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The Initial-Boundary Value Poblem for the Korteweg-de Vries Equation on the Positive Quarter-Plane

Pham Loi VU

Institute of Mechanics, Vietnam Academy of Science and Technology, 264 Doi Can Street, Hanoi, Vietnam
E-mail: vuphamloi@yahoo.com

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Abstract

The paper deals with a problem of developing an inverse-scattering transform for solving the initial-boundary value problem (IBVP) for the Korteweg-de Vries equation on the positive quarter-plane:

$$p_t - 6pp_x + p_{xxx} = 0, \;\; x \geqslant 0, t \geqslant 0, \tag{a}$$

with the given initial and boundary conditions:

$$p(x,0) = p(x), \;\; p(x) \text{ is a real-valued rapidly decreasing function,} \tag{b}$$
$$p(0,t) = f(t), \;\; f(t) \text{ is a real-valued continuous function.} \tag{c}$$

The Sturm-Liouville scattering problem (SP) in the interval $(0,b)$ (b is a large positive number) generated by the linear Schrödinger equation (LSEq) with the zero boundary conditions (BCs) at $x = 0$ and at $x = b$ is regarded as the linear problem associated with the IBVP (a)-(c). The time dependencies of the scattering data set of the SP are determined by the unknown boundary values (BVs) evaluated at $x = 0$ of the Jost solution of the LSEq. To overcome the difficulty we derive the asymptotic equation for the normalization eigenfunction of the Sturm-Liouville SP. This allows one to show the approximate time-independence of the scattering phase. Then, from the evolution equation for the scattering phase we deduce the asymptotic formulas for caculating the unknown BVs. We prove that the potential $p(x,t)$ in the LSEq is uniquely found from the solution of the inverse SP in terms of the given data (b) and (c) and therefore, $p(x,t)$ is a solution of the IBVP (a)-(c). Every solution of the problem (a)-(c) corresponds to an unique scattering data set and evolves from the continuous and discrete spectrum of the SP.

1 Introduction

The inverse-scattering transform method (ISM) is an effective tool for studying nonlinear integrable equations [14]. It allows one to construct large classes of exact solutions and to investigate in detail the Cauchy problems. As for the mixed problem for the same equations, here the success of the method is not so impressive. For instance, the ISM applied to the initial-boundary value problem (IBVP) in its general formulation is not sufficiently effective (see., for example [4]), except some special kinds of boundary conditions [1, 6, 7]. The problem of searching boundary conditions consistent with the integrability property of the equation given as well as the problem of finding proper modifications of analytical integration procedure for the corresponding IBVPs is undoubtedly important.

To deal efficiently with the presence of unknown boundary values (BVs), Degasperis, Manakov and Santini proposed two alternative methods in Fourier space: the analiticity approach and the Elimination by Restriction approach [3].

The author deal with a problem of developing an inverse-scattering transformation for solving problems for a system of nonlinear equations on a half-line:

$$\begin{aligned} iq_{1t} &= q_{1xx} - 2q_1^2 q_2, \\ -iq_{2t} &= q_{2xx} - 2q_1 q_2^2, \end{aligned} \qquad 0 \leqslant x < \infty,\ -\infty < t < \infty.$$

The attractive and repulsive nonlinear Schrödinger (NLS) equations are obtained from the above system when $q_1 = -\overline{q}_2$ and $q_1 = \overline{q}_2$, respectively. The IBVPs for this system and for the attractive NLS have been studied [11]. The IBVPs for the cubic nonlinear equations on a half-line have been solved [12]. The Dirichlet IBVPs for the sine-Gordon (sG) and sinh-Gordon equations have been studied [13]. The difficulty in solving the IBVPs in [11]-[13] is that the time-dependencies of the scattering data are determined by the unknown BVs at $x = 0$ of the Jost solutions. To overcome the difficulty we derived linear evolutionary equations (EEqs) for the unknown BVs. The solutions of the derived EEqs, i.e., the unknown BVs are expressed in terms of the given data.

The IBVP for the Korteweg-de Vries (KdV) equation on the positive quarter-plane has been open for a long time. In recent years there has been a lot of interest in solving it. The quarter-plane problem for the KdV equation considered by Bona and Winther may serve as a model for unidirectional propagation of plane waves generated by a wavemaker in a uniform medium [2]. The analitycity approach is inspired by Fokas' recent discovery of the global relation. The use of global relation to study the IBVPs is applied to the NLS, to the sG, and to the KdV [5].

The IBVP for the KdV equation on the negative quarter-plane:

$$p_t + 6pp_x + p_{xxx} = 0, \quad x < 0, t > 0$$

is considered in [7]. Two boundary conditions (BCs) are required at $x = 0$ for the negative quarter-plane problem, in contrast to the one BC needed at $x = 0$ for the positive quarter-plane problem [7]:

$$p_t - 6pp_x + p_{xxx} = 0, \quad x \geqslant 0,\ t \geqslant 0. \tag{1.1}$$

In the present paper we develop an inverse-scattering transform for solving the problem for equation (1.1) with the given initial and boundary conditions:

$$p(x,0) = p(x), \quad p(x) \text{ is a real-valued rapidly decreasing function,} \tag{1.2}$$

$$p(0,t) = f(t), \quad f(t) \text{ is a real-valued continuous function.} \tag{1.3}$$

It is to be noticed that the solution $p(x,t)$ of the KdV equation (1.1) is uniquely fixed by $p(x,0)$ and $p(0,t)$ in the positive quarter plane, while in this quarter-plane the solution of the KdV equation

$$p_t + 6pp_x - p_{xxx} = 0, \quad x > 0, t > 0$$

requires the specification of $p(x,0)$ and of two BCs, for instance $p(0,t)$ and $p_x(0,t)$ (or $p_{xx}(0,t)$) [6].

The difficulty in solving the IBVP (1.1)-(1.3) is that the time-dependencies of the scattering data of the scattering problem (SP) are determined by the unknown BVs at $x = 0$ of the Jost solutions of the linear Schrödinger equation (LSEq) on a half-line. Unlike the cases considered in [11, 12, 13], it was impossible to derive linear EEqs for the unknown BVs. To overcome the difficulty we consider the Sturm-Liouville SP in the interval $(0, b)$ (b is a large positive number) generated by the LSEq with the zero BCs at $x = 0$ and at $x = b$. We derive the asymptotic equation for the normalization eigenfunction of this Sturm-Liouville SP. This allows one to show the approximate time-independence of the scattering phase. Then the asymptotic formulas for calculating the unknown BVs are deduced from the EEq for the scattering phase.

