Journal of Nonlinear Mathematical Physics Volume 14, Number 2 (2007), 290–310 ARTICLE

The Riccati and Ermakov-Pinney hierarchies

Marianna EULER[1], *Norbert EULER*[2] *and Peter LEACH*[3]

Institut Mittag-Leffler, The Royal Swedish Academy of Sciences
Auravägen 17, SE-182 60 Djursholm, Sweden

Abstract

The concept and use of recursion operators is well-established in the study of evolution, in particular nonlinear, equations. We demonstrate the application of the idea of recursion operators to ordinary differential equations. For the purposes of our demonstration we use two equations, one chosen from the class of linearisable hierarchies of evolution equations studied by Euler *et al* (*Stud Appl Math* **111** (2003) 315-337) and the other from the class of integrable but nonlinearisible equations studied by Petersson *et al* (*Stud Appl Math* **112** (2004) 201-225). We construct the hierarchies for each equation. The symmetry properties of the first hierarchy are considered in some detail. For both hierarchies we apply the singularity analysis. For both we observe intersting behaviour of the resonances for the different possible leading order behaviours. In particular we note the proliferation of subsidiary solutions as one ascends the hierarchy.

1 Introduction

It is known that one can construct integrable partial differential equations (or system of partial differential equations) by the use of so-called recursion operators, $R[u]$, which generate an infinite number of Lie-Bäcklund symmetries [20, 7, 8]. Those type of equations are usually described as being *symmetry integrable*. The main problem is to find the recursion operator for a given system or to show that an infinite number of Lie-Bäcklund symmetries exists or that it does not exist (the latter being the more demanding task). Since the recursion operator can in general contain nonlocal variables even for equations linearisable by a (nonlocal) coordinate transformation (Petersson *et al* [22]), the procedure is not an easy one especially for higher order equations and for systems.

In their paper Euler *et al* [5] report a large collection of recursion operators for second-order evolution equations. In particular eight classes of second-order linearisable evolution equations and their recursion operators are given one of which is Class VIII, namely

$$u_t = u_{xx} + \lambda_8 u_x + h_8 u_x^2. \tag{VIII}$$

Copyright © 2006 by M Euler, M Euler and PGL Leach

[1]Permanent address: Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden, Email: marianna@sm.luth.se

[2]Author to whom correspondence should be addressed. Permanent address: Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden, Email: norbert@sm.luth.se

[3]Permanent address: School of Mathematical Sciences, Howard College, University of KwaZulu-Natal, Durban 4041, Republic of South Africa, Email: leachp@ukzn.ac.za

Here h_8 is an arbitrary C^∞-functions depending on u and λ_8 an arbitrary constant. Equation (VIII) admits the following recursion operator

$$R_8[u] = D_x + h_8 u_x.$$

In their paper Petersson *et al* [22] report the classification of the symmetry integrable class of equations

$$u_t = u^\alpha u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + r(u)u_{xx} + p(u)u_x^2 + q(u)u_x + s(u). \tag{1.1}$$

The equation

$$u_t = u^3 u_{xxx} + \lambda_1 u^3 u_x + \lambda_2 u^{-1} u_x + \lambda_3 u_x \tag{1.2}$$

(λ's are arbitrary constants) is a member of the class (1.1) [22]. Equation (1.2) is of interest to us in the context of ordinary differential equations due to its reduction to an Ermakov-Pinney equation. For this purpose we write (1.2) in potential form through the change of dependent variable $v_x = u^{-2}$. The potential equation is

$$v_t = v_x^{-3/2} v_{xxx} - \frac{3}{2} v_x^{-5/2} v_{xx}^2 - 2\lambda_1 v_x^{-1/2} + \frac{2\lambda_2}{3} v_x^{3/2} + \lambda_3 v_x + C. \tag{1.3}$$

(C is an arbitrary constant) which is a slight generalisation of the *Cavalcante-Tenenblat equation* [2] and admits the recursion operator

$$\begin{aligned} R[v] \quad = \quad & v_x^{-1} D_x^2 - \frac{3}{2} v_x^{-2} v_{xx} D_x - \frac{1}{2} v_x^{-2} v_{xxx} + \frac{3}{4} v_x^{-3} v_{xx}^2 + \lambda_1 v_x^{-1} + \frac{\lambda_2}{3} v_x \\ & - \frac{1}{4}\left(v_x^{-3/2} v_{xxx} - \frac{3}{2} v_x^{-5/2} v_{xx}^2 - 2\lambda_1 v_x^{-1/2} + \frac{2\lambda_2}{3}\right) D_x^{-1} v_x^{-3/2} v_{xx}. \end{aligned} \tag{1.4}$$

In this paper we apply the notion of recursion operators to ordinary ordinary differential equations in a very natural way. We take a $1+1$ evolution equation with a recursion operator and suppress the time dependence. This produces an ordinary differential equation with a recursion operator provided the recursion operator in the partial differential equation is free of t. The linearisable second-order evolution equations provide a good source of hierarchies for our study. Our interest is primarily the connection of the integrability of the members of the hierarchies to the symmetry and singularitity properties of the equations. Although it would be of interest to see what happens for all the eight classes listed above, we confine our present attentions to just one class. This class, given an extensive treatment in §2, is derived from Class VIII since the two early memebers of the hierarchy are the Riccati [26] and the Painlevé–Ince equation ([21][(9) p 33] and [13][p 332]), both of which find frequent mention in the literature. We see in §2 that this hierarchy possesses some very attractive features.

In §3 we make a parallel study of the Ermakov–Pinney Equation [4, 23] and the hierarchy generated from it. Naturally there are some differences in emphasis in large measure dictated by the greater complexity of the members of the hierarchy as the recursion moves in steps of two. The contrast of the properties under the singularity analysis is noted, but the most interesting feature is the nature of the relationship between the first integrals of higher members of the hierarchy which we find to be rather more subtle than a simple-minded expectation would envisage. Finally in §4 we conclude with some observations.

