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q-Euler numbers and polynomials associated with p-adic q-integrals

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Abstract

The main purpose of this paper is to present a systemic study of some families of multiple q-Euler numbers and polynomials. In particular, by using the q-Volkenborn integration on $\mathbb{Z}_p$, we construct p-adic q-Euler numbers and polynomials of higher order. We also define new generating functions of multiple q-Euler numbers and polynomials. Furthermore, we construct Euler q-Zeta function.

1 Introduction

For any complex number z, it is well known that the familiar Euler polynomials $E_n(z)$ are defined by means of the following generating function, see Refs. [3, 5, 6, 9, 13]:

$$F(z,t) = \frac{2}{e^t+1}e^{zt} = \sum_{n=0}^{\infty} E_n(z)\frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

We note that, by substituting $z = 0$ into (1.1), $E_n(0) = E_n$ is the familiar n-th Euler number defined by [4, 5]

$$G(t) = F(0,t) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}, \quad (|t| < \pi).$$

By the meaning of the generalization of E_n, Frobenius-Euler numbers and polynomials are also defined by [16]

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}, \text{ and } \frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(u,x)\frac{t^n}{n!} \quad (u \in \mathbb{C} \text{ with } |u| > 1).$$

Over five decades ago, Calitz [2, 3] defined q-extension of Frobenius-Euler numbers and polynomials and proved properties analogous to those satisfied $H_n(u)$ and $H_n(u,x)$. Recently, Satoh [14, 15] used these properties, especially the so-called distribution relation for the q-Frobenius-Euler polynomials, in order to construct the corresponding q-extension of the p-adic measure and to define a q-extension of p-adic l-function $l_{p,q}(s,u)$.

Let p be a fixed odd prime in this paper. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$, denote the ring of rational integers, the ring of p-adic integers, the field of p-adic numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\nu_p(p)$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one speaks of q-extension, q can be regarded as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$; it is always clear from the context. If $q \in \mathbb{C}$, then one usually assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then one usually assumes that $|q-1|_p < p^{-\frac{1}{p-1}}$, and hence $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. In this paper, we use the below notation [6, 7, 8, 9, 10, 11, 14]

$$[x]_q = \frac{1-q^x}{1-q}, \quad (a:q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Note that $\lim_{q\to 1}[x]_q = x$ for any x with $|x|_p \le 1$ in the p-adic case. For a fixed positive integer d with $(p,d)=1$, set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, X_1 = \mathbb{Z}_p, X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$, (see Refs. [10, 11]). We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x,y) \to (a,a)$ [11]. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression [7, 8, 9, 11]

$$\frac{1}{[p^N]_q} \sum_{0\le j<p^N} q^j f(j) = \sum_{0\le j<p^N} f(j)\mu_q(j+p^N\mathbb{Z}_p),$$

which represents a q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit of those sums(as $n \to \infty$) if this limit exists. The q-Volkenborn integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_X f(x)d\mu_q(x) = \int_{X_d} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x)q^x.$$

Recently, we considered another construction of a q-Eulerian numbers, which are different than Carlitz's q-Eulerian numbers as follows [6, 12, 13]:

$$F_q(x,t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}.$$

Thus we have

$$E_{n,q} = E_{n,q}(0) = \frac{[2]_q}{(1-q)_n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{l+1}}, E_{n,q}(x) = \frac{[2]_q}{(1-q)_n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{l+1}} q^{lx},$$

where $\binom{n}{l}$ is a binomial coefficient [13].

Note that $\lim_{q\to 1} E_{n,q} = E_n$ and $\lim_{q\to 1} E_{n,q}(x) = E_n(x)$. In Ref. [12], we also proved that q-Eulerian polynomial $E_{n,q}(x)$ can be represented by q-Volkenborn integral as follows:

$$\int_{X_d} [x + x_1]_q^k d\mu_{-q}(x_1) = \int_{\mathbb{Z}_p} [x + x_1]_q^k d\mu_{-q}(x) = E_{k,q}(x), \quad \text{for } k, d \in \mathbb{N},$$

where $\mu_{-q}(x + p^N\mathbb{Z}_p) = \frac{q^x[2]_q}{1+q^{p^N}}(-1)^x$.

