quad \dot{x} = \alpha[y-\delta x+m f_1(x)], \tag{11}$$

that leads to the system of equations for x, y in the form

$$l(y) = l_1(x) = \alpha y + (\gamma - \alpha\delta)x + \alpha m f_1(x). \tag{12}$$

Let us suppose that

$$\gamma_1 = \gamma - \alpha\delta \tag{13}$$

and let us consider the following cases.
CASE 1 Let us suppose that

$$\gamma_1 \neq 0, \quad 1 + \frac{\alpha m}{\gamma_1} > 0. \tag{14}$$

Then system (12) will take the form

$$\frac{l(y) - \alpha y}{\gamma_1} = \frac{l_1(x) - \alpha y}{\gamma_1} = x + \frac{\alpha m}{\gamma_1} f_1(x). \tag{15}$$

Taking one of equations from (15) one can obtain the expressions for x:

$$x = \frac{l(y) - \alpha y}{\gamma_1} \qquad \text{for} \qquad |x| \leq 1, \tag{16}$$

$$x = \left(1 + \frac{\alpha m}{\gamma_1}\right)^{-1} \left(\frac{l(y) - \alpha y}{\gamma_1} - 1\right) + 1 \quad \text{for} \quad x \geq 1, \tag{17}$$

$$x = \left(1 + \frac{\alpha m}{\gamma_1}\right)^{-1} \left[\frac{l(y) - \alpha y}{\gamma_1} + 1\right] - 1 \quad \text{for} \quad x \le -1. \tag{18}$$

After the substitution of (16) into another equation from (15) which we will denote by (15*) we obtain an equation for y:

$$l_1 \left(\frac{l(y) - \alpha y}{\gamma_1}\right) = l(y),$$

which is possible to transform to the following form

$$L[y] = 0 \quad \text{for} \quad |\frac{l(y) - \alpha y}{\gamma_1}| \le 1. \tag{19}$$

After the substitution of (17) into (15*) we obtain the equation for y:

$$l_1 \left[\left(1 + \frac{\alpha m}{\gamma_1}\right)^{-1} \left(\frac{l(y) - \alpha y}{\gamma_1} - 1\right) + 1\right] = l(y),$$

which is possible to transform to the following form

$$L[y] = \alpha m(l(y) - \gamma) \quad \text{for} \quad \frac{l(y) - \alpha y}{\gamma_1} \ge 1. \tag{20}$$

After the substitution of (18) into (15*) we obtain the equation for y:

$$L[y] = \alpha m(l(y) + \gamma) \quad \text{for} \quad \frac{l(y) - \alpha y}{\gamma_1} \le -1. \tag{21}$$

Thus the system of equations (15) is equivalent to the system of equations (18) – (21). In order to obtain the equation for y from (18) – (21) we will consider equations (18) – (21) as the formulas for the substitution which replace variables $\ddot{y}$ for x in the corresponding domains $|x| \le 1,\ x \ge 1,\ x \le -1$ of the phase space $x, y, \dot{y}$ of system (15). Under this change the domain $|x| \le 1,\ x \ge 1,\ x \le -1$ of the space $x, y, \dot{y}$ is transformed into the domain

$$|\frac{l(y) - \alpha y}{\gamma_1}| \le 1, \quad \frac{l(y) - \alpha y}{\gamma_1} \ge 1, \quad \frac{l(y) - \alpha y}{\gamma_1} \le -1 \tag{22}$$

of the space of the variables $y, \dot{y}, \ddot{y}$. Domains (22) are intersected by the planes which correspond to boundaries

$$\Gamma_{-1} : \frac{l(y) - \alpha y}{\gamma_1} = -1 \quad \text{and} \quad \Gamma_1 : \frac{l(y) - \alpha y}{\gamma_1} = 1. \tag{23}$$

Exterior to these planes equations (19) – (21) uniquely define vector fields for $y, \dot{y}, \ddot{y}$. Equations (19) – (21) define two vector fields at the points Γ_{-1} and Γ_1. The right hand side of system (15) satisfies the Lipshits condition and thus the vector fields at the points Γ_{-1} and Γ_1 have the same direction with respect to Γ_{-1} and Γ_1. Omitting one of the vector fields at Γ_{-1} and Γ_1 we can define uniquely the vector field for $y, \dot{y}, \ddot{y}$ in the hole space. Hence we will get the following equation for y

$$\begin{aligned} L[y] &= 0 && \text{for} \quad |\tfrac{l(y)-\alpha y}{\gamma_1}| \leq 1, \\ L[y] &= \alpha m(l(y)-\gamma) && \text{for} \quad \tfrac{l(y)-\alpha y}{\gamma_1} > 1, \\ L[y] &= \alpha m(l(y)+\gamma) && \text{for} \quad \tfrac{l(y)-\alpha y}{\gamma_1} < -1. \end{aligned} \tag{24}$$