2 A Riccati hierarchy

We commence with the simplest of the recursion operators, *videlicet* that of Class VIII which is

$$R_8[u] = D_x + h_8(u)u_x. \tag{2.1}$$

The basic evolution equation is

$$u_t = u_{xx} + \lambda_8 u_x + h_8 u_x^2 \tag{2.2}$$

(here and below we suppress the variable dependence in h_8 unless it is necessary for contextual clarity), where λ_8 is some parameter. By the t-translation symmetry the corresponding ordinary differential equation is

$$u'' + h_8 u'^2 + \lambda_8 u' = 0 \tag{2.3}$$

and successive members of the hierarchy can be obtained by the action of $R_8[u] + C$ on the left hand side of (2.3). The prime denotes x-derivatives. The two members following are

$$u''' + 3h_8 u'u'' + \left(\dot{h}_8 + h_8^2\right) u'^3 + (\lambda + C)\left(u'' + h_8 u'^2\right) + \lambda C u' = 0 \tag{2.4}$$

and

$$\begin{aligned} &u'''' + 4h_8 u'u''' + 3h_8 u''^2 + 6\left(\dot{h}_8 + h_8^2\right) u'^2 u'' + \left(\ddot{h}_8 + 3h_8\dot{h}_8 + h_8^3\right) u'^4 \\ &+(\lambda + 2C)\left[u''' + 3h_8 u'u'' + \left(\dot{h}_8 + h_8^2\right) u'^3\right] + C(2\lambda + C)\left(u'' + h_8 u'^2\right) + \lambda C^2 u' = 0 \end{aligned} \tag{2.5}$$

in which we maintain the notation that the overdot on h_8 represents differentiation with respect to its argument. Two points are immediately apparent. The first is that the members of the hierarchy quickly become very complicated equations. The second is that substructures in the higher order repeat the structure of the lower order with some suggestive variations. In particular we observe members reminescent of the Riccati and Painlevé–Ince Equations, *videlicet*

$$w' + w^2 = 0 \quad \text{and} \quad w'' + 3ww' + w^3 = 0 \tag{2.6}$$

in $\dot{h}_8 + h_8^2$ and $\ddot{h}_8 + 3h_8\dot{h}_8 + h_8^3$. We also note that for h_8 a constant (2.3) is a Riccati Equation in the variable u'.

These observations lead us to make some adjustments to our definition. In the process we depart from our initial point of the Class VIII equation. This departure is possible due to the transition from an evolution partial differential equation to an ordinary differential equation. To enable a clear discussion we set $\lambda = 0 = C$ and $h_8 = 1$. We return to the general case below.

We write the hierarchy in potential form by replacing u' with y so that the first few members of the hierarchy in the simplified form are

$$y' + y^2 = 0 \tag{2.7}$$

$$y'' + 3yy' + y^3 = 0 \tag{2.8}$$

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 = 0 \tag{2.9}$$

$$y'''' + 5yy''' + 10y'y'' + 10y^2y'' + 15yy'^2 + 10y^3y' + y^5 = 0 \tag{2.10}$$

with the recursion operator

$$R = D + y, \tag{2.11}$$

where we replace D_x with D since there is just the single independent variable.

Successive application of (2.11) to (2.7) generates the higher members of the hierarchy, which we may as well call the Riccati hierarchy. By construction (2.8), (2.9) and (2.10) are symmetry coefficients of (2.7); (2.9) and (2.10) are symmetry coefficients of (2.8) and (2.10) a symmetry coefficient of (2.9).

A different origin of the hierarchy can be obtained by means of the operator adjoint to R, *videlicet*

$$R^* = D - y, \tag{2.12}$$

by seeking the function, f, which R^* annihilates. Then

$$R^*f = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}f}{\mathrm{d}x} - fy = 0 \quad \Longrightarrow \quad f = \exp\left[\int y\mathrm{d}x\right]. \tag{2.13}$$

Then R^2f, R^3f, R^4f *etc* generate the hierarchy[4]

$$\begin{aligned} y' + 2y^2 &= 0 && (2.14)\\ y'' + 6yy' + 4y^3 &= 0 && (2.15)\\ y''' + 8yy'' + 6y'^2 + 24y^2y' + 8y^4 &= 0 && (2.16)\\ y'''' + 10yy''' + 20y'y'' + 40y^2y'' + 60yy'^2 + 80y^3y' + 16y^5 &= 0. && (2.17)\end{aligned}$$

The resemblance to the creation and annihilation operators of the quantum mechanical simple harmonic oscillator should not be pushed too far. Once the adjoint operator has produced the generating function, it does not act as a lowering operator on the higher members of the hierarchy. Rather it produces its own line of equations. For example, when R^* acts repeatedly on the element Rf, one obtains the hierarchy of elementary linear equations

$$y^{(n)} = 0, \quad n = 1, 2, \ldots. \tag{2.18}$$

In this generation of a second hierarchy we have the departure from the usual situation in Quantum Mechanics. The failure to have the second hierarchy of solutions to exist in Quantum Mechanics is due not to the lack of a suitable operator but to the imposition of some boundary condition such as the vanishing of the solution at infinity. Before we leave this brief digression we note that one could construct a parallel hierarchy using R to produce the generating function and R^* to develop the hierarchy. Thus

$$Rf = 0 \quad \Leftrightarrow \quad f' + fy = 0 \quad \Longrightarrow \quad f = \exp\left[-\int y\mathrm{d}x\right] \tag{2.19}$$

and $R^{*2}f$, $R^{*3}f$ and $R^{*4}f$ lead to

$$\begin{aligned} y' - 2y^2 &= 0 && (2.20)\\ y'' - 6yy' + 4y^3 &= 0 && (2.21)\\ y''' - 8yy'' - 6y'^2 + 24y^2y' - 8y^4 &= 0 && (2.22)\end{aligned}$$

[4]The equation generated by Rf, *videlicet* $y = 0$, is a bit trivial.

which mimic the elements of the other hierarchy and become identical when y is replaced by $-y$.

A simplification of the members of this hierarchy, (2.20) to (2.22), can be achieved by multiplying each equation by two and replacing $2y$ with y. Thus the first few elements of the hierarchy generated by R become identical to (2.8) to (2.10) above which were generated from (2.7). In (2.7) and (2.8) the Riccati and Painlevé-Ince Equations are quite evident. Henceforth we confine our attention to this representation of the hierarchy.