Our paper is constructed as follows. We present in section 2 the solution of the inverse SP for the LSEq on a half-line. The solution of this problem is well known, although this solution will be used in sections 3 and 4 to find the solution of the IBVP (1.1)-(1.3). Theorem 3.1 in section 3 plays a crusial role in solving the IBVP (1.1)-(1.3). This theorem shows the time-dependencies of the scattering data of the SP and the determination of the unknown BVs in terms of the given data (1.2) and (1.3). In section 4 by virtue of theorem 3.1 we prove theorem 4.1 on the solution of the IBVP (1.1)-(1.3). This solution is constructed from the solution of the inverse SP in terms of the given data (1.2) and (1.3).

2 The inverse SP for the LSEq on a half-line

We consider the following fundamental problem arising in the quantum theory of scattering. The solution of this problem satisfies the LSEq [10]:

$$-y_{xx} + p(x)y = \lambda y, \quad 0 \leqslant x < \infty, \tag{2.1}$$

and the zero BC at the origin $x = 0$:

$$y(\rho, 0) = 0, \quad \rho^2 = \lambda, \tag{2.2}$$

where the potential $p(x)$ is a real-valued function satisfying the inequality:

$$\int_0^\infty x|p(x)|dx < \infty. \tag{2.3}$$

The solutions $e(\rho, x)$ and $\omega(\rho, x)$ of (2.1) are determined by the BCs:

$$\lim_{x\to\infty} e^{-i\rho x} e(\rho, x) = 1, \tag{2.4}$$

$$\omega(\rho, 0) = 0, \quad \omega_x(\rho, 0) = 1. \tag{2.5}$$

The problems (2.1), (2.4) and (2.1), (2.5) are equivalent to the integral equations:

$$e(\rho,x)=e^{i\rho x}+\int_x^\infty \frac{\sin\rho(s-x)}{\rho}p(s)e(\rho,s)ds, \tag{2.6}$$

$$\omega(\rho,x)=\frac{\sin\rho x}{\rho}+\int_0^x \frac{\sin\rho(x-s)}{\rho}p(s)\omega(\rho,s)ds. \tag{2.7}$$

The Jost solution of (2.6) exists uniquely and may be represented in the form:

$$e(\rho,x)=e^{i\rho x}+\int_x^\infty K(x,\xi)e^{i\rho\xi}d\xi, \quad \mathrm{Im}\rho\geqslant 0, \quad x\geqslant 0. \tag{2.8}$$

Substituting (2.8) into (2.6) we obtain the integral equation for the kernel $K(x,\xi)$:

$$K(x,\xi)=\frac{1}{2}\int_{\frac{x+\xi}{2}}^\infty p(u)du+\frac{1}{2}\int_x^\infty p(u)du\int_{\xi-(u-x)}^{\xi+(u-x)}K(u,v)dv. \tag{2.9}$$

Due to (2.3) the solution $K(x,\xi)$ of (2.9) exists uniquely and satisfies the estimate:

$$|K(x,\xi)|\leqslant\frac{1}{2}\sigma_1\Big(\frac{x+\xi}{2}\Big)\exp\Big\{\sigma_2(x)-\sigma_2\Big(\frac{x+\xi}{2}\Big)\Big\}, \tag{2.10}$$

where $\sigma_1(x)=\int_x^\infty |p(\xi)|d\xi$, $\sigma_2(x)=\int_x^\infty \sigma_1(u)du$.
In view of (2.9) and (2.10) the function $K(x,\xi)$ possesses the partial derivatives of first order with both variables and relates to the potential by the formula:

$$\frac{d}{dx}K(x,x)=-\frac{1}{2}p(x). \tag{2.11}$$

Lemma 2.1. *The Jost solution $e(\rho,x)$ of the problem* (2.1), (2.4) *is analytic in ρ in the upper half-plane* $\mathrm{Im}\rho>0$ *and continuous on the real axis. The estimates*

$$|e(\rho,x)|\leqslant\exp\{-\mathrm{Im}\rho x+\sigma_2(x)\}, \tag{2.12}$$

$$|e(\rho,x)-e^{i\rho x}|\leqslant\Big\{\sigma_2(x)-\sigma_2\Big(x+\frac{1}{|\rho|}\Big)\Big\}\exp\{-\mathrm{Im}\rho x+\sigma_2(x)\}, \tag{2.13}$$

$$|e_x(\rho,x)-i\rho e^{i\rho x}|\leqslant\sigma_1(x)\exp\{-\mathrm{Im}\rho x+\sigma_2(x)\}, \tag{2.14}$$

hold in the whole upper half-plane $\mathrm{Im}\rho\geqslant 0$.

Without further mention we shall write k for ρ whenever ρ is real. Due to (2.13) and (2.14) the Wronskian of $e(k,x)$ and $e(-k,x)$ is equal to

$$W[e(k,x),e(-k,x)]=e_x(k,x)e(-k,x)-e(k,x)e_x(-k,x)=2ik \text{ for real } k\neq 0. \tag{2.15}$$

Hence, $e(k,x)$ and $e(-k,x)$ for $k\neq 0$ form a fundamental system of solutions of (2.1). In view of (2.3) the equation (2.7) for all values of ρ has a solution $\omega(\rho,x)$ satisfying the conditions as $x\to 0$:

$$\omega(\rho,x)=x(1+o(1)), \quad \omega_x(\rho,x)=1+o(1). \tag{2.16}$$

The solution $\omega(k,x)$ is an entire function of ρ and has the representation:

$$\omega(k,x) = \frac{1}{2ik}\Big[e(-k)e(k,x) - e(k)e(-k,x)\Big] \quad \text{for all real } k \neq 0. \tag{2.17}$$

$e(\rho)$ denotes the BVs at $x = 0$ of $e(\rho, x)$. By virtue of (2.13) the estimate:

$$e(\rho) = 1 + o(1). \quad \text{Im}\rho \geqslant 0, \tag{2.18}$$

holds for large $|\rho|$. From the reality of the potential $p(x)$ we see that the kernel $K(x,\xi)$ of (2.9) is a real-valued function, therefore,

$$e(k,x) = \overline{e(-k,x)} \quad \text{for all } k. \tag{2.19}$$

$e(k) \neq 0$ for all $k \neq 0$. Indeed, if $e(k) = 0$, then due to (2.19), (2.17) would imply that $\omega(k,x) \equiv 0$, and this is imposible. In view of (2.12)-(2.14) and (2.16)-(2.18), we can prove that $e(\rho)$ can only have a finite number of zeros on the imaginary axis at $\rho = i\rho_j$, $\rho_j > 0$, $j = 1, \ldots, n$, and the function $\rho[e(\rho)]^{-1}$ is bounded in some neighbourhood of zero.

