Characterizations of probability distributions via bivariate regression of generalized order statistics

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Let $X_{i,n,m,k}$, $i = 1,...,n$ are n generalized order statistics (gos) based on an absolutely continuous distribution function F. Suppose that $\phi(x)$ is an absolutely continuous and monotonically increasing function in (a,b), $-\infty \le a < b \le \infty$, with finite $\phi(a^+)$, $\phi(b^-)$ and $E(\phi(X))$. We give characterization of distributions based on the regression of linear combinations of gos. Using our results can obtain characterizations of some distributions.

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1. Introduction

The random variables (r.v's) $X_{r,n,\widetilde{m},k}$, $1 \le r \le n$, $n \in N$, $k \ge 1$, $\widetilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$ are said to be gos from a distribution function F if their joint probability density function (pdf) is of the form (see Kamps [11])

$$f_{1,2,\cdots,n}(x_1, x_2, \ldots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\prod_{i=1}^{n-1}\left(\overline{F}(x_i)\right)^{m_i} f(x_i)\left(\overline{F}(x_n)\right)^{k-1} f(x_n),$$
$$F^{-1}(0) < x_1 < x_2 < \ldots < x_n < F^{-1}(1), \tag{1.1}$$

where $\overline{F}(.) = 1 - F(.)$, $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j > 0$ and $\gamma_n = k$.

The model of gos was introduced by Kamps [11], contains many models of ordered r.v's as special cases, e.g. order statistics, record values and sequential order statistics and progressive type II order statistics. Let ϕ be an absolutely continuous and monotonically increasing function in (a,b) with finite $\phi(a^+)$, $\phi(b^-)$ and finite expectation $E(\phi(X))$. The main problem considered in this paper is to characterize the some distributions for which the relations

$$E\left(\phi^{\tau}(X_{s,n,\widetilde{m},k})|X_{s-r,n,\widetilde{m},k}=x,X_{s+\ell,n,\widetilde{m},k}=z\right)=A(x,z), \tag{1.2}$$

and

$$E\left(\psi^{(\ell+r-1)}(\phi(X_{s,n,\widetilde{m},k}))|X_{s-r,n,\widetilde{m},k}=x,X_{s+\ell,n,\widetilde{m},k}=z\right)=B(x,z), \tag{1.3}$$

where $\psi^{\ell+r-1}(x)$ is continuous in (a,b); $A(.,.)$ and $B(.,.)$ are a real valued function satisfying certain regularity conditions. Bairamov et al. [5] introduced the characterization of the exponential type via regression on pairs of record values. Bairamov and Özkal [6] introduced the characterization of distributions through the properties of conditional expectations of order statistics. Akhundov and Nevzorov [4] characterized distributions via bivariate regression on differences of records. Yanev [14] proved that the exponential distribution is the only one which satisfies a regression identity. Yanev et al. [16] introduced characterizations of probability distributions via bivariate regression of record values. Yildiz and Bairamov [17] characterized several distributions by properties of conditional expectations of gos $X_{j,n,m,m+1}$ given $X_{j-p,n,m,m+1}=x$ and $X_{j+q,n,m,m+1}=y$. Yanev and Ahsanullah [15] characterized the exponential distribution based on regression of linear combinations of record values. The characterization of distributions via linearity of regression was obtained by many researchers, see for example, Ahsanullah and Beg [1], Balakrishnan and Akhundov [7], Bieniek and Szynal [9], Dembińska and Wesołowski [10] and Nagaraja [12]. Some articles on characterization of distributions by conditional expectations of functions are found in Ahsanullah and Hamedani [2], Ahsanullah and Raqab [3], Beg and Ahsanullah [8], Samuel [13] and Yanev [14].

In this paper, introducing characterize several distributions by properties of conditional expectations of gos $X_{s,n,m,k}=y$ given $X_{s-r,n,m,k}=x$ and $X_{s+\ell,n,m,k}=z$. We will also characterize some distributions in terms of order statistics, sequential order statistics and record values as special cases of gos.