The linearisability of the Class VIII nonlinear evolution partial differential equations, which was established in [5], implies integrability in the sense of an infinite number of Lie–Bäcklund symmetries. In the case of ordinary differential equations the criteria for linearisability depend, as always, upon the type of transformation admitted. The Riccati hierarchy treated here is distinguished by the possession of the maximal number of Lie point symmetries possible at that relevant order[5]. Thus (2.8) has eight Lie point symmetries with the algebra $sl(3, R)$. The third-order member, (2.9), has seven Lie point symmetries. There are ten contact symmetries with the algebra $sp(5)$ [1]. Thereafter the sequence becomes more orderly with the nth-order member having $n+4$ Lie point symmetries with the algebra $A_{3,8} \oplus_s \{A_1 \oplus_s nA_1\}$, where we use the Mubarakzyanov classification scheme [16, 17, 18, 19]. The subalgebra $A_{3,8}$ (also popularly known as $sl(2, R)$) is characteristic of scalar nth-order ordinary differential equations of maximal symmetry [15]. The interesting feature about the possession of the maximal number of Lie point symmetries is that the most convenient linearising transformation, that based on the Riccati Equation and the direct counterpart of the linearising transformation for the Class VIII nonlinear evolution partial differential equation, is a nonlocal transformation. If we multiply (2.7) by the integrating factor $\exp\left[\int y\mathrm{d}x\right]$, (2.7) may be written as $\left(\exp\left[\int y\mathrm{d}x\right]\right)'' = 0$ so that the transformation

$$w = \left(\exp\left[\int y\mathrm{d}x\right]\right)' \tag{2.23}$$

immediately produces the required linear equation, $w' = 0$. We note that w is just the first member of the hierarchy obtained by action with R on the generating function.

[5]Although the statement is true for *the* Riccati Equation (2.7), it is not a useful statement. Nevertheless the linearising transformation which works for the higher members of the hierarchy is in fact found using the selfsame Riccati equation.

Thus we have

Proposition I: *The members of the Riccati hierarchy possess the maximal number of Lie point symmetries for an equation of that order* (≥ 2)

and

Proposition II: *The members of the Riccati hierarchy are linearised by the transformation* $x = x, \quad w = \left(\exp\left[\int y\mathrm{d}x\right]\right)' = y\exp\left[\int y\mathrm{d}x\right]$.

A consequence of Proposition II is that there exists a general formula for the solutions of the Riccati hierarchy and we have

Proposition III: *The general solution of the* n*th member of the Riccati hierarchy,* $n \geq 3$*, is given by*

$$y_n = \frac{\left(\sum_{i=0}^{n-1} A_i x^i\right)'}{\sum_{i=0}^{n-1} A_i x^i}, \tag{2.24}$$

where the A_i*,* $i = 1, n(1)$*, are the constants of integration.*

Corollary: *The solution to the original hierarchy, ie without the introduction of the potential form, is*

$$u_n = \log\left[\sum_{i=0}^{n-1} A_i x^i\right]. \tag{2.25}$$

The Riccati Equation is well-known [13][290ff] to be the only first-order equation of the form $y' = f(x, y)$, where f is rational in y and analytic in x, to possess the Painlevé Property. The Painlevé–Ince Equation, (2.8), not only possesses the Painlevé Property but is also distinguished as being one of the few equations, *ie* a fraction of those equations of the same order possessing the Painlevé Property, which has both a Left Painlevé series and a Right Painlevé Series [6]. One immediately wonders if this be a property generic to the hierarchy. The application of the Painlevé Test is a routine matter and we need not dwell upon its details. Rather we summarise the application of the Painlevé Test to the first several members of the hierarchy, (2.7) to (2.10). For a discussion of the application of the singularity analysis to ordinary differential equations we refer the reader to the works of Tabor [29] and Ramani *et al* [25] for the techniques, Conte [3] for some deeper analysis and Feix *et al* [10] for a broader discussion of the philosophy. To determine the leading order behaviour we set $y = \alpha\chi^p$, where $\chi = x - x_0$ and x_0 is the location of the putative singularity which in our case is always a simple pole, *ie* $p = -1$, since each member of the hierarchy is invariant under the action of the similarity symmetry $-x\partial_x + y\partial_y$ [6]. The value of α is found from the solution of a polynomial equation. The solution is then written as a Laurent series commencing with the leading order term and the powers at which the remaining arbitrary coefficients enter are found by substituting $y = \alpha\chi^{-1} + \mu\chi^{r-1}$ and equating the coefficient of μ to zero. Provided that all is satisfactory to this point, the consistency of the presumed arbitrary constants needs to be checked.

With this deliberately potted version of the Painlevé Test we summarise the results of the application of the test for the earlier members of the hierarchy in Table 1. Of particular interest is the value of the resonances for the different roots of the polynomial equation determining α.

Table 1. Summary of the results of the Painlevé Test applied to the earlier members of the Riccati hierarchy. We commence the numbering of the members from the Riccati equation, (2.7).

Member	Characteristic equations for α and r	Roots
I	$\alpha^2-\alpha=0$ $r+1=0$	$\alpha=0,1$ $r=-1$
II	$\alpha^3-3\alpha^2+2\alpha=0$ $r^2+(3\alpha-3)r+3\alpha^2-6\alpha+2=0$	$\alpha=0,1,2$ $\alpha=1:\ \ r=-1,1$ $\alpha=2:\ \ r=-1,-2$
III	$\alpha^4-6\alpha^3+11\alpha^2-6\alpha=0$ $r^3+(4\alpha-6)r^2+(6\alpha^2-18\alpha+11)r$ $+4\alpha^3-18\alpha^2+22\alpha-6=0$	$\alpha=0,1,2,3$ $\alpha=1:\ \ r=-1,1,2$ $\alpha=2:\ \ r=-1,1,-2$ $\alpha=3:\ \ r=-1,-2,-3$
IV	$\alpha^5-10\alpha^4+35\alpha^3-50\alpha^2+24\alpha=0$ $r^4+5(\alpha-2)r^3+5(2\alpha^2-8\alpha+7)r^2$ $+5(2\alpha^3-12\alpha^2+21\alpha-10)r$ $+5\alpha^4-40\alpha^3+105\alpha^2-100\alpha+24=0$	$\alpha=0,1,2,3,4$ $\alpha=1:\ \ r=-1,1,2,3$ $\alpha=2:\ \ r=-1,1,2,-2$ $\alpha=3:\ \ r=-1,1,-2,-3$ $\alpha=4:\ \ r=-1,-2,-3,-4.$

From Table 1 we can discern the pattern for the higher order equations. For the nth-order member of the Riccati hierarchy the nontrivial values of the coefficient of the leading order term can be $1,2,\ldots,n$. In the case of the Riccati Equation itself we have only the generic resonance. For the Painlevé–Ince Equation the nongeneric resonance corresponding to each of the two possible values of α indicates the existence of a Left Painlevé Series and a Right Painlevé Series, as is well-known [6]. For the higher order equations the different principal branches indicate quite diverse behaviours. A Right Painlevé Series exists for the smallest value of α and a Left

Table 2. Summary of the values of α for the values of r obtained using the Painlevé Test on the fourth member of the Riccati hierarchy. The shifted symmetry about the nonpossible value $r = 0$ is even more obvious if one sets $r = 4$ which is not a value found in the analysis.