Lemma 2.2. *The identity*

$$-\frac{2ik\omega(k,x)}{e(k)} = e(-k,x) - S(k)e(k,x) \tag{2.20}$$

holds for all real $k \neq 0$, where $S(k)$ is continuous for $k \neq 0$, and

$$S(k) = \frac{e(-k)}{e(k)} = \overline{S(-k)} = [S(-k)]^{-1}, \quad |S(k)| = 1, \tag{2.21}$$

$$1 - S(k) = o(1) \quad \textit{for large } |k|. \tag{2.22}$$

We call $S(k)$ the scattering function for the SP (2.1)-(2.2).

Due to (2.12), (2.14) and (2.16) the norma m_j^{-1} of $e(i\rho_j, x)$ is determined by:

$$m_j^{-2} = \int_0^\infty [e(i\rho_j,x)]^2 dx = \frac{ie_x(i\rho_j,0)\dot{e}(i\rho_j,0)}{2\rho_j}\,, \tag{2.23}$$

where $\dot{e}(i\rho_j, x) = \dfrac{d}{d\rho} e(\rho,x)|_{\rho=i\rho_j}$, m_j is called the normalization multiplier.

Since $\displaystyle\int_0^\infty [e(i\rho_j,x)]^2 dx > 0$, then $\dot{e}(i\rho_j, 0) \neq 0$ and the zeros of $e(\rho)$ are simple.

From (2.20) we derive the fundamental equation in the inverse SP:

$$F(x+y) + K(x,y) + \int_x^\infty K(x,\xi)F(\xi+y)d\xi = 0, \quad 0 \leqslant x < y < \infty, \tag{2.24}$$

here the function $F(x)$ is constructed from the scattering data set s:

$$s = \big\{S(k), -\infty < k < \infty;\ i\rho_j, \rho_j > 0;\ m_j, j = 1, \ldots, n\big\} \tag{2.25}$$

by the formula

$$F(x) = F_S(x) + \sum_{j=1}^n m_j^2 e^{-\rho_j x}, \tag{2.26}$$

where

$$F_S(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[1 - S(k)]e^{ikx}dk. \tag{2.27}$$

Moreover, the function $F_S(x)$ is represented in the form:

$$F_S(x) = F_S^{(1)}(x) + F_S^{(2)}(x), \tag{2.28}$$

where $F_S^{(1)}(x) \in L_1(-\infty,\infty)$, $F_S^{(2)}(x)$ is bounded and belongs to $L_2(-\infty,\infty)$.
The equation (2.24) connecting the set (2.25) with the kernel $K(x,y)$ of the transformation operator enables one to reconstruct the potential by formula (2.11).

From the differentiability of the kernel $K(x,y)$ and the estimates (2.3), (2.10) it follows that there exist the derivatives $F'(x)$ and $F'_S(x)$ and

$$\int_0^{\infty} x|F'(x)|dx < \infty, \quad \int_0^{\infty} x|F'_S(x)|dx < \infty. \tag{2.29}$$

Using (2.8) and (2.24), we can show that $S(k)$ is continuous at zero, in addition [10]:

$$S(0) = \begin{cases} 1, & \text{if } e(0,0) \neq 0, \\ -1. & \text{if } e(0,0) = 0. \end{cases} \tag{2.30}$$

Putting $e(k) = |e(k)|e^{i\delta(k)}$, we call $\delta(k)$ the scattering phase of problem (2.1) - (2.2). It follows from (2.19), (2.21) and (2.22) that

$$\ln S(k) = -2i\delta(k), \quad \delta(k) = -\delta(-k), \ \delta(\pm\infty) = 0. \tag{2.31}$$

Applying the principle of the argument to $e(\rho)$ we find the relation

$$n = \begin{cases} \dfrac{2}{2\pi}[\delta(+\infty) - \delta(+0)], & \text{if } e(0,0) \neq 0, \\ \dfrac{2}{2\pi}[\delta(+\infty) - \delta(+0)] - \dfrac{1}{2}, & \text{if } e(0,0) = 0. \end{cases} \tag{2.32}$$

Theorem 2.1. *The scattering data* (2.25) *of the SP* (2.1)-(2.2) *with the real-valued potential satisfying condition* (2.3) *possess the following properties:*
I. *The scattering function $S(k)$ is continuous on the full-line and possesses properties* (2.21), (2.30) *and* (2.31). *The function $1-S(k)$ satisfies estimate* (2.22) *and is the Fourier transform of the function* (2.27). *This function is represented in the form* (2.28) *and has the derivative $F'_S(x)$, $0 < x < \infty$, satisfying condition* (2.29)*;*
II. *The full increment of the scattering phase $\delta(k)$ is related with the number n of negative eigenvalues of the SP* (2.1)-(2.2) *by formula* (2.32).

Marchenko [10] proved that I and II not only are necessary, but also are sufficient conditions for the given quantities of (2.25) to be scattering data for the SP.

3 The time-dependencies of the scattering data

Let the potential in equation (2.1) depend on an additional parameter t (time). We consider the time-dependent SP for the LSEq:

$$-y_{xx} + p(x,t)y = \lambda y, \quad x \geqslant 0,\ t \geqslant 0,\ \lambda = \rho^2, \tag{3.1}$$

with the zero BC at the origin $x = 0$:

$$y(\rho, 0; t) = 0, \quad t \geqslant 0, \tag{3.2}$$

where the potential $p(x,t)$ is a real-valued function satisfying the inequality:

$$\int_0^\infty x|p(x,t)|dx < \infty \quad \text{for any } t \geqslant 0. \tag{3.3}$$

By $L(t)$ we mean the energy operator defined in $L_2(0,\infty)$ by the LSEq (3.1) and the BC (3.2). The time-dependence of the solution of (3.1) is given by the EEq:

$$y_t(\rho,x;t) = 2[2\rho^2 + p(x,t)]y_x(\rho,x;t) - [4i\rho^3 + p_x(x,t)]y(\rho,x;t),\ x \geqslant 0, t \geqslant 0. \tag{3.4}$$

Assuming that the eigenvalues of $L(t)$ are time-independent, i.e., $\lambda_t = 0$, then from (3.1) and (3.4) we obtain

$$y_{txx} = [(4i\rho^3 - p_x)(p-\rho^2) - p_{xxx} + 6pp_x]y + (4\rho^2 + 2p)(p-\rho^2)y_x, \tag{3.5}$$

$$y_{xxt} = [(4i\rho^3 - p_x)(p-\rho^2) + p_t]y + (4\rho^2 + 2p)(p-\rho^2)y_x. \tag{3.5'}$$

Therefore, (3.5) and (3.5') are compatible, i.e., $y_{txx} = y_{xxt}$, if and only if the potential $p(x,t)$ in (3.1) satisfies equation (1.1). Conversely, if equation (1.1) for $p(x,t)$ is satisfied, then necessarily the eigenvalue of $L(t)$ must be time-independent.