2. Main Results

We will need the following Lemma (see Yanev et al. [16]) to prove Theorem 2.2.

Lemma 2.1. *(Yanev et al. [16]) Let $h(x)$ be a given function and for integer $i,j\geq 1$ define $M(u,v),M^i(u,v),M_j(u,v)$, and $M^i_j(u,v)$ as in*

$$M(u,v)=\frac{h(v)-h(u)}{v-u},\quad M^i_j(u,v)=\frac{\partial^{i+j}}{\partial u^i\partial v^j}\left(\frac{h(v)-h(u)}{v-u}\right),\quad (u\neq v). \tag{2.1}$$

If $h(x)$ has a continuous derivative of order $\max\{i,j\}$ over the interval (a,b), then for $a<u<v<b$

$$M_j(u,v)=\frac{h^{(j)}(v)-jM_{j-1}(u,v)}{v-u},\quad M^j(u,v)=\frac{jM^{j-1}(u,v)-h^{(j)}(u)}{v-u}, \tag{2.2}$$

and

$$M^i_j(u,v)=\frac{iM^{i-1}_j(u,v)-jM^i_{j-1}(u,v)}{v-u}, \tag{2.3}$$

where $M_1(u,v)$ and $M^1(u,v)$ are given in (2.2) and $M^1_1(u,v)=(M_1(u,v)-M^1(u,v))/(v-u)$.

If $m_1 = m_2 = \ldots = m_{n-1} = m$, then the joint pdf of the (r,s,ℓ)th gos, $X_{r,n,m,k}$, $X_{s,n,m,k}$ and $X_{\ell,n,m,k}$, $r < s < \ell$, as

$$f_{r,s,\ell,n,m,k}(x,y,z) = \frac{c_{\ell-1}}{(r-1)!(s-r-1)!(\ell-s-1)!} g_m^{r-1}\left(F(x)\right)\left(\overline{F}(z)\right)^{\gamma_\ell-1} \times W_{y,x}^{s-r-1} W_{z,y}^{\ell-s-1}\left(\overline{F}(x)\overline{F}(y)\right)^m f(x)f(y)f(z), \tag{2.4}$$

where $W_{y,x} = h_m(F(y)) - h_m(F(x))$, $g_m(z) = h_m(z) - h_m(0)$, $0 \le z < 1$ and

$$h_m(z) = \begin{cases} -(1-z)^{m+1}/(m+1), \; m \ne -1, \\ -\ln(1-z), \qquad m = -1. \end{cases} \tag{2.5}$$

Moreover the conditional density of $X_{s,n,m,k}$ given $X_{r,n,m,k}$ and $X_{\ell,n,m,k}$, $1 \le r < s < \ell \le n$, have the following form

$$f_{s|r,\ell,n,m,k}(y|x,z) = \frac{(\ell-r-1)!}{(s-r-1)!(\ell-s-1)!} \frac{W_{y,x}^{s-r-1} W_{z,y}^{\ell-s-1}}{W_{z,x}^{\ell-r-1}} \left(\overline{F}(y)\right)^m f(y), \quad x < y. \tag{2.6}$$

Since

$$\lim_{m \to -1}\left(\frac{1-x^{m+1}}{m+1}\right) = -\ln(x), \tag{2.7}$$

by using (2.6) for all values of $m \to -1$ and $k = 1$, we get the conditional density in the record values case. Our characterization results are based on the following two theorems.