Value of the resonance	Corresponding values for α
$r = 3$	$\alpha = 0, 1, (1 \pm i\sqrt{7})/2$
$r = 2$	$\alpha = 0, 1(2), 2$
$r = 1$	$\alpha = 0, 1, 2, 3$
$r = -1$	$\alpha = 1, 2, 3, 4$
$r = -2$	$\alpha = 2, 3(2), 4$
$r = -3$	$\alpha = 3, 4, (7 \pm i\sqrt{7})/2$
$r = -4$	$\alpha = 4(2), 4 \pm i\sqrt{5}.$

Painlevé Series for the largest value of α. For the intermediate values mixed behaviour occurs. One can have both Left Painlevé Series and Right Painlevé Series. However, each series does not have the requisite number of arbitrary constants to represent a general solution and the equation formally fails the Painlevé Test as it has usually been presented [25, 29]. Yet we have demonstrated the explicit analytic solution in (2.23). This is not the first occasion that solutions with an incomplete quota of constants of integration have been reported [14, 24]. These partial solutions are analytic by construction. Moreover they are embedded in the parameter space of the general solution and so the argument that the solution ceases to be analytic once one leaves the surface in parameter space is demonstrably nontenable. These solutions, termed 'subsidiary' by Rajasekar [24] which seems to be more acceptable than the previously used terms 'partial' and 'particular', do not invalidate the possession by the equations concerned of the Painlevé Property.

We conclude the discussion of the special instance of the Riccati hierarchy with an observation. This concerns the values of α corresponding to the values of the resonance revealed by Step Two of the Painlevé Test. The results are summarised in Table 2.

The polynomial used for the calculation is the characteristic equation for the resonances for member IV in Table 1. One notes that for the value $r = 4$, which is not a resonance of the equation, we obtain $\alpha = 0(2), \pm i\sqrt{5}$ so that there is some pattern in the roots for α in the polynomial as this irrelevant detail helps to indicate.

We have spent a considerable space on the special case in which $\lambda_8 = C = 0$ and $h_8 = 1$ in (2.1) and (2.3). In the first instance it is a simpler case to treat and yet reveals several interesting properties. Two questions are relevant. Do these properties persist for general forms of (2.1) and (2.3)? If this be not the case, what properties can we adduce for the general forms? For the remainder of this section we address these questions.

Consider (2.3) with $\lambda_8 = 0$, *ie*

$$u'' + h_8(u)u'^2 = 0. \tag{2.26}$$

This is not the derivative form of a Riccati equation for $h_8(u)$ not a constant. Indeed unlike (2.3) with h_8 a constant (2.26) is truly a second-order equation. We rewrite h_8 as

$$h_8 = \frac{H''(u)}{H'(u)} \tag{2.27}$$

so that (2.26) becomes

$$H''(u)u'^2 + H'(u)u'' = 0 \tag{2.28}$$

which is obviously

$$\frac{\mathrm{d}^2 H}{\mathrm{d}x^2} = 0 \quad \Longrightarrow \quad H = A + Bx. \tag{2.29}$$

From (2.27) it is evident that

$$H(u) = \int \exp\left[\int h_8(u)\mathrm{d}u\right]\mathrm{d}u. \tag{2.30}$$

In general the right hand side of (2.30) is some function of u the global inversion of which is certainly to be problematical, but local inversion is guaranteed almost everywhere by consequence of the Implicit Function Theorem. Thus we can write

$$u(x) = F^{-1}(A + Bx), \tag{2.31}$$

where F^{-1} is the inverse function of the right side of (2.30).

When we turn to (2.4) with $\lambda_8 = C = 0$, we have the equation

$$u''' + 3h_8u'u'' + \left(\dot{h}_8 + h_8^2\right)u'^3 = 0. \tag{2.32}$$

We recall that (2.32) has a similarity symmetry for h_8 a constant. This cannot be expected to persist for some general $h_8(u)$. We assume a symmetry of the form[6]

$$\Gamma = P(u)\partial_u, \tag{2.33}$$

where P is some function of u. The invariants of Γ are

$$r = x \quad \text{and} \quad s = \frac{u'}{P(u)}. \tag{2.34}$$

Under this reduction of order (2.32) becomes

$$\frac{\mathrm{d}^2 s}{\mathrm{d}r^2} = 0 \tag{2.35}$$

[6]The motivation for so doing is that in our previous explorations the independent variable, x, has played a subsidiary role.

provided that the coefficient function in Γ is given by

$$P(u) = \exp\left[-\int h_8(u)\mathrm{d}u\right]. \tag{2.36}$$

Through (2.34) we have a linearisation of (2.32) in (2.35). The solution of (2.35) is just of the form of that given in (2.29) with H replaced by s. On reverse substitution and an integration we obtain

$$u = F^{-1}\left(A + Bx + Cx^2\right), \tag{2.37}$$

where F^{-1} is the same inverse function as we introduced above.

It is evident that we have a generalisation of the Corollary to Proposition III in

Proposition IV: *The general solution of the nth-order member of the hierarchy of ordinary differential equations generated from the second-order ordinary differential equation*

$$u'' + h_8(u)u'^2 = 0 \tag{2.38}$$

by the recursion operator $R_8 = D + h_8(u)u'$ given by

$$u_n(x) = F^{-1}\left(\sum_{i=1}^{n-1} A_i x^i\right), \tag{2.39}$$

where F^{-1} is the inverse function of $\int \exp\left[\int h_8(u)\mathrm{d}u\right]\mathrm{d}u$.