Hence, the KdV equation (1.1) on the half-line can be written as the compatibility condition of two linear eigenvalue equations (3.1) and (3.4), i.e., the KdV equation (1.1) admits its Lax pair by equations (3.1) and (3.4).

3.1 The Jost solution

The Jost solution of equation (3.1) with the BC at infinity:

$$\lim_{x\to\infty} e^{-i\rho x} e(\rho,x;t) = 1 \tag{3.6}$$

is represented in the form (2.8):

$$e(\rho,x;t) = e^{i\rho x} + \int_x^\infty K(x,\xi;t)e^{i\rho\xi}d\xi, \quad \mathrm{Im}\rho \geqslant 0, \tag{3.7}$$

where t enters (3.7) as a parameter and the kernel $K(x,\xi;t)$ is a real-valued function satisfying equation (2.9) and estimate (2.10).

For $k \neq 0$ the solutions $e(k,x;t)$ and $e(-k,x;t)$ of problem (3.1), (3.6) form a fundamental system of solutions of this problem and evolve according to EEq (3.4):

$$e_t(k,x;t) = 2[2k^2 + p(x,t)]e_x(k,x;t) - [4ik^3 + p_x(x,t)]e(k,x;t), \tag{3.8}$$

$$e_t(-k,x;t) = 2[2k^2 + p(x,t)]e_x(-k,x;t) + [4ik^3 - p_x(x,t)]e(-k,x;t). \tag{3.9}$$

Taking estimates (3.3), (2.13) and (2.14) into account we show that $e_t(\rho,x;t)=o(1)$, $\mathrm{Im}\rho>0$, and $e_t(\pm k,x;t)=o(1)$ as $x\to\infty$. Thus, the asymptotic behavior of the Jost solutions $e(\rho,x;t)$, $\mathrm{Im}\rho\geqslant 0$ as $x\to\infty$ does not depend on time.

The BVs $e(\rho,0;t)$ is denoted by $e(\rho;t)$. The next lemma on the properties of the Jost solution and its BVs is deduced from the above discussion.

Lemma 3.1. *The Jost solution $e(\rho,x;t)$ of problem* (3.1), (3.6) *with the real- valued potential $p(x,t)$ satisfying estimate* (3.3) *is analytic in ρ in the upper half-plane* $\mathrm{Im}\rho>0$ *and continuous on the real axis. This solution possesses the properties:*

$$e(k,x;t)=\overline{e(-k,x;t)}\quad \textit{for } k\neq 0 \textit{ and any } (x,t)\in[0,\infty)\times[0,\infty), \tag{3.10}$$
$$e(k;t)\neq 0\quad \textit{for all } k\neq 0 \textit{ and } t\geqslant 0,$$

and obeys the asymptotic estimate as $x\to\infty$:

$$e(\rho,x;t)=e^{i\rho x}+o(1),\ e_x(\rho,x;t)=i\rho e^{i\rho x}+o(1),\ e_t(\rho,x;t)=o(1),\ \mathrm{Im}\rho\geqslant 0. \tag{3.11}$$

The BV $e(\rho;t)$ at the origin $x=0$ of the Jost solution behaves asymptotically like:

$$e(\rho;t)=1+o(1)\quad \textit{for large } |\rho|.\quad \mathrm{Im}\rho\geqslant 0, \tag{3.12}$$

and has a finite number of simple zeros on the imaginary axis at $\rho=i\rho_j$, $\rho_j>0$, $j=1,\dots,n$. The function $\rho[e(\rho;t)]^{-1}$ is bounded in some neighbourhood of zero.

The time-dependence of the Jost solutions $e(\rho,x;t)$, $\mathrm{Im}\rho>0$ *and $e(\pm k,x;t)$ are given by EEqs* (3.4) *and* (3.8), (3.9)*, respectively.*

3.2 The normalization eigenfunctions and the Lax pair

Consider the time-dependent normalization eigenfunctions of the SP (3.1)-(3.2):

$$\Omega(\rho,x;t)=-\frac{2i\rho\omega(\rho,x;t)}{e(\rho;t)},\quad \mathrm{Im}\rho\geqslant 0,\ x\geqslant 0, t\geqslant 0, \tag{3.13}$$

where $\omega(\rho,x;t)$ is the solution of equation (3.1) satisfying BCs (2.16), therefore $\Omega(\rho,x;t)$ satisfies the BCs at the origin $x=0$:

$$\Omega(\rho,0;t)=0,\quad \Omega_x(\rho,0;t)=-\frac{2i\rho}{e(\rho;t)},\quad \mathrm{Im}\rho\geqslant 0. \tag{3.14}$$

From the first of BCs (3.14) it follows that

$$\Omega_t(\rho,0;t)=0\quad \text{for any } t\geqslant 0,\quad \mathrm{Im}\rho\geqslant 0. \tag{3.15}$$

Consider first the eigenfunction $\Omega(k,x;t)$ corresponding to the continuous spectrum of the SP (3.1), (3.14). We put

$$e(k;t)=|e(k;t)|e^{i\delta(k;t)}, \tag{3.16}$$

where $|e(k;t)|\neq 0$ for $k\neq 0$, $t\geqslant 0$, $\delta(k;t)$ is the scattering phase or phase shift of the SP (3.1)-(3.2).

According to (2.31) and (2.20) the scattering phase possesses the properties: $\delta(k;t) = -\delta(-k,t)$, $\delta(\pm\infty,t) = 0$ and the eigenfunction $\Omega(k,x;t)$ is represented by

$$\Omega(k,x;t) = e(-k,x;t) - S(k;t)e(k,x;t) \quad \text{for } k \neq 0, t \geqslant 0, \tag{3.17}$$

where $S(k;t)$ is the scattering function of the SP (3.1)-(3.2). For every $t \geqslant 0$ $S(k;t)$ possesses the properties assembled in theorem 2.1.