Theorem 2.1. *Suppose that $\phi(x)$ is an absolutely continuous and monotonically increasing function x in (a,b), $-\infty \le a < b \le \infty$, with finite $E[\phi(X)]$ and for all positive integer s, r, ℓ, such that $1 \le r \le s-1$ and $\ell, \tau \ge 1$,*

$$E\left(\phi^\tau(X_{s,n,m,k}) | X_{s-r,n,m,k} = x, X_{s+\ell,n,m,k} = z\right) = \phi^\tau(x) \sum_{i=0}^{\tau} c_i \left(\frac{\phi(z)}{\phi(x)} - 1\right)^i, \tag{2.8}$$

if and only if

$$\overline{F}(x) = \begin{cases} \left(\dfrac{\phi(b^-) - \phi(x)}{\phi(b^-) - \phi(a^+)}\right)^{\frac{1}{m+1}}, & \text{for} \quad \phi(b^-) < \infty, \quad m > -1, \\ \exp\left(-\dfrac{\phi(x) - \phi(a)}{\phi(x_0) - \phi(a)}\right), & \text{for} \quad \phi(b^-) = \infty, \quad m = -1, \end{cases} \tag{2.9}$$

where $c_i = \binom{\tau}{i} \dfrac{(r+i-1)\cdots(r)}{(\ell+r+i-1)\cdots(\ell+r)}$, *$i > 1$ and $c_0 = 1$.*

Proof. To prove that (2.9) implies (2.8), from (2.6) that the conditional density of $X_{s,n,m,k} = y$ given $X_{s-r,n,m,k} = x$ and $X_{s+\ell,n,m,k} = z$ is

$$\begin{aligned}\xi_\tau(x,z) &= E\left[\phi^\tau(X_{s,n,m,k})|X_{s-r,n,m,k} = x, X_{s+\ell,n,m,k} = z\right] \\ &= \int_x^z \phi^\tau(y) f_{s|s-r,s+\ell,n,m,k}(y|x,z)dy \\ &= \frac{(\ell+r-1)!}{(r-1)!(\ell-1)!W_{z,x}^{\ell+r-1}} \int_x^z \phi^\tau(y) W_{y,x}^{r-1} W_{z,y}^{\ell-1} \left(\overline{F}(y)\right)^m f(y)dy. \qquad (2.10)\end{aligned}$$

If F is given by (2.9), with $m > -1$, it is easy to verify that equation (2.10) reduces to

$$\begin{aligned}\xi_\tau(x,z) = &\frac{(\ell+r-1)!}{(r-1)!(\ell-1)!} [\phi(z) - \phi(x)]^{-(\ell+r-1)} \int_x^z \phi^\tau(y) \\ &\times [\phi(y) - \phi(x)]^{r-1} [\phi(z) - \phi(y)]^{\ell-1} \phi'(y)dy. \qquad (2.11)\end{aligned}$$

Making the transformation $t = \dfrac{\phi(y) - \phi(x)}{\phi(z) - \phi(x)}$, we get

$$\xi_\tau(x,z) = \frac{(\ell+r-1)!}{(r-1)!(\ell-1)!} \int_0^1 t^{r-1} \left(\phi(x) + (\phi(z) - \phi(x))t\right)^\tau (1-t)^{\ell-1} dt, \qquad (2.12)$$

which reduce to

$$\xi_\tau(x,z) = \frac{\phi^\tau(x)}{B(r,\ell)} \sum_{i=0}^{\tau} \binom{\tau}{i} B(r+i,\ell) \left(\frac{\phi(z) - \phi(x)}{\phi(x)}\right)^i. \qquad (2.13)$$

By using the following relation

$$\frac{B(r+i,\ell)}{B(r,\ell)} = \frac{(r+i-1)\cdots(r)}{(\ell+r+i-1)\cdots(\ell+r)}, \qquad (2.14)$$

equation (2.8) holds. Then to prove the converse, since (2.8) is true for all positive integer s, r, ℓ, such that $1 \le r \le s-1$ and $\ell, \tau \ge 1$, let's assume that $\tau = r = \ell = 1$, we have

$$2\int_x^z \phi(y) \left(\overline{F}(y)\right)^m f(y)dy = W_{z,x} \left(\phi(z) + \phi(x)\right). \qquad (2.15)$$

Differentiating both sides of (2.15) with respect to z, and simplifying we get

$$\frac{\left(\overline{F}(z)\right)^m f(z)}{W_{z,x}} = \frac{\phi'(z)}{\phi(z) - \phi(x)}. \qquad (2.16)$$