Let us denote by χ_1 and σ_1 the following functions:

$$\chi_1(x) = \begin{cases} 1 & \text{for} \quad |x| > 1, \\ 0 & \text{for} \quad |x| \leq 1, \end{cases} \qquad \sigma_1(x) = \begin{cases} 1 & \text{for} \quad x > 1, \\ 0 & \text{for} \quad |x| \leq 1, \\ -1 & \text{for} \quad x < -1. \end{cases}$$

Then equations (24) for y will take the form

$$L[y] = \alpha m \left[\chi_1 \left(\frac{l(y) - \alpha m}{\gamma_1} \right) - \gamma \sigma_1 \left(\frac{l(y) - \alpha y}{\gamma_1} \right) \right]. \tag{25}$$

CASE 2 Let us suppose that

$$\gamma_1 \neq 0, \quad 1 + \frac{\alpha m}{\gamma_1} < 0. \tag{26}$$

The system of equations (12) has the form of the system (15).

Formulas (16) – (18) express x in terms of y and define the equation for y which is analogous to (19) – (21). Formula (16) being used as the change of variables replacing x for $\ddot{y}$ transforms the domain x of the phase space $x, y, \dot{y}$ of the system (15) into the domain

$$|\frac{l(y) - \alpha y}{\gamma_1}| \leq 1 \tag{27}$$

of the space $y, \dot{y}, \ddot{y}$. Formula (17) being used as the change of variables replacing x for $\ddot{y}$ transforms the domain $x \geq 1$ of the phase space $x, y, \dot{y}$ of system (15) into the domain Π_1

$$\frac{l(y) - \alpha y}{\gamma_1} \leq 1 \tag{28}$$

of the space $y, \dot{y}, \ddot{y}$. Formula (18) transforms the domain $x \geq 1$ into the domain Π_{-1}

$$\frac{l(y) - \alpha y}{\gamma_1} \geq -1 \tag{29}$$

of the space $y, \dot{y}, \ddot{y}$. One can see from (27) – (29) that the domain Π_0 is the common part of the domains Π_0, Π_1, Π_{-1}. We will construct a three-sheeted domain of the space of $y, \dot{y}, \ddot{y}$ formed by sewing together of the domains Π_{-1} and Π_0 with the help of the plane Γ_{-1}, and the domains Π_1 and Π_0 with the help of the plane Γ_1. According to (24) the differential equation for y in this domain takes the form

$$\begin{array}{lll} L[y]=0 & \text{for} & |\frac{l(y)-\alpha y}{\gamma_1}|\le 1, \\ L[y]=\alpha m(l(y)-\gamma) & \text{for} & \frac{l(y)-\alpha y}{\gamma_1}<1, \\ L[y]=\alpha m(l(y)+\gamma) & \text{for} & \frac{l(y)-\alpha y}{\gamma_1}>-1. \end{array} \tag{30}$$

Equations (30) define uniquely the vector field exterior to the domain Π_0 along with three vector fields corresponding to each sheet. The motion starting from the lower sheet (defined by the third equation (30)) to the upper sheet (defined by the second equation (30)) will necessarily pass through the middle sheet (defined by the first equation (30)). In the case under consideration the trajectories of system (30) can form the mode in Π_0.

CASE 3. Let us suppose that

$$\frac{\alpha m}{\gamma_1}=-1. \tag{31}$$

System of equations (15) takes the form

$$\frac{l(y)-\alpha y}{\gamma_1}=\frac{l_1(x)-\alpha y}{\gamma_1}=x-f_1(x)=\begin{cases} 1 & \text{for} \quad x\ge 1, \\ x & \text{for} \quad |x|\le 1, \\ -1 & \text{for} \quad x\le -1 \end{cases} \tag{32}$$

and in the domain Π_0 leads to the equation

$$L[y]=0 \quad \text{for} \quad |\frac{l(y)-\alpha y}{\gamma_1}|\le 1. \tag{33}$$

On the boundary of the domain Π_0 we will have

$$\frac{l(y)-\alpha y}{\gamma_1}=1 \quad \text{on} \quad \Gamma_1, \tag{34}$$

$$\frac{l(y)-\alpha y}{\gamma_1}=-1 \quad \text{on} \quad \Gamma_{-1}. \tag{35}$$