The representation of $S(k;t)$ is deduced from (2.21) and (3.16):

$$S(k;t) = \frac{e(-k;t)}{e(k;t)} = e^{-2i\delta(k;t)}. \tag{3.18}$$

In view of (3.10) the eigenfunction (3.13) with $\mathrm{Im}\rho = 0$ is rewritten as follows:

$$\Omega(k,x;t) = -\frac{2i}{e(k;t)}\,\mathrm{Im}\big\{e(-k;t)e(k,x;t)\big\} \quad \text{for } k \neq 0, t \geqslant 0. \tag{3.19}$$

Substituting (3.7) with $\mathrm{Im}\rho = 0$ and (3.16) into (3.19), then taking into account that $|e(k;t)| \neq 0$ for $k \neq 0$, $t \geqslant 0$, we obtain the important formula for the representation of the normalization eigenfunction corresponding to the continuous spectrum of the SP (3.1), (3.14):

$$\begin{aligned}\Omega(k,x;t) = {}& -2ie^{-i\delta(k;t)}\Big\{\sin(kx - \delta(k;t)) \\ &+ \int_x^\infty K(x,\xi;t)\sin(k\xi - \delta(k;t))d\xi\Big\} \quad \text{for } k \neq 0 \text{ and } x \geqslant 0, t \geqslant 0.\end{aligned} \tag{3.20}$$

In formula (3.20) the improper integral depending on the parameter t is regularly convergent. Indeed, the kernel $K(x,\xi;t)$ obeys equation (2.9), in which $p(x)$ is replaced with the potential $p(x,t)$ satisfying inequality (3.3). Hence, according to (2.10) the kernel $K(x,\xi;t)$ satisfies the estimate:

$$|K(x,\xi;t)| \leqslant \frac{1}{2}\sigma_1\Big(\frac{x+\xi}{2};t\Big)\exp\Big\{\sigma_2(x;t) - \sigma_2\Big(\frac{x+\xi}{2};t\Big)\Big\} \quad \text{for all} \quad t \geqslant 0, \tag{3.21}$$

where $\sigma_1(x;t) = \int_x^\infty |p(\xi,t)|d\xi$, $\sigma_2(x;t) = \int_x^\infty \sigma_1(u;t)du$.

The regular convergence of the improper integral in (3.20) is deduced from estimate (3.21) and the boundedness of the function $\sin(k\xi - \delta(k;t))$.

Now we derive the EEq for the normalization eigenfunction corresponding to the continuous spectrum of the SP (3.1), (3.14). Differentiate (3.17) with respect to t:

$$\Omega_t(k,x;t) = e_t(-k,x;t) - S_t(k;t)e(k,x;t) - S(k;t)e_t(k,x;t). \tag{3.22}$$

Substituting (3.8) and (3.9) into (3.22), then using (3.17), we get

$$\begin{aligned}\Omega_t(k,x;t) = {}& [4ik^3 - p_x(x,t)]\Omega(k,x;t) + 2[2k^2 + p(x,t)]\Omega_x(k,x;t) \\ &+ [8ik^3 S(k;t) - S_t(k;t)]e(k,x;t).\end{aligned} \tag{3.23}$$

Due to (3.14) and (3.15), the passage in equation (3.23) to the limit as $x \to 0$ leads to the EEq for the scattering function $S(k;t)$ of the SP (3.1), (3.14):

$$S_t(k;t) = 8ik^3 S(k;t) - 4ik\frac{2k^2 + p(0,t)}{e^2(k;t)}, \quad k \neq 0,\ t \geqslant 0. \tag{3.24}$$

Substituting (3.24) into (3.23) we obtain the EEq for the normalization eigenfunction $\Omega(k,x;t)$ corresponding to the continuous spectrum of the SP (3.1), (3.14):

$$\begin{aligned}\Omega_t(k,x;t) &= [4ik^3 - p_x(x,t)]\Omega(k,x;t) + 2[2k^2 + p(x,t)]\Omega_x(k,x;t)\\ &\quad + 4ik\frac{2k^2 + p(0,t)}{e^2(k;t)}e(k,x;t) \quad \text{for } k \neq 0 \text{ and } x \geqslant 0,\ t \geqslant 0.\end{aligned} \tag{3.25}$$

Since $\Omega(k,x;t)$ and $\Omega_x(k,x;t)$ in the right-hand side of (3.25) satisfy the BCs (3.14), then $\Omega_t(k,x;t)$ in the left-hand side of (3.25) obeys condition (3.15). The EEq (3.25) is the t part of the Lax pair for the KdV equation on the positive quarter-plane. Indeed, we write the second derivative with respect to x of (3.22):

$$\Omega_{txx}(k,x;t) = e_{txx}(-k,x;t) - S_t(k;t)e_{xx}(k,x;t) - S(k;t)e_{txx}(k,x;t). \tag{3.26}$$

Computing the second derivative with respect to x of (3.17), then differentiating the obtained equation with respect to t we receive

$$\Omega_{xxt}(k,x;t) = e_{xxt}(-k,x;t) - S_t(k;t)e_{xx}(k,x;t) - S(k;t)e_{xxt}(k,x;t). \tag{3.27}$$

Assuming that the spectrum of the operator $L(t)$ is time-independent, i.e., $\lambda_t = 0$, then due to (3.8) and (3.9) we obtain that equations (3.26) and (3.27) are compatible: $\Omega_{txx} = \Omega_{xxt}$, if and only if the potential $p(x,t)$ in equation (3.1) satisfies the KdV equation (1.1) on the positive quarter-plane. Hence, the two linear eigenvalue equations (3.1) and (3.25) constitute as the x part and the t part, respectively, of the Lax pair.

Remark 3.1. If the third term in the right-hand side of (3.25) is cancelled, then this equation become the known EEq for the eigenfunction of the Cauchy problem for equation (3.1) on the full-line: $-\infty < x < \infty$ [14].

We thus arrive at the following lemma.

Lemma 3.2. *The normalization eigenfunction corresponding to the continuous spectrum of the SP* (3.1), (3.14) *with the real-valued potential satisfying estimate* (3.3) *has the representation* (3.20) *and evolves according to the EEq* (3.25).

The KdV equation (1.1) *on the positive quarter-plane can be written as the compatibility condition of two eigenvalue equations* (3.1) *and* (3.25). *The LSEq* (3.1) *and the EEq* (3.25) *constitute the Lax pair for the KdV equation* (1.1) *and are usually referred to as x part and the t part, respectively.*

Remark 3.2. The BVs $e(\rho;t)$ evaluated at $x = 0$ of the Jost solution give the most complete characterization of the SP (3.1), (3.2). The time-dependence of the scattering data set (2.25) is determined by the unknown BVs $e(\rho;t)$. Therefore, the main difficulty in solving the IBVP (1.1)-(1.3) lies in the determination of the unknown BVs in terms of the given data (1.2) and (1.3). We deal with this difficulty in the next two subsections.