Thus we see that the introduction of a nonconstant $h_8(u)$ does not destroy the integrability[7] of the hierarchy. For a general function $h_8(u)$ one would not expect the members of the hierarchy to be distinguished by any particular Lie point symmetry apart from the obvious ∂_x. *A fortiori* the possession of the Painlevé Property is generally unlikely. Nevertheless the nth-order member of the hierarchy does possess n functionally independent first integrals for one can rewrite (2.39) as

$$\sum_{i=1}^{n-1} A_i x^i = F(u) \tag{2.40}$$

(we drop the subscript n from u for an obvious reason). This and the $n-1$ derivatives of (2.40) with a nonvanishing left side provides a regular system of n linear equations for the constants of integration $A_i,\ i = 1, n$, and the solution of this system for the coefficients A_i gives the set of n independent first integrals which is an alternate definition of integrability[8]. Indeed the route proffered by (2.40) and its derivatives may well be generally more attractive than the solution (2.39). The integral $\int \exp\left[\int h_8(u)\mathrm{d}u\right]\mathrm{d}u$ can be expected to be rather better defined than its inverse.

[7]We use the word 'integrability' in the formal sense of being able to express the solution as in (2.39) rather than in the more precise sense of a single-valued analytic function.

[8]This alternate definition is often of great use in Mechanics. A system may have a complete set of first integrals of moderately attractive functional form, but the elimination of the derivatives to obtain the solution can be an exercise in the futile attempted inversion of refractory functions.

The formal origin of the n first integrals is easy to see. If $I = I(x, u, u', \ldots, u^{(n-1)})$ is to be an integral of the nth-order member of the hierarchy invariant under the symmetry $\Gamma = \partial_x$, two equations must be satisfied, *videlicet*

$$\Gamma^{[n-1]} I = 0 \quad \text{and} \quad \frac{\mathrm{d}I}{\mathrm{d}x} = 0, \tag{2.41}$$

where $\Gamma^{[n-1]}$ is the nth extension of Γ, with the differential equation taken into account for the second in (2.41). The former equation in (2.41) eliminates x from I and so the latter provides $n-1$ first integrals. If one takes a general nth-order autonomous ordinary differential equation and applies this procedure, one would not expect to be able to find all of these $n-1$ autonomous first integrals. In our case they are easily obtained by eliminating x from the n integrals obtained as described above. The secret lies in the linearisability of the hierarchy. There is in fact always a sufficient number of Lie symmetries to give the nth-order differential equation a structure commensurate with integrability.

For nonzero λ the nth-order equation in H becomes

$$\frac{\mathrm{d}^n H}{\mathrm{d}x^n} + \lambda \frac{\mathrm{d}^{n-1} H}{\mathrm{d}x^{n-1}} = 0 \tag{2.42}$$

which has the solution

$$H = \sum_{i=0}^{n-2} A_i x^i + A_{n-1} \exp\left[-\lambda x\right] \tag{2.43}$$

and so we have

Proposition V: *The general solution of the nth member of the hierarchy of ordinary differential equations generated from the second-order differential equation*

$$u'' + h_8(u) u'^2 + \lambda u' = 0 \tag{2.44}$$

by the recursion operator $R_8 = D + h_8(u)u'$ is given by

$$u(x) = F^{-1}\left(\sum_{i=0}^{n-2} A_i x^i + A_{n-1} \exp\left[-\lambda x\right]\right), \tag{2.45}$$

where F^{-1} is as defined above.

Finally we turn to a situation in which the hierarchy of equations is generated by a polynomial in R_8 of specific structure, namely that in which the recursion operator from one order to the next is $R_8[u] + C$, where C is a constant, *ie* the sequence of equations beginning with (2.4) and (2.5). To give a flavour of what happens we consider (2.4) in the H representation which is

$$\frac{\mathrm{d}^3 H}{\mathrm{d}x^3} + (\lambda + C)\frac{\mathrm{d}^2 H}{\mathrm{d}x^2} + \lambda C \frac{\mathrm{d}H}{\mathrm{d}x} = 0 \tag{2.46}$$

with the obvious solution

$$H = A + B\mathrm{e}^{-\lambda x} + C\mathrm{e}^{-Cx}\,, C \neq \lambda, \tag{2.47}$$

and

$$H = A + (B + Cx)\mathrm{e}^{-\lambda x},\ C = \lambda, \tag{2.48}$$

ie, there is a new option as to the direction of the evolution of a solution for higher members of the hierarchy. A few moments spent with (2.5), which in terms of H is

$$\frac{\mathrm{d}^4 H}{\mathrm{d}x^4} + (\lambda + 2C)\frac{\mathrm{d}^3 H}{\mathrm{d}x^3} + C(\lambda + C)\frac{\mathrm{d}^2 H}{\mathrm{d}x^2} + \lambda C^2 \frac{\mathrm{d}H}{\mathrm{d}x} = 0 \tag{2.49}$$

with the solutions

$$H = A + B\mathrm{e}^{-\lambda x} + (C + Dx)\mathrm{e}^{-Cx}, C \neq \lambda, \tag{2.50}$$

and

$$H = A + (B + Cx + Dx^2)\mathrm{e}^{-\lambda x}, C = \lambda, \tag{2.51}$$

leads one immediately to

Proposition VI: *For the same as Proposition V with the exception that now we use the recursion operator $D + h_8(u)u' + C$ the solution of the nth member of the hierarchy is*

$$u(x) = F^{-1}\left(A + B\mathrm{e}^{-\lambda x} + \left(\sum_{i=0}^{n-3} C_i x^i\right)\mathrm{e}^{-Cx}\right),\ C \neq \lambda, \tag{2.52}$$

and

$$u(x) = F^{-1}\left(A + \left(\sum_{i=0}^{n-2} B_i x^i\right)\mathrm{e}^{-\lambda x}\right),\ C = \lambda, \tag{2.53}$$

where again F^{-1} is the inverse function defined through (2.30).

We have completed the formal construction of the solutions of the hierarchy of ordinary differential equations derived from the Class VIII nonlinear evolution partial differential equations presented by Euler *et al* [5]. We dwelt in detail upon the simplest members of the hierarchy of nonlinear ordinary differential equations since it commences with two equations, the Riccati Equation and the Painlevé–Ince Equation, which arise so often in theory and application. This family displays a richness in terms of both the symmetry and the singularity analyses which can only be described as unfortunately exceptional. That the members of the family are linearisable through a point transformation for all functions $h_8(u)$ and the parameters λ and C means that all members of the hierarchy possess the symmetry algebra of the linearised version. The simplest version of the hierarchy comprised equations of maximal Lie point symmetry and possessed the Painlevé Property with a richness of detail which is pedagogically useful even for practitioners in the field. Obviously our final question of this class of equations is the persistence or otherwise of these properties for general functions $h_8(u)$ and parameters λ and C.