The purpose of this paper is to present a systemic study of some families of multiple q-Euler numbers and polynomials. In particular, by using the q-Volkenborn integration on $\mathbb{Z}_p$, we construct p-adic q-Euler numbers and polynomials of higher order. We also define new generating function of these q-Euler numbers and polynomials of higher order. Furthermore, we construct Euler q-ζ-function. From section 2 to section 5, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$.

2 q-Euler numbers and polynomials associated with an invariant p-adic q-integrals on $\mathbb{Z}_p$

Let $h \in \mathbb{Z}$, $k \in \mathbb{N} = \{1, 2, 3, \cdots\}$, and let us consider the extended higher-order q-Euler numbers as follows:

$$E_{m,q}^{(h,k)} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} [x_1 + x_2 + \cdots + x_k]_q^m q^{x_1(h-1)+\cdots+x_k(h-k)} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).$$

Then we have

$$E_{m,q}^{(h,k)} = \frac{[2]_q^k}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} \frac{(-1)^l}{(-q^{h+l} : q^{-1})_k}.$$

From the definition of $E_{m,q}^{(h,k)}$, we can easily derive the below:

$$E_{m,q}^{(h,k)} = E_{m,q}^{(h-1,k)} + (q-1)E_{m+1,q}^{(h-1,k)}, \quad (m \ge 0).$$

It is easy to show that

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k+1 \text{ times}} q^{\sum_{j=1}^{k+1}(m-j)x_j} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_{k+1}) =$$

$$\sum_{j=1}^{m} \binom{m}{j} (q-1)^j \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k+1 \text{ times}} [\sum_{l=1}^{k+1} x_l]_q^j \; q^{-\sum_{j=1}^{k} jx_j} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_{k+1}), \tag{2.1}$$

and we also get

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k+1 \text{ times}} q^{\sum_{j=1}^{k+1}(m-j)x_j} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_{k+1}) = \frac{[2]_q^{k+1}}{(-q^m : q^{-1})_{k+1}}. \tag{2.2}$$

From (2) and (2.2), we can derive the below proposition.

Proposition 1. *For $m,k\in\mathbb{N}$, we have*

$$\sum_{j=0}^{m}\binom{m}{j}(q-1)^j E_{j,q}^{(0,k+1)}=\frac{[2]_q^{k+1}}{(-q^m:q^{-1})_{k+1}}, E_{m,q}^{(h,k)}=$$

$$\frac{[2]_q^k}{(1-q)^m}\sum_{l=0}^{m}\binom{m}{l}\frac{(-1)^l}{(-q^{h+l}:q^{-1})_k}.$$

Remark. Note that $E_{n,q}^{(1,1)}=E_{n,q}$, where $E_{n,q}$ are the q-Euler numbers (see Ref. [13]).

From the definition of $E_{n,q}^{(h,k)}$, we can derive

$$\sum_{j=0}^{i}\binom{i}{j}(q-1)^j E_{m-i+j,q}^{(h-1,k)}=\sum_{j=0}^{i-1}\binom{i-1}{j}(q-1)^j E_{m+j-i,q}^{(h,k)}$$

for $m\geq i$. By simple calculation, we easily see that

$$\sum_{j=0}^{m}\binom{m}{j}(q-1)^j E_{j,q}^{(h,1)}=\int_{\mathbb{Z}_p} q^{mx}q^{(h-1)x}d\mu_{-q}(x)=\frac{[2]_q}{[2]_{q^{m+h}}}.$$

Furthermore, we can give the following relation for the q-Euler numbers, $E_{m,q}^{(0,h)}$,:

$$\sum_{j=0}^{m}\binom{m}{j}(q-1)^j E_{j,q}^{(0,k)}=\frac{[2]_q^k}{(-q^m:q^{-1})_k}. \tag{2.3}$$