Integrating both sides with respect to x from a^+ to b^-, (2.16) follows:

$$\frac{\left(\overline{F}(x)\right)^{m+1}}{\left(\overline{F}(x)\right)^{m+1} - 1} = \frac{\phi(b^-) - \phi(x)}{\phi(a^+) - \phi(x)}, \qquad (2.17)$$

and hence

$$\overline{F}(x) = \left(\frac{\phi(b^-) - \phi(x)}{\phi(b^-) - \phi(a^+)}\right)^{\frac{1}{m+1}}, \quad \forall x \in (a,b), \quad m > -1. \qquad (2.18)$$

With $k=1$, for $m_1=\cdots=m_{n-1}=m=-1$, taking limit $m\to -1$ and letting $x\to a^+$, we obtain from (2.15),

$$\int_{a^+}^{z}\phi(y)\frac{f(y)}{\overline{F}(y)}dy=\frac{-1}{2}\ln(1-F(z))\left(\phi(z)+\phi(a^+)\right). \tag{2.19}$$

Differentiating both sides of (2.19) with respect to z, and simplifying we get

$$\frac{f(z)}{(1-F(z))\,(-\ln(1-F(z)))}=\frac{\phi'(z)}{\phi(z)-\phi(a^+)}. \tag{2.20}$$

Integrating (2.20) with respect to z and simplifying, we get

$$F(z)=1-\exp\left(c\left(\phi(z)-\phi(a^+)\right)\right). \tag{2.21}$$

Since $F(z)$ is a decreasing function with $F(a^+)=0$ and $F(b^-)=1$, there exists a point x_0 such that $\overline{F}(x_0)=e^{-1}$. Thus $c=\phi(a^+)-\phi(x_0)$ and hence

$$F(z)=1-\exp\left(-\frac{\phi(z)-\phi(a^+)}{\phi(x_0)-\phi(a^+)}\right). \tag{2.22}$$

The proof is complete. □

Remark 2.1. Setting $r=1$ and $\ell=1$ in (2.8), we have

$$\begin{aligned}E\left(\phi^{\tau}(X_{s,n,\widetilde{m},k})|X_{s-1,n,\widetilde{m},k}=x,X_{s+1,n,\widetilde{m},k}=z\right)&=\frac{1}{\tau+1}\sum_{i=0}^{\tau}\phi^{\tau-i}(z)\phi^{i}(x)\\&=\frac{1}{\tau+1}\frac{\phi^{\tau+1}(z)-\phi^{\tau+1}(x)}{\phi(z)-\phi(x)}.\end{aligned} \tag{2.23}$$

Remark 2.2. Setting $\tau=1$ and for record values ($m_1=\ldots=m_{n-1}=m=-1$ and $k=1$), equation (2.23) reduce to

$$E\left(\phi(X_{(s)})|X_{(s-1)}=x,X_{(s+1)}=z\right)=\frac{1}{2}\left(\phi(x)+\phi(x)\right), \tag{2.24}$$

which agrees with Bairamov et al. [5] [Theorem 4. p. 546].

Remark 2.3. Setting $\tau=1$, $1\le r\le s-1$ and $\ell\ge 1$, equation (2.8) reduce to

$$E\left(\phi(X_{s,n,m,k})|X_{s-r,n,m,k}=x,X_{s+\ell,n,m,k}=z\right)=\frac{r\phi(z)+\ell\phi(x)}{r+\ell}. \tag{2.25}$$

Theorem 2.2. *Suppose that $\phi(x)$ is an absolutely continuous in $[a,b]$ such that $\psi^{\ell+r-1}(x)$ is continuous in (a,b), $-\infty\le a<b\le\infty$ and for all positive integer s, r, ℓ, such that $1\le r\le s-1$ and $\ell,\tau\ge 1$,*

$$E\left(\psi^{(\ell+r-1)}(\phi(X_{s,n,m,k}))|X_{s-r,n,m,k}=x,X_{s+\ell,n,m,k}=z\right)=a_{\ell,r}M_{r-1}^{\ell-1}\left(\phi(x),\phi(z)\right), \tag{2.26}$$

if and only if the continuous r.v. X has the distribution (2.9), where

$$a_{\ell,r}=\frac{(\ell+r-1)!}{(r-1)!(\ell-1)!}\quad\text{and}\quad M_j^i\left(\phi(x),\phi(z)\right)=\frac{\partial^{i+j}}{\partial x^i\partial z^j}\left(\frac{\psi(\phi(z))-\psi(\phi(x))}{\phi(z)-\phi(x)}\right). \tag{2.27}$$