As far as Lie point symmetries are concerned, we cannot commence with the first member of the hierarchy, (2.3), since it is transformable to a linear second-order equation by a point transformation and so always possesses the eight-element algebra $sl(2,R)$. For $n \geq 3$ the linearised equation, and so the original equation, possesses at least $n+2$ Lie point symmetries comprising n solution symmetries, the homogeneity symmetry and ∂_x [15]. Two additional symmetries, which with ∂_x constitute a representation of the three element $sl(2,R)$, exist if the coefficients of the equation in normal form are related in a suitable way[9]. For an nth-order linear equation in normal form with constant coefficients, *videlicet*

$$w^{(n)} + \sum_{i=0}^{n-2} B_i w^{(i)} = 0, \tag{2.54}$$

the symmetries related to $sl(2,R)$ are found from the solution of the third-order equation,

$$\frac{(n+1)!}{(n-2)!4!} a^{(3)} + a^{(1)} B_{n-2} + \tfrac{1}{2} a B_{n-2}^{(1)} = 0, \tag{2.55}$$

where the symmetry has the form $a(x)\partial_x + \frac{1}{2}(n-1)a^{(1)} y\partial_y$, which involves just the coefficient B_{n-2}. The consistency of the rest of the equation with (2.55) is expressed as the requirement that the coefficients in (2.54) satisfy the sequence of equations

$$\begin{aligned} &\frac{(n+1)!(i-1)}{2(n-i)!(i+1)!} a^{(i+1)} + i a^{(1)} B_{n-i} + a B_{n-i}^{(1)} \\ &\qquad + \sum_{j=2}^{n-1} B_{n-j} \frac{(n-j)![n(i-j-1)+i+j-1]}{2(n-i)!(i-j+1)!} = 0, \quad i = 3, n. \end{aligned} \tag{2.56}$$

The coefficients in the normal form of the nth member of the hierarchy are autonomous. In this case the first few equations of the sequence (2.56) lead to the conditions

$$\begin{aligned} B_{n-3} &= 0 \\ B_{n-4} &= \frac{1}{2.5(n+1)!}(n-2)!(5n+7)(n-2)(n-3)B_{n-2}^2 \\ B_{n-5} &= 0 \\ B_{n-6} &= \frac{(n-2)!^3(35n^2+110n+93)}{2.3.5.7(n-6)!(n+1)!^2} B_{n-2}^3 \\ B_{n-7} &= 0 \\ B_{n-8} &= \frac{(n-2)!(175n^3+945n^2+1769n+1143)}{2^2.3.5^2.7(n-8)!(n+1)!} B_{n-2}^4. \end{aligned} \tag{2.57}$$

Our task is to relate these results to the nth-order equation generated by $R_8 + C$ from (2.3) in its linear equivalent.

In the case that $n = 3$ the linear equation corresponding to (2.4) is

$$\frac{\mathrm{d}^3 H}{\mathrm{d}x^3} + (\lambda + C)\frac{\mathrm{d}^2 H}{\mathrm{d}x^2} + \lambda C \frac{\mathrm{d}H}{\mathrm{d}x} = 0 \tag{2.58}$$

and this has the normal form

$$w''' - \tfrac{1}{3}\left(\lambda^2 - \lambda C + C^2\right) w' + \tfrac{1}{27}(\lambda + C)(2\lambda - C)(\lambda - 2C) w = 0 \tag{2.59}$$

[9]A thorough discussion is found in Mahomed *et al* [15]. Here we are simply quoting the relevant results.

when we set $H = w(x)\exp\left[-\frac{1}{3}(\lambda + C)\,x\right]$. From (2.57) the only requirement that (2.58) be a third-order equation of maximal symmetry is that the coefficient of w be zero, *ie*

$$\lambda = -C,\ 2C,\ C/2. \tag{2.60}$$

For λ and C otherly related (2.4) has just the five Lie point symmetries[10].

In the case of (2.5) the normal form of the equation is

$$w'''' - \tfrac{1}{8}\left(9\lambda^2 + 16\lambda C + 35C^2\right)w'' + \tfrac{1}{8}\lambda^2(\lambda - 2C)w' - \tfrac{1}{256}\lambda + 2C)(3\lambda - 2C)(\lambda - 2C)^2 w = 0. \tag{2.61}$$

For the coefficient of w' to be zero we require that $\lambda = 0,\ 2C$ so that $B_0 = -C^2/32,\ 0$, respectively. However, from (2.57) we have that

$$\begin{aligned} B_0 &= \tfrac{9}{6400}\left(9\lambda^2 + 16\lambda C + 35C^2\right) \\ &= \tfrac{9.35^2C^4}{6400},\ \tfrac{9.101^2C^4}{6400}, \end{aligned}$$

respectively. Neither expression is zero for C nonzero. Even though we had a common relation of $\lambda = 2C$ for both third-order and fourth-order equations, the additional constraints on the fourth-order equation reduced the hierarchy to the specific instance of the Riccati hierarchy.

We conclude that the general hierarchy is not of maximal Lie point symmetry.

3 The Ermakov–Pinney equation

All Ermakov–Pinney equations can be transformed by a point transformation to the basic form

$$y'' = y^{-3}. \tag{3.1}$$

This admits the rescaling symmetry $x\partial_x + \frac{1}{2}y\partial_y$ and does not have the Painlevé property since the leading order exponent is $\frac{1}{2}$. Under the transformation

$$u(x) = \frac{1}{y^2(x)}$$

we obtain

$$uu'' - \frac{3}{2}(u')^2 + 2u^4 = 0, \tag{3.2}$$

which has the Painlevé Property. The Riccati transformation

$$u(x) = \alpha\frac{w'(x)}{w(x)}, \qquad \alpha^2 = -\frac{1}{4},$$

brings us to the third order equation

$$w'w''' - \tfrac{3}{2}(w'')^2 = 0$$

which is the Kummer-Schwarz equation.

[10]As a linear third-order equation with fewer than the maximal number of Lie point symmetries (2.4) does not have any intrinsically contact symmetries.