3 Polynomials $E_{n,q}^{(0,k)}(x)$

We now define the polynomials $E_{n,q}^{(0,k)}(x)$ (in q^x) by

$$E_{n,q}^{(0,k)}(x)=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{ times}}[x_1+x_2+\cdots+x_k]_q^m q^{\sum_{j=1}^{k}jx_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).$$

Thus, we have

$$(q-1)^m E_{m,q}^{(0,k)}(x)=[2]_q^k\sum_{j=0}^{m}\binom{m}{j}q^{jx}(-1)^{m-j}\frac{1}{(-q^j:q^{-1})_k}. \tag{3.1}$$

It is not difficult to show that

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{ times}} q^{\sum_{j=1}^{m}(m-j)x_j+mx}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=q^{mx}\frac{[2]_q^k}{(-q^m:q^{-1})_k},$$

and

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{ times}} q^{\sum_{j=1}^{m}(m-j)x_j+mx}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=\sum_{j=0}^{m}\binom{m}{j}(q-1)^j E_{j,q}^{(0,k)}(x).$$

Therefore we obtain the following.

Lemma 1. *For $m, k \in \mathbb{N}$, we have*

$$\sum_{j=0}^{m} \binom{m}{j} (q-1)^j E_{j,q}^{(0,k)}(x) = \frac{q^{mx}[2]_q^k}{(-q^m : q^{-1})_k}, E_{m,q}^{(0,k)}(x) =$$
$$\frac{[2]_q^k}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} q^{jx}(-1)^j \frac{1}{(-q^j : q^{-1})_k}. \tag{3.2}$$

Let $l \in \mathbb{N}$ with $l \equiv 1 \pmod 2$. Then we get easily

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} \left[x + \sum_{j=1}^{k} x_j\right]_q^m q^{-\sum_{j=1}^{k} jx_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) =$$
$$\frac{[l]_q^m}{[l]_{-q}^k} \sum_{i_1, \cdots, i_k = 0}^{l-1} q^{-\sum_{j=2}^{k}(j-1)i_j} \cdot (-1)^{\sum_{j=1}^{k} i_j} \times$$
$$\times \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} \left[\frac{x + \sum_{j=1}^{k} i_j}{l} + \sum_{j=1}^{k} x_j\right]_{q^l}^m q^{-l\sum_{j=1}^{k} jx_j} d\mu_{-q^l}(x_1) \cdots d\mu_{-q^l}(x_k).$$

From this, we can derive the following "multiplication formula":

Theorem 1. *Let l be an odd positive integer. Then*

$$E_{m,q}^{(0,k)}(x) = \frac{[l]_q^m}{[l]_{-q}^k} \sum_{i_1, \cdots, i_k = 0}^{l-1} q^{-\sum_{j=2}^{k}(j-1)i_j} (-1)^{\sum_{l=1}^{k} i_l} E_{m,q^l}^{(0,k)}\left(\frac{x + i_1 + \cdots + i_k}{l}\right). \tag{3.3}$$

Moreover,

$$E_{m,q}^{(0,k)}(lx) = \frac{[l]_q^m}{[l]_{-q}^k} \sum_{i_1, \cdots, i_k = 0}^{l-1} q^{-\sum_{j=2}^{k}(j-1)i_j} (-1)^{\sum_{l=1}^{k} i_l} E_{m,q^l}^{(0,k)}\left(x + \frac{i_1 + \cdots + i_k}{l}\right). \tag{3.4}$$

From (2.3) and (3.1), we can also derive the below expression for $E_{n,q}^{(0,k)}(x)$:

$$E_{m,q}^{(0,k)}(x) = \sum_{i=0}^{m} \binom{m}{i} E_{i,q}^{(0,k)} [x]_q^{m-i} q^{ix}, \tag{3.5}$$

whence also

$$E_{m,q}^{(0,k)}(x+y) = \sum_{j=0}^{m} \binom{m}{j} [y]_q^{m-i} q^{jy} E_{j,q}^{(0,k)}(x). \tag{3.6}$$