3.3 The scattering phase and the determination of the unknown

By using (3.10), (3.16) and (3.18), from (3.24) we derive the EEq for the scattering phase of the SP (3.1), (3.14):

$$\delta_t(k;t) = -4k^3 + 2k\frac{2k^2 + p(0,t)}{|e(k;t)|^2}, \quad \text{for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.28}$$

Denote by b a large positive number. Consider the Sturm-Liouville SP in the interval $(0, b)$ generated by equation (3.1) with the BC (3.14) at $x = 0$ and the BC at $x = b$, [8, 9]:

$$y(\rho, b;t) = 0 \quad \text{for any } t \geqslant 0. \tag{3.29}$$

In view of lemma 3.2 the SP (3.1), (3.14) is associated with the IBVP (1.1)-(1.3). We show that the Sturm-Liouville SP (3.1), (3.14), (3.29) can be regarded as the linear problem associated with this IBVP. Indeed, the normalization eigenfunction $\Omega(k, x;t)$ of the SP (3.1), (3.14) has been determined by formula (3.20). Consider this defined eigenfunction (3.20) of the SP (3.1), (3.14) with the following BC at $x = b$:

$$\Omega(k, b;t) = 0 \quad \text{for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.30}$$

The equation (3.30) is written as the zero BC at $x = b$ of the normalization eigenfunction $\Omega(k, x;t)$ corresponding to the continuous spectrum of the Sturm-Liouville SP (3.1), (3.14), (3.29). t enters this equation as parameter and $\Omega(k, b;t)$ is the function of k. Therefore, the positive eigenvalues of the Sturm-Liouville SP (3.1), (3.14), (3.30) in essence are the roots of equation (3.30). Since the improper integral in (3.20) is regularly convergent, then we can write equation (3.30) in the asymptotic form:

$$\sin(kb - \delta(k;t)) = o(1) \quad (b \to \infty) \quad \text{for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.31}$$

It is evident that, the roots of (3.31) are determined by the asymptotic equation:

$$kb - \delta(k;t) = \text{Arcsin}(o(1)) = j\pi + o(1) \quad (b \to \infty) \text{ for } k \neq 0 \text{ and } t \geqslant 0, \quad j = 0, \pm 1, \pm 2, \ldots \tag{3.32}$$

It follows from (3.32) that the characterization of the SP (3.1), (3.14), (3.30) on the continuous spectrum is described by the approximately time-independent scattering phase. This means that

$$\delta_t(k;t) = o(1) \quad (b \to \infty) \text{ for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.33}$$

The asymptotic relation between $\delta(k;t)$ and $\delta(k)$ is deduced from (3.32) and (3.33):

$$\delta(k;t) = \delta(k) + o(1) \quad (b \to \infty) \text{ for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.34}$$

Then the scattering function (3.18) of the SP (3.1), (3.14), (3.30) is represented by the asymptotic formula:

$$S(k;t) = e^{-2i[\delta(k)+o(1)]} = S(k)[1 + o(1)] \quad (b \to \infty) \text{ for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.35}$$

Thus, the scattering function of the SP (3.1), (3.14), (3.30) is approximately time-independent. From (3.32) and (3.34) we obtain the asymptotic formula for finding the positive eigenvalues of the Sturm-Liouville SP (3.1), (3.14), (3.30):

$$\lambda = k^2 = \Big(\frac{\delta(k) + j\pi}{b}\Big)^2 + o\Big(\frac{1}{b^2}\Big) \quad (b \to \infty) \text{ for } k \neq 0 \;\; j = 0, \pm 1, \pm 2, \dots \tag{3.36}$$

Due to (3.33) the EEq (3.28) for the scattering phase is led to the asymptotic form:

$$-2k^2 + \frac{2k^2 + p(0,t)}{|e(k;t)|^2} = o(1) \quad (b \to \infty) \text{ for } k \neq 0 \text{ and } t \geqslant 0.$$

Since $e(k;t) \neq 0$ for $k \neq 0$ and $t \geqslant 0$, then the last asymptotic equation can be rewritten as follows

$$|e(k;t)|^2 = 1 + \frac{p(0,t)}{2k^2} + o(1) > 0 \quad (b \to \infty) \text{ for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.37}$$

By virtue of the obtained relations (3.34) and (3.37) we get the important asymptotic formulas for finding the unknown time-dependent BV (3.16):

$$\begin{aligned} e(k;t) &= |e(k;t)| e^{i\delta(k)} [1 + o(1)], \\ |e(k;t)| &= \sqrt{1 + \frac{p(0,t)}{2k^2} + o(1)} \quad (b \to \infty) \text{ for } k \neq 0 \text{ and } t \geqslant 0, \end{aligned} \tag{3.38}$$

where the scattering phase $\delta(k)$ of the SP (2.1), (2.2) is calculated from the given initial condition (1.2), therefore the unknown $e(k;t)$ is found in terms of the given initial and BCs (1.2) and (1.3).

Remark 3.3. If the given potential function $p(x,t)$ in (3.1) is so decreasing fast that

$$p(x,t) \equiv 0 \quad \text{for } x \geqslant b \text{ and } \int_0^b x|p(x,t)|dx < \infty \quad \text{for any } t \geqslant 0. \tag{3.39}$$

Then, from (3.21) it follows that the kernel $K(x,\xi;t)$ satisfies the condition:

$$K(x,\xi;t) \equiv 0 \text{ for } \xi \geqslant x \geqslant b \text{ and } t \geqslant 0.$$

Consequently, under the conditions (3.39) the zero BC at $x = b$ (3.30) of the normalization eigenfunction (3.20) is of the form:

$$\Omega(k,b;t) = -2ie^{-i\delta(k;t)} \sin(kb - \delta(k;t)) = 0 \quad \text{for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.40}$$

The roots of equation (3.40) are determined by

$$kb - \delta(k;t) = j_1\pi \quad \text{for } k \neq 0 \text{ and } t \geqslant 0, j_1 = 0, \pm 1, \pm 2, \dots$$

Hence, in the case of the potential satisfying conditions (3.39) the scattering phase is time-independent, i.e., $\delta_t(k;t) = 0$. Due to this fact the exact formulas for determining the unknown BV (3.16) for case (3.39) are deduced from (3.38):

$$e(k;t) = |e(k;t)| e^{i\delta(k)}, \quad |e(k;t)| = \sqrt{1 + \frac{p(0,t)}{2k^2}} \quad \text{for } k \neq 0 \text{ and } t \geqslant 0. \tag{3.41}$$

Now we take up the study of the time-dependence of the normalization multipliers (2.23).

3.4 The time-dependent scattering data set $s(t)$

The BV $e(k;t)$ calculated by (3.38) has a bounded analytic continuation $e(\rho;t)$ into the upper half-plane $\mathrm{Im}\rho > 0$ possessing the properties assembled in lemma 3.1. Knowing the function $e(\rho;t)$, $\mathrm{Im}\rho > 0$ we consider the normalization eigenfunction $\Omega(\rho,x;t)$, $\mathrm{Im}\rho > 0$, corresponding to the discrete spectrum of the Sturm-Liouville SP (3.1), (3.14), (3.29):

$$\Omega(\rho,x;t) = -\frac{2i\rho\omega(\rho,x;t)}{e(\rho;t)}\,, \quad \mathrm{Im}\rho > 0,$$

where $\omega(\rho,x;t)$ is an entire function for all values of ρ satisfying the BCs (2.16).