Since the vector field of the system defined on the planes Γ_1 and Γ_{-1} belong to these planes the sliding modes in the system are possible after the moment than the solution reaches these planes [9]. To obtain the equation of motion for y in this case we should add conditions for slopping of the sliding mode. That means that we should add the conditions which will guarantee leaving of the planes Γ_1 and Γ_{-1} for the domain Π_0. To obtain them we will use the equations for x from system (32). This leads to the equations for y of the form

$$\begin{array}{lll} L[y]=0 & \text{for} & |\frac{l(y)-\alpha y}{\gamma_1}|<1, \\ \frac{l(y)-\alpha y}{\gamma_1}=1 & \text{for} & \Phi(y,\dot y,y_0,\dot y_0,c)\ge 1, \\ \frac{l(y)-\alpha y}{\gamma_1}=1 & \text{for} & \Phi(y,\dot y,y_0,\dot y_0,c)\le -1, \end{array} \tag{36}$$

where Φ is a known function; $y_0, \dot{y}_0$ are the values of $y(\tau), \dot{y}(\tau)$ at the point $c = 1$ in the moment $\tau = \tau_1$ which corresponds to the instant at which the trajectory will come out of the domain Π_0 to the boundary Γ_1 and for $c = -1$ to the boundary Γ_{-1}; for $c > 1$ they will be the initial values of the trajectories remaining on Γ_1 for $\tau \in R$ (remaining on Γ_1 for $\tau \in R, \quad c < -1$).

CASE 4 Let us suppose that

$$\gamma_1 = 0. \tag{37}$$

System (20) takes the form

$$\frac{l(y) - \alpha y}{\alpha m} = \frac{l_1(x) - \alpha y}{\alpha m} = f_1(x) = \begin{cases} x - 1 & \text{for} \quad x \geq 1, \\ 0 & \text{for} \quad |x| \leq 1, \\ x + 1 & \text{for} \quad x \leq -1 \end{cases} \tag{38}$$

and can be reduced to the equations in Π_1 and Π_{-1} having the form

$$\begin{array}{ll} L[y] = \alpha m(l(y) - \gamma) & \text{for} \quad \frac{l(y)-\alpha y}{\alpha m} \geq 0, \\ L[y] = \alpha m(l(y) + \gamma) & \text{for} \quad \frac{l(y)-\alpha y}{\alpha m} \leq 0, \end{array} \tag{39}$$

along with the equation $\frac{l(y)-\alpha y}{\alpha m} = 0$ on the boundary Γ_0 of the domains Π_1 and Π_{-1}.

Analogously to Case 2 both equations (39) define the vector field on the plane Γ_0. We will form a three-sheeted domain of the space $y, \dot{y}, \ddot{y}$ by sewing together of the domains Π_1 and Π_{-1} with the help of the plane Γ_0. In this domain the differential equation for y takes the form

$$\begin{array}{lll} L[y] = \dot{\alpha} m(l(y) - \gamma) & \text{for} & \frac{l(y)-\alpha y}{\alpha m} > 0, \\ L[y] = \alpha m(l(y) + \gamma) & \text{for} & \frac{l(y)-\alpha y}{\alpha m} < 0, \\ \frac{l(y)-\alpha y}{\alpha m} = 0 & \text{for} & |F(y, \dot{y}, y_0, \dot{y}_0, c)| \leq 1 \end{array}$$

and the condition given by the function F has the same meaning as in Case 3.

We should note that all equations for y, except Case 3, are homogeneous for $\gamma = 0$.

Equation for x

System (6) implies

$$y = \frac{\dot{x}}{\alpha} + \delta x - m f_1(x), \quad z = \frac{\ddot{x} + \dot{x}}{\alpha} + \delta(\dot{x} + x) - x +$$

$$(-m)[f_1(x) + \chi_1(x)\dot{x}], \quad \dot{z} = -\beta y - \gamma z.$$

So for $|x| \neq 1$ we will get the equation for x of the form

$$\frac{\dddot{x} + \ddot{x}}{\alpha} + \delta(\ddot{x} + \dot{x}) - \dot{x} - m\chi_1(x)(\ddot{x} + \dot{x}) + \frac{\gamma}{\alpha}(\ddot{x} + \dot{x}) + \gamma\delta(\dot{x} + x) -$$

$$\gamma x - m\gamma[f_1(x) + \chi_1(x)\dot{x}] + \frac{\beta}{\alpha}\dot{x} + \beta(\delta x - mf_1(x)) = 0$$

and thus we obtain

$$L[y] = \alpha m[\chi_1(x)l(x) - \beta_1\sigma_1(x)] \quad \text{for} \quad |x| \neq 1. \tag{40}$$