4 Polynomials $E_{m,q}^{(h,1)}(x)$

Let us define

$$E_{m,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} [x + x_1]_q^m \; q^{x_1(h-1)} d\mu_{-q}(x_1). \tag{4.1}$$

Then we have

$$E_{m,q}^{(h,1)}(x) = \frac{[2]_q}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} (-1)^l q^{lx} \frac{1}{(1+q^{l+h})}.$$

By simple calculation of q-Volkenvorn integral, we note that

$$q^x \int_{\mathbb{Z}_p} [x + x_1]_q^m \; q^{x_1(h-1)} d\mu_{-q}(x_1) =$$

$$(q-1) \int_{\mathbb{Z}_p} [x + x_1]_q^{m+1} \; q^{x_1(h-2)} d\mu_{-q}(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_q^m \; q^{x_1(h-2)} d\mu_{-q}(x_1).$$

Thus, we have

$$q^x E_{m,q}^{(h,1)}(x) = (q-1) E_{m+1,q}^{(h-1,1)}(x) + E_{m,q}^{(h-1,1)}(x). \tag{4.2}$$

It is easy to show that

$$\int_{\mathbb{Z}_p} [x + x_1]_q^m \; q^{(h-1)x_1} d\mu_{-q}(x_1) = \sum_{j=0}^{m} \binom{m}{j} [x]_q^{m-j} q^{jx} \int_{\mathbb{Z}_p} [x_1]_q^j q^{(h-1)x_1} d\mu_{-q}(x_1).$$

This is equivalent to

$$E_{m,q}^{(h,1)}(x) = \sum_{j=0}^{m} \binom{m}{j} [x]_q^{m-j} q^{jx} E_{j,q}^{(h,1)} = \left(q^x E_q^{(h,1)} + [x]_q \right)^m, \quad \text{for } m \geq 1,$$

where we use the technique method notation by replacing $(E_q^{(h,1)})^n$ by $E_{n,q}^{(h,1)}$, symbolically. From (4.1), we can derive

$$q^h E_{m,q}^{(h,1)}(x+1) + E_{m,q}^{(h,1)}(x) = [2]_q [x]_q^m. \tag{4.3}$$

For $x = 0$ in (4.3), this gives

$$q^h \left(q E_{m,q}^{(h,1)} + 1 \right)^m + E_{m,q}^{(h,1)} = \delta_{0,k}, \tag{4.4}$$

where $\delta_{0,k}$ is Kronecker symbol. By the simple calculation of q-Volkenborn integration, we easily see that

$$\int_{\mathbb{Z}_p} q^{x_1(h-1)} d\mu_{-q}(x_1) = \frac{[2]_q}{[2]_{q^h}}.$$

Thus, we have $E_{0,q}^{(h,1)} = \frac{[2]_q}{[2]_{q^h}}$. From the definition of q-Euler polynomials, we can derive

$$\int_{\mathbb{Z}_p} [1 - x + x_1]_{q^{-1}}^m \; q^{-x_1(h-1)} d\mu_{-q}(x_1) = q^{m+h-1} (-1)^m E_{m,q}^{(h,1)}(x).$$

Therefore we obtain the below “complementary formula”:

Theorem 2. *For $m \in \mathbb{N}$, $n \in \mathbb{Z}$, we have*

$$E_{m,q^{-1}}^{(h,1)}(1-x) = (-1)^m q^{m+h-1} E_{m,q}^{(h,1)}(x). \tag{4.5}$$

In particular, for $x = 1$, we see that

$$E_{m,q-1}^{(h,1)}(0) = (-1)^m q^{m+h-1} E_{m,q}^{(h,1)}(1) = (-1)^{m-1} q^{m-1} E_{m,q}^{(h,1)}, \quad \text{for } m \geq 1. \tag{4.6}$$