The time-dependence of the Jost solution $e(\rho,x;t)$, $\mathrm{Im}\rho > 0$ of problem (3.1), (3.6) is given by EEq (3.4):

$$e_t(\rho,x;t) = 2[2\rho^2 + p(x,t)]e_x(\rho,x;t) - [4i\rho^3 + p_x(x,t)]e(\rho,x;t), \quad \mathrm{Im}\rho > 0, \tag{3.42}$$

where the BV $e(\rho;t)$ has the simple zeros on the imaginary axis at $\rho = i\rho_j$, $\rho_j > 0$:

$$e(i\rho_j,0;t) = e(i\rho_j;t) = 0, \quad j = 1,\dots,n. \tag{3.43}$$

According to (2.23) the normalization multipliers $m_j(t)$ are represented in the form:

$$m_j^2(t) = \frac{2\rho_j}{ie_x(i\rho_j;t)\dot{e}(i\rho_j;t)}\,, \quad j = 1,\dots,n, \tag{3.44}$$

where $e_x(i\rho_j;t) = e_x(i\rho_j,0;t)$, $\dot{e}(i\rho_j;t) = \dfrac{\partial}{\partial\rho}e(\rho,0;t)\big|_{\rho=\rho_j}$, $j = 1,\dots,n$.
By using EEq (3.42) we can get rid of $e_x(i\rho_j;t)$ in formula (3.44). Indeed, suppose that in EEq (3.42) $x = 0$ and $\rho = i\rho_j$, then due to (3.43) $e_x(i\rho_j;t)$ is expressed in terms of the known data by the formula:

$$e_x(i\rho_j;t) = \frac{e_t(i\rho_j;t)}{2[p(0,t) - 2\rho_j^2]}\,, \quad j = 1,\dots,n, \tag{3.45}$$

where $e_t(i\rho_j;t) = e_t(i\rho_j,0;t)$.
Substituting (3.45) into (3.44) we obtain the representation of $m_j^2(t)$:

$$m_j^2(t) = \frac{4\rho_j[p(0,t) - 2\rho_j^2]}{i\dot{e}(i\rho_j;t)e_t(i\rho_j;t)}\,, \quad j = 1,\dots,n, \tag{3.46}$$

where $\dot{e}(i\rho_j;t)$ and $e_t(i\rho_j;t)$ are computed from the bounded analytic continuation $e(\rho;t)$ of the calculated BV (3.38) into the upper half-plane $\mathrm{Im}\rho > 0$. Therefore, the right-hand side of (3.46) is determined in terms of the given data (1.2) and (1.3). Furthermore, $m_j^2(0) = m_j^2$ and m_j^{-2} is defined by (2.23) from the given data (1.2).

The scattering function $S(k)$ and the multipliers $m_j(t)$ characterise the Sturm-Liouville SP (3.1), (3.14), (3.29) on the continuous and on the discrete spectrum, respectively. Thereby, the time-dependent scattering data set $s(t)$ for this problem is constructed:

$$s(t) = \big\{S(k), -\infty < k < \infty; i\rho_j, \rho_j > 0, m_j(t), j = 1,\dots,n\big\}, \tag{3.47}$$

where the set $s(t)$ for every $t \geqslant 0$ satisfies the properties I-II written in theorem 2.1.

The results obtained in this section play a crucial role in the following development and can be stated as follows.

Theorem 3.1. *The Sturm-Liouville SP in the interval* $(0, b)$ *(b is a large positive number) generated by equation* (3.1) *with the real-valued potential* $p(x, t)$ *satisfying estimate* (3.3) *and with the zero BCs* (3.14) *at* $x = 0$ *and* (3.29) *at* $x = b$ *is regarded as the linear problem associated with the IBVP* (1.1)-(1.3) *on the positive quarter-plane.*

1. *The characterization of the SP on the continuous spectrum is described by the approximately time-independent scattering phase and scattering function, which are represented by the asymptotic formulas* (3.34) *and* (3.35)*, respectively;*

2. *The unknown time-dependent BV* $e(k; t)$ *evaluated at* $x = 0$ *of the Jost solution of problem* (3.1), (3.6) *is found by the asymptotic formulas* (3.38) *in terms of the given initial and boundary data* (1.2) *and* (1.3) *of the IBVP* (1.1)-(1.3)*;*

3. *The characterization of the SP on the discrete spectrum is given by the time-dependent multipliers* $m_j(t)$*, which are calculated by the formulas* (3.46) *in terms of the given BC* (1.3) *and the bounded analytic continuation* $e(\rho; t)$ *of the calculated BV* $e(k; t)$ *into upper half-plane* $\mathrm{Im}\rho > 0$*;*

4. *If the potential* $p(x, t)$ *satisfies conditions* (3.39)*, then the unknown BV* $e(k; t)$ *is found by the exact formulas* (3.41).

4 The initial-boundary value problem (1.1)-(1.3)

The objective of the present paper is to solve the IBVP (1.1)-(1.3). We consider this problem in the class of real-valued functions satisfying estimate (3.3).

As shown in section 2, we construct from the given initial condition (1.2) a data set of the form (2.25) of the quantities satisfying the properties enumerated in theorem 2.1. In accordance with theorem 3.1 a time-dependent data set (3.47) is constructed from the set (2.25) and the given BC (1.3). Then the time-dependent function (2.26) is constructed from the set (3.47) by the formula:

$$F(x; t) = F_S(x) + \sum_{j=1}^{n} m_j^2(t) e^{-\rho_j x}, \quad x \geqslant 0,\ t \geqslant 0, \tag{4.1}$$

where the function $F_S(x)$ is represented in the form (2.28) and construcred by (2.27), (3.35): $F_S(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \{1 - S(k)[1 + o(1)]\} e^{ikx} dk$, $(b \to \infty)$.

As shown in sections 2 and 3, the problem of solving the IBVP (1.1)-(1.3) is reduced to that of solving the time-dependent equation of the form (2.24) for $0 \leqslant x < y < \infty$:

$$F(x + y; t) + K(x, y; t) + \int_x^{\infty} K(x, \xi; t) F(\xi + y; t) d\xi = 0,\ x \geqslant 0,\ t \geqslant 0. \tag{4.2}$$

Note that both x and t enter equation (4.2) merely as parameters, $F(x; t)$, $F_S(x)$ are known functions constructed from the given data set (3.47) and $K(x, y; t)$ is an unknown function of y for every $(x, t) \in [0, \infty) \times [0, \infty)$.