We should add to equation (40) conditions for a jump of the function $\ddot{x}$ which it will get passing through the value $|x| = 1$. To find these conditions we obtain the equality

$$\ddot{x}(\tau) = \alpha[\dot{y}(\tau) - \delta\dot{x}(\tau) + m\chi_1(x(\tau)\dot{x}(\tau)] \tag{41}$$

from which we find the value of the jump $\Delta\ddot{x}|_{\tau=\tau_1} = \ddot{x}(\tau_1 + 0) - \ddot{x}(\tau_1 - 0)$ for $x(\tau_1 - 0) < -1, \;\; x(\tau_1) = -1, \;\; x(\tau_1 + 0) > 1$

$$\Delta\ddot{x}|_{\tau=\tau_1} = \alpha m\dot{x}(\tau)[\chi_1(x(\tau_1 + 0)) - \chi_1(x(\tau_1 - 0))] = -\alpha m\dot{x}(\tau_1). \tag{42}$$

For $x(\tau_1 - 0) > -1, \;\; x(\tau_1) = -1, \;\; x(\tau_1 + 0) < -1$ (41) implies

$$\Delta\ddot{x}|_{\tau=\tau_1} = \alpha m\dot{x}(\tau_1). \tag{43}$$

Since for the case (42) we have $\dot{x}(\tau_1) \geq 0$ and for (43) conditions (42), (43) will take the form

$$\Delta\ddot{x}|_{x=-1} = (-1)\alpha m|\dot{x}|, \tag{44}$$

where $\dot{x}$ stands for the value of the function $\dot{x}(\tau)$ at the point $\tau = \tau_1$ such that $x(\tau_1) = -1$.

For $x(\tau_2 - 0) < 1, \;\; x(\tau_2) = 1, \;\; x(\tau_2 + 0) > 1$ (41) implies

$$\Delta\ddot{x}|_{\tau=\tau_2} = \alpha m\dot{x}(\tau_2), \tag{45}$$

for $x(\tau_2 - 0) > 1, \;\; x(\tau_2) = 1, \;\; x(\tau_2 + 0) < 1$ (41) implies

$$\Delta\ddot{x}|_{\tau=\tau_2} = -\alpha m\dot{x}(\tau_2). \tag{46}$$

Taking into account the signs of $\dot{x}(\tau_2)$ we can write conditions (53), (54) in the form

$$\Delta\ddot{x}|_{x=1} = \alpha m|\dot{x}|, \tag{47}$$

where $\dot{x}$ stands for the value of the function $\dot{x}(\tau)$ at the point $\tau = \tau_2$ such that $x(\tau_2) = 1$.

One can combine the formulas (46), (47) into the condition

$$\Delta\ddot{x}|_{|x|=1} = \alpha m|\dot{x}| \text{ sign } x,$$

where variables x and $\dot{x}$ have the meaning defined above.

In accordance with the notations of [10] we obtain for x the equation with the impulse action – the dynamic system with an impulse action of the form

$$L[x] = \alpha m[\chi_1(x)l(x) - \beta_1\sigma_1(x)] \quad \text{for} \quad |x| \neq 1,$$

$$\Delta\ddot{x}|_{|x|=1} = \alpha m|\dot{x}| \text{ sign } x. \tag{48}$$

Trajectories of this system have discontinuities on the planes $|x| = 1$ with the jumps directed along the $\ddot{x}$ axis and with the values $\alpha m|\dot{x}|_{\text{sign } x}$ at the points of a discontinuity.

At the case $\beta_1 = 0$ the equation for x is homogeneous.

2. Relationship between the coefficients and the roots of the characteristic equation

A characteristic equation for the differential equation of components of the canonical Chua's circuit has the form:

in the strip Π_0

$$L[\lambda] \stackrel{\Delta}{=} \lambda l(\lambda) + \alpha[\delta l(\lambda) - l_1(\lambda)] = 0, \tag{49}$$

exterior to this strip

$$L[\mu] = \alpha m l(\mu). \tag{50}$$

Let us denote by p_1, p_2, p_3 and q_1, q_2, q_3 coefficients of the equations (49), (50). Thus we obtain six equations for five parameters of the circuit:

$$p_1 = \gamma + 1 + \delta_1, \quad p_2 = \beta_1 + \delta_1(\gamma + 1) - \alpha, \quad p_3 = \beta_1\delta_1 - \alpha\gamma_1,$$

$$q_1 = p_1 - m_1, \quad q_2 = p_2 - m_1(\gamma + 1), \quad q_3 = p_3 - m_1\beta_1, \tag{51}$$

where we denote

$$\delta_1 = \alpha\delta, \quad m_1 = \alpha m.$$

These parameters satisfy condition (1.8) and this implies the inequality

$$p_1 - q_1 \neq 0. \tag{52}$$

In spite of the inequality (52) it is possible to solve (51) in the form

$$m_1 = p_1 - q_1, \quad \gamma + 1 = \frac{p_2 - q_2}{p_1 - q_1}, \quad \beta_1 = \frac{p_3 - q_3}{p_1 - q_1},$$