Proof. Using (2.6) and (2.9) with $m > -1$, we have

$$\begin{aligned}\eta(x,z) &= E\left(\psi^{(\ell+r-1)}(\phi(X_{s,n,m,k}))|X_{s-r,n,m,k}=x, X_{s+\ell,n,m,k}=z\right)\\ &= \frac{(\ell+r-1)!}{(r-1)!(\ell-1)!}\left[\phi(z)-\phi(x)\right]^{-(\ell+r-1)}\int_x^z \psi^{(\ell+r-1)}(\phi(y))\\ &\times \left[\phi(y)-\phi(x)\right]^{r-1}\left[\phi(z)-\phi(y)\right]^{\ell-1}\phi'(y)dy. \qquad (2.28)\end{aligned}$$

Let $t=\phi(y)$, $u=\phi(x)$ and $v=\phi(z)$, we get

$$\eta(x,z) = \frac{(\ell+r-1)!}{(r-1)!(\ell-1)!}\left[v-u\right]^{-(\ell+r-1)}\int_u^v \psi^{(\ell+r-1)}(t)\left[t-u\right]^{r-1}\left[v-t\right]^{\ell-1}dt. \qquad (2.29)$$

From Lemma 2.1 (see Yanev et al. [16]), we have

$$\int_u^v h^{(\ell+r-1)}(t)\left[t-u\right]^{r-1}\left[v-t\right]^{\ell-1}dt = (v-u)^{\ell+r-1}M_{r-1}^{\ell-1}(u,v). \qquad (2.30)$$

From (2.30) in (2.29), then (2.26) holds. For the sufficient condition, differentiating both sides of (2.26) with respect to x, we have for $m > -1$,

$$\begin{aligned}&M_{r-1}^{\ell}(\phi(x),\phi(z))W_{z,x}^{\ell+r-1} - (\ell+r-1)W_{z,x}^{\ell+r-2}\left(\overline{F}(x)\right)^m f(x)M_{r-1}^{\ell-1}(\phi(x),\phi(z))\\ &= -(r-1)\left(\overline{F}(x)\right)^m f(x)\int_x^z \psi^{(\ell+r-1)}(\phi(y))W_{y,x}^{r-2}W_{z,y}^{\ell-1}\left(\overline{F}(y)\right)^m f(y)dy. \qquad (2.31)\end{aligned}$$

Making use of Theorem 1 (see Yenev et al. [16]), we have

$$\grave{M}_{r-2}^{\ell-1}(\phi(x_2),\phi(z))W_{z,x_2}^{\ell+r-2} = \int_{x_2}^z \psi^{(\ell+r-1)}(\phi(y))W_{y,x_2}^{r-2}W_{z,y}^{\ell-1}\left(\overline{F}(y)\right)^m f(y)dy, \qquad (2.32)$$

where

$$\grave{M}(x,z) = \left(\frac{\psi'(z)-\psi'(x)}{z-x}\right). \qquad (2.33)$$

Using (2.31), (2.32) and let $x_2 \to x$, we have

$$\left(\overline{F}(x)\right)^m f(x) = \frac{W_{x,z}M_{r-1}^{\ell}(\phi(x),\phi(z))}{(r-1)\grave{M}_{r-2}^{\ell-1}(\phi(x),\phi(z)) - (\ell+r-1)M_{r-1}^{\ell-1}(\phi(x),\phi(z))}. \qquad (2.34)$$