The constructed functions (4.1) satisfy the conditions I of theorem 2.1, therefore they have the derivatives with respect to x: $F'(x; t)$ and $F'_S(x)$ satisfying the estimates for any $t \geqslant 0$:

$$\int_0^{\infty} x[F'(x; t)| dx < \infty, \quad \int_0^{\infty} x|F'_S(x)| dx < \infty. \tag{4.3}$$

The formula (4.1), equation (4.2) and estimates (4.3), together with the properties of the function $S(k)$ written in theorem 2.1, show that the data set (3.47) calculated from the given data (1.2) and (1.3) satisfy the conditions I and II of theorem 2.1. Arguing as in section 2, we prove that the following facts are deduced from the conditions I and II of theorem 2.1:
1º the integral equation (4.2) has for every $(x,t) \in [0,\infty) \times [0,\infty)$ an unique solution $K(x,y;t) \in L_1(0,\infty)$;
2º the function of the form (3.7) constructed from the solution $K(x,y;t)$ of (4.2) satisfies an equation of the form (3.1) with the reconstructed potential:

$$p(x,t) = -2\frac{d}{dx}K(x,x;t), \quad x \geqslant 0,\ t \geqslant 0, \tag{4.4}$$

where $p(x,t)$ is a real-valued function satisfying estimate (3.3);
3º the data set of the type (3.47) constructed from the given data (1.2) and (1.3) is the scattering data of the Sturm-Lioyville SP in the interval $(0,b)$ for an equation of the form (3.1) with the constructed potential (4.4) and with the BCs (3.14), (3.29), i.e.,

$$e(-k,0;t) - S(k)[1+o(1)]e(k,0;t) = 0 \quad (b \to \infty),\ -\infty < k < \infty,\ t \geqslant 0,$$
$$e(i\rho_j,0;t) = 0, \quad t \geqslant 0, \quad \rho_j > 0, \quad j = 1,\dots,n;$$

4º a correspondence between the data set (3.47) and the constructed potential (4.4) is one-to-one, therefore the constructed potential (4.4) satisfies the KdV equation (1.1) on the positive quarter-plane with the conditions (1.2) and (1.3).

From the above discussion follows the following theorem.

Theorem 4.1. *Let $p(x)$ in* (1.2) *be a given real-valued function satisfying estimate* (2.3) *and let $f(t)$ in* (1.3) *be a given real-valued continuous function. Then*

1. *the time-dependent data set of the type* (3.47) *constructed by theorems 2.1 and 3.1 from the given data* (1.2) *and* (1.3) *is the scattering data of the Sturm-Liouville SP in the interval* $(0,b)$ *for an equation of the form* (3.1) *with the constructed potential* (4.4) *and with the zero BCs* (3.14) *at $x = 0$ and* (3.29) *at $x = b$;*

2. *a correspondence between the set* (3.47) *of the SP* (3.1), (3.14), (3.29) *and the constructed potential* (4.4) *is one-to-one and therefore, the potential* (4.4) *constructed from the given data* (1.2) *and* (1.3) *is the solution of the IBVP* (1.1)-(1.3) *in the class of real-valued functions satisfying estimate* (3.3). *Every solution of the IBVP* (1.1)-(1.3) *corresponds to an unique scattering data set of the type* (3.47) *and evolves from the continuous and discrete spectrum of the associated SP. Hence, a pure soliton-solution of the IBVP* (1.1)-(1.3) *does not exists.*

5 Conclusions

In this paper we demonstrate the developments of the inverse-scattering transform method for solving the IBVP (1.1)-(1.3). The developments consist of the following:

• Regarding the Sturm-Liouville SP in the interval $(0,b)$ (b is large positive number) generated by equation (3.1) and the zero BCs (3.14) at $x = 0$ and (3.29) at $x = b$, as the linear problem associated with the IBVP (1.1)-(1.3). The asymptotic formula (3.31) for the zero BC at $x = b$ of the normalization eigenfunction of the considered Sturm-Liouville SP is derived. This allows one to show the approximate time-independence of the scattering phase. Then, the asymptotic formulas (3.38) for calculating the unknown BV $e(k;t)$ in terms of the given data (1.2) and (1.3) are deduced from the EEq for the scattering phase;

• Establishing a one-to-one correspondence between the scattering data set (3.47) and the potential (4.4) constructed in term of the solution of the inverse-scattering problem. Therefore, the constructed potential is a solution of the IBVP (1.1)-(1.3) and every solution of the IBVP (1.1)-(1.3) corresponds to an unique scattering data set of the type (3.47).

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References

[1] Ablowitz M J and Segur H J, The inverse scattering transform: semi-infinite interval, *J. Math. Phys.* **16** (1975), 1054–1056.

[2] Bona J and Winther R, The Korteweg-de Vries equation, posed in a quarter-plane, *SIAM J. Math. Anal.* **14** (1983), 1056–1106.

[3] Degasperis A, Manakov S V and Santini P M, Initial-boundary value problems for linear PDEs: The analyticity approach, *arXiv:nlin.SI/0210058*, **1** (2002), 1–22.

[4] Fokas A S and Its A R, An initial-boundary value problem for the sine-Gordon equation in laboratory coordinates, *Teoret. Matem. Fizika* **92** (1992), 386–403.

[5] Fokas A S, Integrable Nonlinear Evolution Equations on the Half-Line, *Comm. Math. Phys.* **230** (2002), 1–39.

[6] Habibullin L T and Vil'danov A N, The KdV equation on a half-line, `arXiv: solv-int/9910002`.

[7] Marchant T R and Smyth N F, The initial-boundary problem for the Korteweg-de Vries equation on the negative quarter-plane, *Pro. Roy. Soc. London*, A **458** (2002), 1–15.

[8] Levitan B M, The Inverse Sturm-Liouville Problems, Nauka, Moscow, 1984.

[9] Levitan B M and Sargsjan I S, Introduction to Spectral Theory, Nauka, Moscow, 1970.

[10] Marchenko V A, Spectral Theory of Sturm-Liouville Operators, Naukova Dumka, Kiev, 1972.

[11] Pham Loi Vu, Some problems for a system of nonlinear equations on a half-line, *Inverse Problems* **17** (2001), 1889–1906.

[12] Pham Loi Vu, Some problems for Cubic nonlinear equations on a half-line, *Acta Applicandae Mathematicae* **84** (2004), 97–120.

[13] Pham Loi Vu, The Dirichlet initial-boundary-value problems for sine and sinh-Gordon equations on a half-line, *Inverse Problems* **21** (2005), 1225–1248.

[14] Zakharov E V, Manakov S V, Novikov S P and Pitaievski I P, Theory of Soliton, The Inverse Scattering Method, Nauka, Moscow, 1980.