$$\delta_1 = p_1 - \frac{p_2 - q_2}{p_1 - q_1}, \quad \alpha = -p_2 + \frac{p_2 - q_2}{p_1 - q_1}\left(p_1 - \frac{p_2 - q_2}{p_1 - q_1}\right) + \frac{p_3 - q_3}{p_1 - q_1}, \tag{53}$$

and we obtain the relationship between $p_v,\ q_v$:

$$p_3 = \left(p_1 + 1 - 2\,\frac{p_2 - q_2}{p_1 - q_1}\right)\frac{p_3 - q_3}{p_1 - q_1} + \left(1 - \frac{p_2 - q_2}{p_1 - q_1}\right) \times$$

$$\left[-p_2 + \left(p_1 - \frac{p_2 - q_2}{p_1 - q_1}\right)\frac{p_2 - q_2}{p_1 - q_1}\right]. \tag{54}$$

For

$$p_1 + 1 \neq 2\,\frac{p_2 - q_2}{p_1 - q_1} \tag{55}$$

(this condition is equivalent to the following inequality for $\alpha, \beta, \gamma \quad \gamma_1 = \gamma - -\alpha\delta \neq 0$), one can solve equation (54) with respect to q_3. Otherwise relation (54) is equivalent to the system of equalities

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{p_1 + 1}{2}, \quad p_3 = \frac{p_1 - 1}{2}\left(p_2 + \frac{1 - p_1^2}{4}\right) \tag{56}$$

and equations (38) imply

$$\lambda_1 = -\gamma, \quad \lambda_? = -\frac{\gamma + 1}{2} \pm \sqrt{\frac{(\gamma + 1)^2}{4} + \alpha - \beta}.$$

Thus an arbitrary collection of parameters which satisfy conditions (52), (54) defines the canonical Chua's circuit.

Equality (54) can be used not only for the coefficients but for the roots of characteristic equations $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$. In that case (54) has the form

$$\alpha(\mu_1, \mu_3)\mu_1^3 + \alpha(\mu_1, \mu_3)\mu_2^3 + \alpha(\mu_1, \mu_2)\mu_3^3 + F(\mu_1, \mu_2, \mu_3) = 0, \tag{57}$$

where F is a polynomial of degree less or equal to two with respect to $\mu_\nu, \quad \nu = \overline{1,3}$,

$$\alpha(\mu_1, \mu_2) = \frac{\mu_1^3 + \mu_2^3}{2} + \mu_1^2 + \mu_2^2 - 2\mu_1\mu_2(\mu_1 + \mu_2) - (p_1 - 1)\mu_1\mu_2 +$$
$$(p_1 + p_2)(\mu_1 + \mu_2) + (p_2 + p_3). \tag{58}$$

Condition (52) is equivalent to the following one:

$$\sum_{\nu=1}^{3}(\mu_\nu - \lambda_\nu) \neq 0. \tag{59}$$

In the six-dimensional space of parameters λ, μ equation (57) defines a surface each point of which determines parameters of the canonical Chua's circuit being subordinated to condition (59).

Under the condition

$$\alpha(\mu_1, \mu_2) + \frac{\mu_1^3 + \mu_2^3}{2} \neq 0, \tag{60}$$

equation (58) is cubic with respect to μ_3, so it defines real μ_3 in terms of λ, μ_1, μ_2 as well as the parameters of the circuit.

3. Equation in variations

Differentiating the equation for z_1 we obtain the equation in the form

$$L[\dot{z}_1] = m_1\chi_1(l(z_1))l(\dot{z}_1), \tag{61}$$

that is the equation in variations for the solution $z_1 = z_1(\tau)$ if we consider $\ddot{z}_1$ as a variation of $z_1(\tau)$.

Let I_T be a set of $\tau \in [0, T]$ such that

$$|l(z_1(\tau))| \leq 1 \quad \text{for} \quad \tau \in I_T, \tag{62}$$

J_T is the complement of I_T with respect to the interval $[0, T]$,

$$I = \lim_{T\to\infty} I_T, \quad J = R^+ \backslash I, \quad R^+ = [0, \infty).$$

If I and J do not contain the interval $\tau > \tau_0$ then $I = \overset{\infty}{\underset{\nu=1}{\vee}} I_\nu$, where

$$I_1 = [0, \tau_1], \quad I_2 = [\tau_1 + \theta_1,\ \tau_1 + \theta_1 + \tau_2],$$

$$I_{p+1} = \left[\sum_{\nu=1}^{p}(\tau_\nu + \theta_\nu),\ \sum_{\nu=1}^{p}(\tau_\nu + \theta_\nu) + \tau_{p+1}\right], ...$$

are intervals from R^+; $\delta_\nu,\ \theta_\nu$ are positive numbers,

$$J = R^+ \backslash I = \overset{\infty}{\underset{\nu=1}{\vee}} J_\nu,$$

where

$$J_1 = (\tau_1, \tau_1 + \theta_1), \quad J_2 = (\tau_1 + \theta_1 + \tau_2, \quad \tau_1 + \theta_1 + \tau_2 + \theta_2), ...$$

$$..., J_{p+1} = \left(\sum_{\nu=1}^{p}(\tau_\nu + \theta_\nu) + \tau_{p+1},\ \sum_{\nu=1}^{p+1}(\tau_\nu + \theta_\nu)\right),$$

are intervals from $R^+,\ p \geq 2$.