For $l \in \mathbb{N}$ with $l \equiv 1 \pmod 2$, we have

$$\int_{\mathbb{Z}_p} q^{(h-1)x_1}[x+x_1]_q^m \; q^{x_1(h-1)} d\mu_{-q}(x_1) =$$
$$\frac{[l]_q^m}{[l]_{-q}} \sum_{i=0}^{l-1} q^{hi}(-1)^i \int_{\mathbb{Z}_p} \left[\frac{x+i}{l} + x_1\right]_{q^l}^m q^{x_1(h-1)l} d\mu_{-q^l}(x_1).$$

Thus, we can also obtain the following:

Theorem 3. *(Multiplication formula) For $l \in \mathbb{N}$ with $l \equiv 1 \pmod 2$, we have*

$$\frac{[2]_q}{[2]_{q^l}}[l]_q^m \sum_{i=0}^{l-1} q^{hi}(-1)^i E_{m,q^l}^{(h,1)}(\frac{x+i}{l}) = E_{m,q}^{(h,1)}(x).$$

Furthermore,

$$\frac{[2]_q}{[2]_{q^l}}[l]_q^m \sum_{i=0}^{l-1} q^{hi}(-1)^i E_{m,q^l}^{(h,1)}(x+\frac{i}{l}) = E_{m,q}^{(h,1)}(lx).$$

5 Polynomials $E_{m,q}^{(h,k)}(x)$ and $h = k$

It is now easy to combine the above results and define the new polynomials as follows:

$$E_{m,q}^{(h,k)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} [x + x_1 + \cdots + x_k]_q^m q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).$$

Thus, we note that

$$(q-1)^m E_{m,q}^{(h,k)}(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} q^{xj} \frac{[2]_q^k}{(-q^{j+h} : q^{-1})_k}. \tag{5.1}$$

We may now mention the following formulas which are easy to prove.

$$q^h E_{m,q}^{(h,k)}(x+1) + E_{m,q}^{(h,k)}(x) = [2]_q E_{m,q}^{(h-1,k-1)}(x), \tag{5.2}$$

and

$$q^x E_{m,q}^{(h+1,k)}(x) = (q-1) E_{m+1,q}^{(h,k)}(x) + E_{m,q}^{(h,k)}(x). \tag{5.3}$$

Let $l \in \mathbb{N}$ with $l \equiv 1 \pmod 2$. Then we note that

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} [x + \sum_{j=1}^{k} x_j]_q^m q^{\sum_{j=1}^{k}(h-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) =$$
$$\frac{[l]_q^m}{[l]_{-q}^k} \sum_{i_1, \cdots, i_k=0}^{l-1} q^{h\sum_{j=1}^{k} i_j - \sum_{j=2}^{k}(j-1)i_j} (-1)^{\sum_{j=1}^{k} i_j} \times$$
$$\times \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} \left[\frac{x + \sum_{j=1}^{k} i_j}{l} + \sum_{j=1}^{k} x_j \right]_{q^l}^m (q^l)^{\sum_{j=1}^{k}(h-j)x_j} d\mu_{-q^l}(x_1) \cdots d\mu_{-q^l}(x_k).$$

Therefore we obtain the following:

Theorem 4. *(Distribution for q-Euler polynomials) For* $l \in \mathbb{N}$ *with* $l \equiv 1 \pmod 2$. *Then we have*

$$E_{m,q}^{(h,k)}(lx) = \frac{[l]_q^m}{[l]_{-q}^k} \sum_{i_1, \cdot, i_k=0}^{l-1} q^{h\sum_{j=1}^{k} i_j - \sum_{j=2}^{k}(j-1)i_j} (-1)^{\sum_{j=1}^{k} i_j}$$
$$E_{m,q^l}^{(h,k)} \left(x + \frac{i_1 + \cdots + i_k}{l} \right). \tag{5.4}$$