For the construction of a Ermakov-Pinney hierarchy we use (3.2) as the basic equation. We note that (3.2) is the Cavalcant-Tenenblat [2] equation (1.3) with

$$v_t = 0, \qquad \lambda_2 = 3, \qquad \lambda_1 = C = 0,$$

namely

$$v_x^{-3/2} v_{xxx} - \frac{3}{2} v_x^{-5/2} v_{xx}^2 + 2 v_x^{3/2} = 0. \tag{3.3}$$

This becomes (3.2) when we multiply by $v_x^{5/2}$ and put $v_x = u(x)$. To find the second member of the hierarchy we use the recursion operator (1.4) of the Cavalcante-Tenenblat equation and obtain

$$v_x^{-11/2} \left(v_x^3 v_{5x} - \frac{15}{2} v_x^2 v_{xx} v_{4x} - 5 v_x^2 v_{3x}^2 + \frac{5}{2} v_x^5 v_{3x} + \frac{245}{8} v_x v_{xx}^2 v_{3x} \right.$$
$$\left. - \frac{315}{16} v_{xx}^4 - \frac{5}{2} v_x^4 v_{xx}^2 + v_x^8 \right) = 0. \tag{3.4}$$

Under $v_x = u(x)$ and multiplication by $v_x^{11/2}$ this equation become the following fourth-order equation

$$u^3 u^{(4)} - \frac{15}{2} u^2 u' u''' - 5 u^2 (u'')^2 + \frac{5}{2} u^5 u'' + \frac{245}{8} u (u')^2 u''$$
$$- \frac{315}{16} (u')^4 - \frac{5}{2} u^4 (u')^2 + u^8 = 0. \tag{3.5}$$

Remark: The Cavalcante–Tenenblat equation as generalised, (1.3), is an evolution equation and the applications of the recursion operator and its inverse require that the equation have the form $v_t = \ldots$. We maintain that structure for the ordinary differential equatiions of this hierarchy to avoid confusion. Thus for example we write the integrating factors as in (3.6) and (3.7) below. Were one to separate entirely the treatment of the ordinary differential equations from the partial differential equations once they are generated,(3.3) for example would be written without the common multiplier $v_x^{-11/2}$. The two integrating factors, μ_1 and μ_2, would have to be adjusted accordingly. To avoid confusion we keep just the initial structures. Next we discuss the first integrals of the above two members of the Ermakov-Pinney hierarchy. The recursion operator (1.4) provides us with one integrating factor, μ_1, namely the term to the right of D^{-1}, i.e.

$$\mu_1 = v_x^{-3/2} v_{xx}. \tag{3.6}$$

Higher order integrating factors for the higher order members of this hierarchy can then be obtained by acting the adjoint of $R[v]$ on μ_1. For the first member we obtain the second integrating factor

$$\mu_2 = R[v]^* \mu_1 = \frac{1}{8} v_x^{-9/2} \left(35 v_{xx}^3 - 40 v_x v_{xx} v_{3x} + 8 v_x^2 v_{4x} + 12 v_x^4 v_{xx} \right). \tag{3.7}$$

By the use of μ_1 we obtain the first integral of (3.3), namely

$$I_{11} = \frac{1}{2} v_x^{-3} v_{xx}^2 + 2 v_x, \tag{3.8}$$

and by the same integrating factor a first integral of (3.4) is obtained in the form

$$I_{21} = v_x^{-4} v_{xx} v_{4x} - \frac{1}{2} v_x^{-4} v_{3x}^2 - \frac{7}{2} v_x^{-5} v_{xx}^2 v_{3x} + \frac{105}{32} v_x^{-6} v_{xx}^4 + \frac{5}{4} v_x^{-2} v_{xx}^2 + \frac{1}{2} v_x^2. \tag{3.9}$$

By means of the integrating factor μ_2 we obtain a second first integral for (3.4), namely

$$\begin{aligned} I_{22} \;=\;& \frac{1}{2} v_x^{-5} v_{4x}^2 + \frac{35}{8} v_x^{-7} v_{xx}^3 v_{4x} + \frac{3}{2} v_x^{-3} v_{xx} v_{4x} - 5 v_x^{-6} v_{xx} v_{3x} v_{4x} + \frac{1}{2} v_x^{-3} v_{3x}^2 \\ & - \frac{175}{8} v_x^{-8} v_{xx}^4 v_{3x} + \frac{25}{2} v_x^{-7} v_{xx}^2 v_{3x}^2 - \frac{37}{4} v_x^{-4} v_{xx}^2 v_{3x} + v_{3x} - \frac{5}{8} v_x^{-1} v_{xx}^2 \\ & + \frac{1225}{128} v_x^{-9} v_{xx}^6 + \frac{259}{32} v_x^{-5} v_{xx}^4 + \frac{1}{2} v_x^3. \end{aligned} \tag{3.10}$$

By solving for v_{4x} from I_{21} and inserting it into I_{22} and further replacing v_{3x} from the first member of the hierarchy (3.3), we obtain the following relation between the first integrals of (3.3) and (3.4):

$$I_{22} = \frac{1}{16} I_{11}^3, \qquad I_{21} = -\frac{3}{8} I_{11}^2. \tag{3.11}$$

The use of I_{21} and I_{22} to reduce the fifth-order equation to a third-order equation with the two parameters, I_{21} and I_{22}, is standard. By eliminating v_{xxx} from the resulting equation by the use of the first member of the hierarchy, (3.3), we are demanding a consistency between the solution of (3.3) and the more general (3.4). This is achieved by imposing constraints in the values of I_{21} and I_{22}, *ie.* they are turned from integrals to configurational invariants [12, 27] and consequently act as constraints. If one thinks of the extended phase space of (3.4), it is a six-dimensional space spanned by x, v, v_x, v_{xx}, v_{xxx} and v_{xxxx}. Each integral represents an hypersurface in this six-dimensional space and its location is a function of the actual value of the integral. The knowledge of five independent integrals leaves the trajectory as the curve of common intersection of these integrals. This is the general situation. Here we look for solutions of (3.4) compatible with (3.3). Naively one might think that this compatibility could be achieved simply be putting I_{21} and I_{22} each equal to zero which is a simple projection onto a space of dimension two lesser. However, in (3.11) we find a somewhat richer result in that the constraints on the integrals are expressed as hypersurfaces determined by specific relationships with an integral of (3.3). This is reminescent of the result for a generalisation of the Kepler Problem [28] in which the subspace for the zero value of the energy, *ie* a configurational invariant, was not a plane as in the case of the Kepler Problem but a paraboloid [11].