Let us suppose that

$$\chi(\tau) = \begin{cases} 0 & \text{for} \quad |l(\dot{z}_1(\tau))| \leq 1, \\ 1 & \text{for} \quad |l(\dot{z}_1(\tau))| > 1. \end{cases} \tag{63}$$

Using the function χ we rewrite equation (61) in the form

$$L[\dot{z}_1] = m_1\chi(\tau)l(\dot{z}_1). \tag{64}$$

The properties of the solution of (64) depend essentially on properties of the set I connected with its distribution on R^+. The following value serves as an integral characteristic for the function $\chi(\tau)$

$$S_{T_\nu} = \frac{1}{T_\nu}\int_0^{T_\nu} \chi(\tau)d\tau = \frac{T_\nu - \text{ mes } I_{T_\nu}}{T_\nu} = 1 - \frac{\text{mes } I_{T_\nu}}{T_\nu},$$

which under a proper choice of T_ν is equal to

$$S_p = 1 - \frac{\sum_{\nu=1}^p \tau_\nu}{\sum_{\nu=1}^p (\tau_\nu + \theta_\nu)} = \frac{1}{1 + \sum_{\nu=1}^p \tau_\nu / \sum_{\nu=1}^p \theta_\nu}. \tag{65}$$

If S_p has a limit for $p \to \infty$

$$S = \lim_{p\to\infty} S_p \tag{66}$$

then this limit can be interpreted as a probability of staying of the point $z_1 = z_1(\tau)$, $\dot z_1 = \dot z_1(\tau)$, $\ddot z_1 = \ddot z_1(\tau)$ at the moment τ in the domain $|l(z_1)| > 1$ of the phase space of system (10).

It is obvious that limit (66) exists for the periodic solution of equation (10) as well as for the quasi-periodic solution of system (6). For the closed invariant curve which contains a saddle point limit (66) is equal to 0 or 1, hence there exist periodic motions in Chua's system having values close to 0 and 1.

We will be interested in the existence of the limit (66) for the recurrent solution of system (6).

The fundamental matrix of solutions of equation (61) is represented explicitly in terms of roots of the characteristic equations λ and μ. If these roots are simple then the fundamental matrix of solutions of equation (61) has the following form

$$Z_{\tau_0}^{\tau} = e^{\lambda(\tau-\tau_0)} B^{-1}(\lambda) \quad \text{for} \quad \tau \in I_{p+1}, \quad \tau_0 = \sum_{\nu=1}^{p} (\tau_\nu + \theta_\nu)$$

and the matrix

$$Z_{\tau_0}^{\tau} = e^{\mu(\tau-\tau_0)} B^{-1}(\mu) \quad \text{for} \quad \tau \in J_{p+1}, \quad \tau_0 = \sum_{\nu=1}^{p} (\tau_\nu + \theta_\nu) + \tau_{p+1} \quad (Z_{\tau_0}^{\tau_0} = E),$$

where $e^{\lambda\tau}(e^{\mu\tau})$ stands for the Wronski matrix for the functions $e^{\lambda_1\tau}$, $e^{\lambda_2\tau}$, $e^{\lambda_3\tau}$ $(e^{\mu_1\tau}, e^{\mu_2\tau}, e^{\mu_3\tau})$; $B(\lambda)(B(\mu))$ stands for the Vandermonde matrix of numbers $\lambda_1, \lambda_2, \lambda_3$ (μ_1, μ_2, μ_3). Having in hand $Z_{\tau_0}^{\tau}$ one can easily write the fundamental matrix of solutions of equation (64):

$$Z_0^{\tau} = \begin{cases} e^{\lambda(\tau-t_p)} B^{-1}(\lambda) Z_0^{t_p} & \text{for} \quad \tau \in I_{p+1} = [t_p, t_p + \tau_{p+1}], \\ e^{\mu(\tau-t_p-\tau_{p+1})} B^{-1}(\mu) Z^{t_p+\tau_{p+1}} & \text{for} \quad \tau \in J_{p+1} = (t_p + \tau_{p+1}, t_{p+1}), \end{cases} \tag{67}$$

where $t_p = \sum\limits_{\nu=1}^{p} (\tau_\nu + \theta_\nu)$.