It is interesting to consider the case $h = k$, which leads to the desired extension of the q-Euler numbers of higher order [1]. We shall denote the polynomials in this special case by $E_{m,q}^{(k)}(x) := E_{m,q}^{(k,k)}(x)$. Then we have

$$(q-1)^m E_{m,q}^{(k)}(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} q^{jx} \frac{[2]_q^k}{(-q^{j+k} : q^{-1})_k}, \tag{5.5}$$

and

$$E_{m,q^{-1}}^{(k)}(k-x) = (-1)^m q^{m+\binom{k}{2}} E_{m,q}^{(k)}(x). \tag{5.6}$$

For $x = k$ in (5.6), we see that

$$E_{m,q^{-1}}^{(k)}(0) = (-1)^m q^{m+\binom{k}{2}} E_{m,q}^{(k)}(k). \tag{5.7}$$

From (5.2), we can derive the below formula:

$$q^k E_{m,q}^{(k)}(x+1) + E_{m,q}^{(k)}(x) = [2]_q E_{m,q}^{(k-1)}(x). \tag{5.8}$$

Putting $x = 0$ in (5.1), we obtain

$$(q-1)^m E_{m,q}^{(k)} = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \frac{[2]_q^k}{(-q^{i+k} : q^{-1})_k}. \tag{5.9}$$

Note that

$$\sum_{i=0}^{m}\binom{m}{i}(q-1)^i \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\ \text{times}}[x_1+\cdots+x_k]_q^i q^{\sum_{j=1}^{k-1}(k-j)x_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=$$

$$\frac{[2]_q^k}{(-q^{m+k}:q^{-1})_k}.$$

From this, we can easily derive

$$\sum_{i=0}^{m}\binom{m}{i}(q-1)^i E_{i,q}^{(k)}=\frac{[2]_q^k}{(-q^{m+k}:q^{-1})_k} \tag{5.10}$$

and so it follows

$$E_{m,q}^{(k)}(x)=(q^xE_q^{(k)}+[x]_q)^m,\quad m\geq 1, \tag{5.11}$$

where we use the technique method notation by replacing $(E_q^{(k)})^n$ by $E_{n,q}^{(k)}$, symbolically. In particular, from (5.8), we have

$$q^k(qE_q^{(k)}+1)^m+E_{m,q}^{(k)}=[2]_qE_{m,q}^{(k-1)}. \tag{5.12}$$

It is easy to see that

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\ \text{times}} q^{(k-1)x_1+\cdots+x_{k-1}}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=\frac{[2]_q^k}{(-q^k:q^{-1})_k}.$$

Thus, we note that $E_{0,q}^{(k)}=\frac{[2]_q^k}{(-q^k:q^{-1})_k}$.

6 Generating function for q-Euler polynomials

An obvious generating function for q-Euler polynomials is obtained, from (5.1), by

$$[2]_q^k e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\frac{(-1)^j}{(-q^{j+h}:q^{-1})_k}q^{jx}\left(\frac{1}{1-q}\right)^j\frac{t^j}{j!}=\sum_{n=0}^{\infty}E_{n,q}^{(h,k)}\frac{t^n}{n!}. \tag{6.1}$$

From (5.1), we can also derive the below formula:

$$q^{h-k}E_{m,q}^{(h,k+1)}(x+1)=[2]_qE_{m,q}^{(h,k)}(x)-E_{m,q}^{(h,k+1)}(x). \tag{6.2}$$

Again from (5.5) and (5.9), we get easily

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\ \text{times}}[x+\sum_{j=1}^{k}x_j]_q^m q^{\sum_{j=1}^{k-1}(k-j)x_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=$$

$$\sum_{j=0}^{m}\binom{m}{j}q^{xj}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\ \text{times}}[x_k]_q^j[x+\sum_{j=1}^{k-1}x_j]_q^{n-j}q^{\sum_{l=1}^{k-1}(k+j-l)x_l}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).$$