Since

$$\begin{aligned}&(r-1)\grave{M}_{r-2}^{\ell-1}(\phi(x),\phi(z)) - (\ell+r-1)M_{r-1}^{\ell-1}(\phi(x),\phi(z))\\ &= (r-1)M_{r-2}^{\ell}(\phi(x),\phi(z)) - \ell M_{r-1}^{\ell-1}(\phi(x),\phi(z)), \qquad (2.35)\end{aligned}$$

then from (2.34) and applying (2.3), we obtain

$$\begin{aligned}\frac{\left(\overline{F}(x)\right)^m f(x)}{W_{x,z}} &= \frac{M_{r-1}^{\ell}(\phi(x),\phi(z))}{(r-1)M_{r-2}^{\ell}(\phi(x),\phi(z)) - \ell M_{r-1}^{\ell-1}(\phi(x),\phi(z))}\\ &= \frac{\phi'(x)}{\phi(x)-\phi(z)}. \qquad (2.36)\end{aligned}$$

Integrating both sides with respect to x from a^+ to b^-, we obtain

$$\frac{\left(\overline{F}(z)\right)^{m+1}}{\left(\overline{F}(z)\right)^{m+1}-1}=\frac{\phi(b)-\phi(z)}{\phi(a)-\phi(z)}, \tag{2.37}$$

and then

$$\overline{F}(z)=\left(\frac{\phi(b)-\phi(z)}{\phi(b)-\phi(a)}\right)^{1/(m+1)}.$$

For $m_1=m_2=\ldots=m_{n-1}=m=-1$, $k=1$ and taking limit $m\to-1$, equation (2.36) reduce to

$$\frac{f(x)/\overline{F}(z)}{\ln\left(\overline{F}(z)\right)-\ln\left(\overline{F}(x)\right)}=\frac{\phi'(x)}{\phi(x)-\phi(z)}.$$

Since $\ln\left(\overline{F}(x)\right)$ is a decreasing function with $\ln\left(\overline{F}(a^+)\right)=0$ and $\ln\left(\overline{F}(b^-)\right)=\infty$, there exists a point x_0 s.t, $\overline{F}(x_0)=e^{-1}$. Then by integrating both sides with respect to x from a^+ to x_0, we obtain

$$\frac{\ln\left(\overline{F}(z)\right)-\ln\left(\overline{F}(x_0)\right)}{\ln\left(\overline{F}(z)\right)}=\frac{\phi(x_0)-\phi(z)}{\phi(a)-\phi(z)},$$

and hence

$$\overline{F}(z)=\exp\left(-\frac{\phi(z)-\phi(a^+)}{\phi(x_0)-\phi(a^+)}\right),\forall z\in(a,b).$$

Thus, the theorem is proved. □

Remark 2.4. For record values, equation (2.26) reduce to

$$E\left(\psi^{(\ell+r-1)}(X_{(s)})|X_{(s-r)}=x,X_{(s+\ell)}=z\right)=\frac{(\ell+r-1)!}{(r-1)!(\ell-1)!}M_{r-1}^{\ell-1}(x,z), \tag{2.38}$$

which agrees with Yanev et al. [16] [Lemma 2].

3. Characterization of distributions

In this section, we suggest some applications based on conditional expectations can easily be discussed arise as special cases of our results.

(1) If $\phi(x)=x^\beta$, β is any constant, then

$$E\left(X_{s,n,m,k}^{\tau\beta}|X_{s-r,n,m,k}=x,X_{s+\ell,n,m,k}=z\right)=x^{\tau\beta}\sum_{i=0}^{\tau}c_i\left(\left(\frac{z}{x}\right)^\beta-1\right)^i, \tag{3.1}$$

if and only if

$$\overline{F}(x)=\left(\frac{b^\beta-x^\beta}{b^\beta-a^\beta}\right)^{\frac{1}{m+1}},a<x<b,\quad m>-1. \tag{3.2}$$

where a and b are restricted by β. By suitably choosing β, a and b we can easily characterize the following distributions.