Let us suppose that

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_3 & -p_2 & -p_1 \end{pmatrix}, \qquad \begin{matrix} z(\tau, z_0) \overset{\Delta}{=} (z_1(\tau), \dot z_1(\tau) \ddot z(\tau)), \\ z_0 \overset{\Delta}{=} (z_1(0), \dot z_1(0) \ddot z(0)). \end{matrix}$$

Then $\dot z(0, z_0) = A_1 z_0$.

So $\dot{z}_1(\tau)$ is the solution of the equation in variations for $z_1(\tau)$ (64), then exists the equation

$$\dot{z}(\tau, z_0) = Z_0^\tau A_1 z_0. \tag{68}$$

After integrating (68) we find the expression for z

$$z(\tau, z_0) = (E + \int_0^\tau Z_0^s ds A_1) z_0. \tag{69}$$

Formula (69) gives the expression for the solution $z_1(\tau)$ of equation (10) in terms of the fundamental matrix of solutions of its equation in variations.

We would like to note that equation in variations (64) can be the same for several solutions of equation (10). For example, it is true for a couple of solutions of equation (10) $z = z(\tau_1 z_0)$ and $z = z(\tau_1 - z_0) = -z(\tau_1 z_0)$ the existence of which is guaranteed by the symmetry of the vector field of the canonical Chua's circuit with respect to the origin.

4. Necessary condition for the existence of periodic solutions

Let us consider equation (64) for the periodic solution $z = z(\tau)$ with the period T. Without the loss of generality we will assume that $z_0 = (z_1(0), \dot{z}_1(0), \ddot{z}_1(0))$ satisfies the condition

$$|(l, z_0)| = |\ddot{z}_1(0) + (\gamma + 1)\dot{z}_1(0) + \beta_1 z_1(0)| = 1$$

and for small $\tau > 0$, $z = z(\tau, z_0)$ satisfies the inequality $|(l, z(\tau, z_0))| < 1$. Then $\chi(\tau)$ has a discontinuity at the point $\tau = 0$ and

$$T = \sum_{\nu=1}^{p} (\tau_\nu + \theta_\nu), \quad p \geq 1. \tag{70}$$

Equations (67), (68) imply the conditions for periodicity of the solution $z_1 = z_1(\tau)$ in the form

$$(Z_0^T - E) A z_0 = 0, \quad \int_0^T Z_0^s ds A_1 z_0 = 0. \tag{71}$$

Matrix

$$Z_0^T = \prod_{\nu=0}^{p} e^{\mu\theta_\nu} B^{-1}(\mu) e^{\lambda\tau_\nu} B^{-1}(\lambda) \tag{72}$$

is a monodromy matrix for equation (64), the first its characteristic value is equal to 1, other two will determine the character of its stability or the critical state (bifurcation). When $\lambda_\nu \neq 0$, $\mu_\nu \neq 0$ $(\nu = 1,2,3)$ for integral (71) we have an expression

$$\int_0^T Z_0^s ds = \sum_{\nu=1}^{p} [A_1^{-1}(e^{\lambda\tau_\nu} B^{-1}(\lambda) - E) + B_1^{-1}(e^{\mu\theta_\nu} B^{-1}(\mu) - E) \times$$

$$e^{\lambda\tau_\nu} B^{-1}(\lambda)] \prod_{j=0}^{\nu=1} e^{\mu\theta_j} B^{-1}(\mu) e^{\lambda\tau_j} B^{-1}(\lambda), \tag{73}$$

where $B_1 = A_1 + m_1 \begin{pmatrix} 0 \\ l \end{pmatrix}$, $l = (\beta_1, \gamma+1, 1)$ is the vector, $e^{\mu\theta_0} B^{-1}(\mu) e^{\lambda\tau_0} \times B^{-1}(\lambda) = E$.

One can see from formulas (72), (73) that equations (71) are not reduced to each other and determine necessary conditions for the existence of periodic solutions for Chua's circuit. For $p = 1$ condition (71) is equivalent to the conditions

$$[e^{\mu\theta_1} B^{-1}(\mu) e^{\lambda\tau_1} B^{-1}(\lambda) - E] A_1 z_0 = 0,$$

$$(B_1 - A_1)[e^{\lambda\tau_1} B^{-1}(\lambda) - E] A_1 z_0 = 0.$$

We would like to note that for $p \geq 1$ the period T of the solution $z = z_1(\tau)$ can be multiple by the period of the function $\chi(\tau)$. This always takes place for the symmetric solution of Chua's equation which is determined by the condition $z(T_1, z_0) = -z_0$. This solution is periodic with the period $T = 2T$ but $\chi(\tau)$ has the period T_1. The inequality which connects the main characteristics of the periodic solution s, p and $\omega = 2\pi/T$ with the parameters of the equation can be obtained in the following way.