Thus, we note that

$$E_{m,q}^{(k)}(x)=\sum_{j=0}^{m}\binom{m}{j}q^{xj}E_{j,q}^{(1)}E_{m-j,q}^{(k+j,k-1)}(x). \tag{6.3}$$

Take $x=0$ in (6.3), we have

$$E_{m,q}^{(k)}=\sum_{i=0}^{m}\binom{m}{i}E_{j,q}^{(1)}E_{m-j,q}^{(k+j,k-1)}. \tag{6.4}$$

So, for $k=2$,

$$E_{m,q}^{(2)}=\sum_{i=0}^{m}\binom{m}{i}E_{j,q}E_{m-j,q}^{(j+2,1)}.$$

It is not difficult to show that

$$\int_{\mathbb{Z}_p}[x]_q^m q^{hx}d\mu_{-q}(x)=\sum_{j=0}^{h}\binom{h}{j}(q-1)^j\int_{\mathbb{Z}_p}[x]_q^{m+j}d\mu_{-q}(x),\ \text{for } h\in\mathbb{N}\ .$$

From this, we can derive the below:

$$E_{m,q}^{(h+1,1)}=\sum_{j=0}^{h}\binom{h}{j}(q-1)^jE_{m+j,q},\quad h\in\mathbb{N}. \tag{6.5}$$

By (6.4) and (6.5), we easily see that

$$E_{m,q}^{(2)}=\sum_{j=0}^{m}\binom{m}{j}E_{j,q}\sum_{i=0}^{j+1}\binom{j+1}{i}(q-1)^iE_{m-j+i,q}. \tag{6.6}$$

By (6.6), for $q=1$, we note that

$$E_m^{(2)}=\sum_{j=0}^{m}\binom{m}{j}E_jE_{m-j},\ \text{where}\ \left(\frac{2}{e^t+1}\right)^k=\sum_{n=0}^{\infty}E_n^{(k)}\frac{t^n}{n!}.$$

It is easy to show that

$$[x+x_1+\cdots+x_k]_q^m=\sum_{j=0}^{m}\binom{m}{j}[x_1+x]_q^{m-j}q^{j(x_1+x)}[x_2+\cdots+x_k]_q^j.$$

By using this, we get easily

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\ \text{times}}[x+\sum_{j=1}^{k}x_j]_q^m q^{\sum_{j=1}^{k-1}(k-j)x_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=$$

$$\sum_{j=0}^{m}\binom{m}{j}q^{jx}\int_{\mathbb{Z}_p}[x+x_1]_q^{m-j}q^{(k+j-1)x_1}d\mu_{-q}(x_1)\times$$

$$\times\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-1\ \text{times}}[x_2+\cdots+x_k]_q^jq^{\sum_{j=2}^{k-1}(k-j)x_j}d\mu_{-q}(x_2)\cdots d\mu_{-q}(x_k).$$

Therefore we obtain the following:

Theorem 5. *For $m, k \in \mathbb{N}$, we have*

$$E_{m,q}^{(k)}(x) = \sum_{j=0}^{m} \binom{m}{j} q^{jx} E_{m-j,q}^{(k+j,1)}(x) E_{j,q}^{(k-1)}. \tag{6.7}$$

Indeed for $x = 0$,

$$E_{m,q}^{(k)} = \sum_{j=0}^{m} \binom{m}{j} E_{m-j,q}^{(k+j,1)} E_{j,q}^{(k-1)} = \tag{6.8}$$

$$\sum_{j=0}^{m} \binom{m}{j} E_{j,q}^{(k-1)} \sum_{j=0}^{k+j} (q-1)^i \binom{k+j-1}{i} E_{m-j+i,q}^{(1)}. \tag{6.9}$$

As for $q = 1$, we get the below formula

$$E_m^{(k)} = \sum_{j=0}^{m} \binom{m}{j} E_j^{(k-1)} E_{m-j}^{(1)}.$$

7 q-Euler zeta function in $\mathbb{C}$

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. From section 4, we note that

$$E_{m,q}^{(h,1)}(x) = \frac{[2]_q}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} q^{lx} (-1)^l \frac{1}{1+q^{l+h}} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{nh} [n+x]_q^n. \tag{7.1}$$

Thus, we can define q-Euler zeta function:

Definition 1. For $s, q \in \mathbb{C}$ with $|q| < 1$, define

$$\zeta_{E,q}^{h}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^{nh}}{[n+x]_q^s},$$

where $x \in \mathbb{R}$ with $0 < x \leq 1$.