(a) When $\beta = 1$, $m_1 = ... = m_{n-1} = m = \alpha - 1$ and $k = \alpha$, then the beta of the first kind distribution for sequential order statistics is given by

$$\overline{F}(x) = \left(\frac{b-x}{b-a}\right)^{\delta}, a \le x \le b, \quad \delta = \frac{1}{\alpha},$$

this includes rectangular distribution for $\alpha = 1$, e.g, $m_1 = ... = m_{n-1} = m = 0$ and $k = 1$ (order statistics).

(b) $\overline{F}(x) = \left(1 - x^{\beta}\right)^{1/(m+1)}, 0 < x < 1$, $\beta > 0$, $m > -1$. For order statistics, we have a power function distribution of the form

$$F(x) = x^{\beta}, \quad 0 < x < 1, \quad \beta > 0, \tag{3.3}$$

and for record values ($m_1 = ... = m_{n-1} = m = -1$ and $k = 1$), we have the Weibull distribution of the form

$$F(x) = 1 - \exp\left(-\theta x^{\beta}\right), \quad 0 < x < \infty, \quad \beta > 0, \quad \theta = x_0^{-\beta}. \tag{3.4}$$

(c) $\overline{F}(x) = \left(\frac{1}{x}\right)^{\beta/(m+1)}, x > 1$, $\beta > 0$, $m > -1$. For order statistics, we have a Pareto distribution of the form $F(x) = 1 - \left(\frac{1}{x}\right)^{\beta}, x > 1$, $\beta > 0$.

(2) If $\phi(x) = \exp(-\beta x)$, β is any constant, then

$$E\left[e^{-\beta X_{s,n,m,k}} | X_{s-r,n,m,k} = x, X_{s+\ell,n,m,k} = z\right] = e^{-\tau\beta x} \sum_{i=0}^{\tau} c_i \left(e^{-\beta(z-x)} - 1\right)^i, \tag{3.5}$$

if and only if

$$\overline{F}(x) = \left(\frac{e^{-b\beta} - e^{-\beta x}}{e^{-b\beta} - e^{-a\beta}}\right)^{\frac{1}{m+1}}, a < x < b, \quad m > -1. \tag{3.6}$$

(a) For order statistics, we get characterization of the exponential distribution,

$$F(x) = 1 - \exp(-\beta x), 0 < x, \quad \beta > 0.$$

(b) For record values and $\beta = -\alpha > 0$, we have characterization of the Gompertz distribution,

$$F(x) = 1 - \exp\left(-\frac{\theta}{\alpha}\left(e^{\alpha x} - 1\right)\right), \quad x > 0, \quad \theta, \alpha > 0, \tag{3.7}$$

where $\theta = \alpha\left(1 - e^{\alpha x_0}\right)^{-1}$.

(3) If $\phi(x) = x$, then

$$E\left(\psi^{(\ell+r-1)}(X_{s,n,m,k}) | X_{s-r,n,m,k} = x, X_{s+\ell,n,m,k} = z\right) = a_{\ell,r} M_{r-1}^{\ell-1}(x,z), \tag{3.8}$$

holds if and only if the continuous r.v. X has the exponential distribution.

(a) For record values, (3.8) becomes

$$E\left(\psi^{(\ell+r-1)}(X_{(s)})|X_{(s-r)}=x,X_{(s+\ell)}=z\right)=\frac{(\ell+r-1)!}{(r-1)!(\ell-1)!}M_{r-1}^{\ell-1}(x,z), \quad (3.9)$$

which agrees with Yanev et al. [16] [Theorem 1.].

(b) If $\psi^{(r+\ell-1)}(x)=x$ and thus $\psi(x)=\dfrac{x^{r+\ell}}{(r+\ell)!}$. Then (3.9) becomes

$$E\left(X_{(s)}|X_{(s-r)}=x,X_{(s+\ell)}=z\right)=\frac{rx+\ell z}{r+\ell},$$

which agrees with Yanev et al. [16] [Theorem 2.].

References

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