The solution of the equation in variations has the Lipshits continuous second order derivative then the series $\sum_{k\neq 0} |f_k| k^2$ converges, where f_k are Fourier coefficients for the function $\dot{z}_1 = \dot{z}_1(\tau)$. For these coefficients the following equality is valid

$$L[ik\omega] f_k = m_1 \sum_{k_1=0} l(ik_1\omega) f_{k_1} \chi_{k-k_1}, \quad k \neq 0,$$

where χ_k are Fourier coefficients

$$\chi_0 = s, \quad \chi_k = \frac{1}{T} \int_0^T \chi(\tau) e^{-ik\omega\tau} d\tau = \frac{1}{T} \sum_{\nu=1}^{p} \int_{J_\nu} e^{-ik\omega\tau} =$$

$$\frac{1}{Tk} \sum_{\nu=1}^{p} \frac{e^{-ik\omega\tau}|_{J_\nu}}{i\omega} = \frac{1}{\pi k} \sum_{\nu=1}^{p} \frac{e^{-ik\omega\tau}|_{J_\nu}}{2i} \quad \text{for} \quad k \neq 0.$$

The estimate

$$|\chi_k| \leq \frac{p}{\pi |k|} \quad \text{for} \quad k \neq 0$$

which is valid for χ_k implies

$$\left|\frac{L(ik\omega)}{l(ik\omega)}\right| \; |l(ik\omega)f_k| \leq |m_1| \sum_{k_1\neq 0} |l(ik_1\omega)f_{k_1}|s_{k_1-k}/|k-k_1|,$$

where $s_0 = s, \quad s_{k-k_1} = \frac{p}{\pi}$ for $k_1 \neq k$. Then

$$\left(\min_{k\neq 0}\left|\frac{L(ik\omega)}{l(ik\omega)}\right|\right)^2 \; |l(ik\omega)f_k|^2 \leq 2|m_1|^2 \sum_{k_1\neq 0} |l(ik\omega)f_{k_1}|^2 \frac{s^2_{k-k_1}}{(k-k_1)^2}$$

and taking the sum over k one can obtain

$$\left(\min_{k\neq 0}\left|\frac{L(ik\omega)}{l(ik\omega)}\right|\right)^2 \sum_{k\neq 0} |l(ik\omega)f_k|^2 \leq 2m_1^2 \sum_{k_1\neq 0} |l(ik_1\omega)f_{k_1}|^2 \sum_{k\neq 0} \frac{s^2_{k-k_1}}{(k-k_1)^2}.$$

The last inequality implies

$$\left(\min_{k\neq 0}\left|\frac{L(ik\omega)}{l(ik\omega)}\right|\right)^2 \leq 2m_1^2\left(s^2 + \frac{p^2}{\pi^2}\sum_{l=1}^{\infty}\frac{2}{l^2}\right) = 2m_1^2(s^2 + p^2/3)$$

and then the inequality takes place

$$\min_{k\neq 0}\left|\frac{L(ik\omega)}{l(ik\omega)}\right| \leq \sqrt{2}\;|m_1|\sqrt{s^2 + \frac{p^2}{3}} \tag{74}$$

which gives the relationship between s, p, ω and parameters of Chua's system.

Taking into account the expression for $L(\lambda)$ we can rewrite inequality (74) in the form:

$$\min_{k\neq 0}\left|ik\omega + \delta_1 - \alpha\frac{l_1(ik\omega)}{l(ik\omega)}\right| \leq \sqrt{2}\;|m_1|\sqrt{s^2 + \frac{p^2}{3}}.$$

The general system of equations in the form

$$\frac{dx}{dt} = ax + a_1y + a_2z + df_1(x), \quad \frac{dy}{dt} = b_1x + by + b_2z, \quad \frac{dz}{dt} = c_1x + c_2y + cz, \tag{75}$$

where $a, a_1, a_2, b, b_1, b_2, c, c_1, c_2$ and d are the parameters, $f_1(x)$ is function (5), has the properties which are similar to the ones mentioned above.

By the linear transformation of the unknown functions x, y, z and the independent variable t system (75) is reduced in the general case (except the cases when system (75) is reduced to the form (75) with $b_1 = c_1 = 0$ or $c_1 = c_2 = 0$) to the system in the following form:

$$\frac{dx}{d\tau} = \alpha x + \beta y + \gamma z + mf_1(x),$$

$$\frac{dy}{d\tau} = x + \delta z, \quad \frac{dz}{d\tau} = y + \eta z, \tag{76}$$

where $\alpha, \beta, \gamma, \delta, \eta$ and m are the new parameters.

It is obvious, that the canonical Chua's circuit can be obtained from system (76) for

$$\beta + \gamma = 0 \tag{77}$$

and that the general properties of system (76) are analogous to the general properties of the canonical Chua's circuit.

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