Note that $\zeta_{E,q}^{h}(-m, x) = E_{m,q}^{(h,1)}(x)$, for $m \in \mathbb{N}$. Let

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}^{(h,1)}(x) \frac{t^n}{n!}.$$

Then we have

$$F_q(t,x) = [2]_q e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{-\frac{q^{n+x}}{1-q}t} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n+x]_q t}, \quad \text{for } h \in \mathbb{Z}.$$

Therefore we obtain the following

Lemma 2. *For $h \in \mathbb{Z}$, we have*

$$F_q(t,x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}^{(h,1)}(x) \frac{t^n}{n!}. \tag{7.2}$$

Let $\Gamma(s)$ be the gamma function. Then we easily see that

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_q(-t,x)dt = \zeta_{E,q}^h(s,x), \quad \text{for } s \in \mathbb{C}. \tag{7.3}$$

From (7.2) and (7.3), we can also derive the below Eq. (7.4):

$$\zeta_{E,q}^h(-n,x) = E_{n,q}^{(h,1)}(x), \quad \text{for} \quad n \in \mathbb{N}. \tag{7.4}$$

References

[1] Carlitz L, q-Bernoulli numbers and polynomials, *Duke Math. J.* **15** (1948), 987–1000.

[2] Carlitz L, q-Bernoulli and Eulerian numbers, *Trans. Amer. Math. Soc.* **76** (1954), 332–350.

[3] Cenkci M and Can M, Some results on q-analogue of the Lerch zeta function, *Advan. Stud. Contemp. Math.* **12** (2006), 212–223.

[4] Deeba E and Rodriguez D, Stirling's series and Bernoulli numbers, *Amer. Math. Monthly* **98** (1991), 423–426.

[5] Howard T, Applications of a recurrences formula for the Bernoulli numbers, *J. Number Theory* **52** (1995), 157–172.

[6] Kim T, Jang L C and Park H K, A note on q-Euler and Genocchi numbers, *Proc. Japan Academy* **77** (2001), 139–141.

[7] Kim T, Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function, *Russ. J. Math. Phys.* **12** (2005), 189–196.

[8] Kim T, Non-Arichimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials, *Russ. J. Math. Phys.* **10** (2003), 91–98.

[9] Kim T, An invariant p-adic integral associated with Daehee numbers, *Integral Transform. Spec. Funct.* **13** (2002), 65–69.

[10] Kim T, On a q-analogue of the p-adic log gamma functions and related integrals, *J. Number Theory* **76** (1999), 320–329.

[11] Kim T, q-Volkenborn integration, *Russ. J. Math. Phys.* **9** (2002), 288–299.

[12] Kim T, A note on q-Volkenborn integration, *Proc. Jangjeon Math. Soc.* **8** (2005), 13–17.

[13] Kim T, On the q-extension of Euler and Genocchi numbers, *J. Math. Anal. Appl.*, to appear.

[14] Satoh J, q-analogue of Riemann's ζ-function and q-Euler numbers, *J. Number Theory* **31** (1989), 346–362.

[15] Simsek Y, Kim D and Rim S-H, On the two-variable Dirichlet q-L-series, *Advan. Stud. Contemp. Math.* **10** (2005), 131–142.

[16] Shiratani J and Yamamoto S, On a p-adic interpolating function for the Euler numbers and its derivatives, *Mem. Fac. Sci. Kyushu Univ. Math.* **39** (1985), 113–125.