We proceed by performing a Painlevé analysis of the Ermakov-Pinney hierarchy. To obtain a better sense of the behaviour of the leading order coefficients and resonances we also consider the next member of the hierarchy which takes the following form:

$$
\begin{aligned}
&v_x^{-17/2}\Big(v_x^5 v_{7x} - 14 v_x^4 v_{xx} v_{6x} + \frac{861}{8} v_{xx}^2 v_x^3 v_{5x} - 28 v_x^4 v_{3x} v_{5x} + \frac{7}{2} v_x^7 v_{5x} - \frac{35}{2} v_x^4 v_{4x}^2 \\
&+\frac{1449}{4} v_x^3 v_{xx} v_{3x} v_{4x} - \frac{9009}{16} v_x^2 v_{xx}^3 v_{4x} - \frac{77}{4} v_x^6 v_{xx} v_{4x} + \frac{651}{8} v_x^3 v_{3x}^3 \\
&-\frac{18249}{16} v_x^2 v_{xx}^2 v_{3x}^2 - \frac{49}{4} v_x^6 v_{3x}^2 + \frac{1043}{16} v_x^5 v_{xx}^2 v_{3x} + \frac{267267}{128} v_x v_{xx}^4 v_{3x} + \frac{35}{8} v_x^9 v_{3x} \\
&-\frac{225225}{256} v_{xx}^6 - \frac{2415}{64} v_x^4 v_{xx}^4 - \frac{35}{16} v_x^8 v_{xx}^2 + \frac{3}{4} v_x^{12}\Big) = 0.
\end{aligned}
\tag{3.12}
$$

Under $v_x = u(x)$ and multiplication by $v_x^{17/2}$ (3.12) becomes the following sixth-order equation

$$
\begin{aligned}
&u^5 u^{(6)} - 14 u^4 u' u^{(5)} + \frac{861}{8} u^3 (u')^2 u^{(4)} - 28 u^4 u'' u^{(4)} + \frac{7}{2} u^7 u^{(4)} - \frac{35}{2} u^4 (u^{(3)})^2 \\
&+\frac{1449}{4} u^3 u' u'' u^{(3)} - \frac{9009}{16} u^2 (u')^3 u^{(3)} - \frac{77}{4} u^6 u' u^{(3)} + \frac{651}{8} u^3 (u'')^3 \\
&-\frac{18249}{16} u^2 (u')^2 (u'')^2 - \frac{49}{4} u^6 (u'')^2 + \frac{1043}{16} u^5 (u')^2 u'' + \frac{267267}{128} u (u')^4 u'' + \frac{35}{8} u^9 u'' \\
&-\frac{225225}{256} (u')^6 - \frac{2415}{64} u^4 (u')^4 - \frac{35}{16} u^8 (u')^2 + \frac{3}{4} u^{12} = 0.
\end{aligned}
\tag{3.13}
$$

We summarise the result of the Painlevé test for these three members of the Ermakov-Pinney hierarchy, (3.2), (3.5) and (3.13), in Table 3 below.

4 Conclusion

We have presented the structure of two hierarchies of ordinary differential equations based upon an initial second-order differential equation possessing a recursion operator. One of the hierarchies contained the Riccati Equation and the Painlevé–Ince Equation, equations frequently encountered in the relevant literature, as early members when certain constraints were placed upon the initial equation and the recursion operator. This particular form of the hierarchy displayed rich features in terms of the symmetry and singularity properties of its members. Although the hierarchy was constructed to be linearisible, there was a dimunition of both symmetry and singularity properties from the particular class to the general class. Generically the possession of the Painlevé Property was lost since the inverse function, F^{-1}, introduced is not invariably analytic. In terms of symmetry the situation was more satisfactory in that generically the members of the hierarchy possess $n+2$ Lie point symmetries. However, in general they do not possess the property of being of maximal symmetry.

We mentioned in the earlier part of §2 that one could contemplate alternate routes for the construction of the hierarchy. Indeed one may be even less diciplined in the construction of higher order equations. Instead of using $R_8[u] + C$ repeatedly one could contemplate the use

Table 3. Summary of the results of the Painlevé Test applied to the earlier members of the Ermakov–Pinney hierarchy. We refer to each member of the hierarchy by its equation number in the text.

Member	Characteristic equations for α and r	Roots
(3.2)	$4\alpha^2+1=0$ $r^2+8\alpha^2+1=0$	$\alpha^2=-1/4$ $r\in\{-1,\ 1\}$
(3.5)	$16\alpha^4+40\alpha^2+9=0$ $18+120\alpha^2+64\alpha^4+5r^2-20r^3-20\alpha^2r$ $+15r+20r^2\alpha^2+8r^4=0$	$\alpha^2=-1/4$ $r\in\{-1,\ 1/2,\ 1,2\}$ or $\alpha^2=-9/4$ $r\in\{-3/2,-1,2,3\}$
(3.13)	$64\alpha^6+560\alpha^4+1036\alpha^2+225=0$ $1152\alpha^6+2128r^4-1120r\alpha^4-2016r^3\alpha^2$ $-1624r^3+8400\alpha^4+560r^2\alpha^4+2968r^2\alpha^2$ $-784r\alpha^2+2025+448r^4\alpha^2-861r^2$ $+12432\alpha^2-896r^5+1890r+128r^6=0$	$\alpha^2=-1/4$ $r\in\{-1,1/2,1,3/2,2,3\}$ or $\alpha^2=-9/4$ $r\in\{-3/2,-1,1,3/2,3,4\}$ or $\alpha^2=-25/4$ $r\in\{-5/2,-3/2,-1,3,$ $4,5\}$

of $\Pi_{i=1}^m(R_8[u]+C_i)$. Obviously th property of linearisability persists, but the algebraic choices become wider as there are more combinations of constants with which to play. When the recursion operator is not self-adjoint, the adjoint operator may also be used to generate higher order equations so that one could employ not only a random sequence of constants, C_i, but of recursion operator and its adjoint. The properties of these alternate approaches to the generation of hierarchies deserve further attention.

The second hierarchy of ordinary differential equations based on the Ermakov–Pinney variant is complicated by the fact that its recursion operator cause the order of the differential equation to increase by two per recursion and so the complexity of the higher members of the hierarchy comes more quickly than for the Riccati hierarchy. Consequently our results must necessarily be somewhat circumscribed. The relationship of the solution of the higher order member to the lower is given in (3.91). It is not as simple as one might imagine.

The singularity analysis, summarised in Table 3, is not as clear-cut as it was for the Riccati hierarchy. Perhaps one should not be surprised at this for the early members of the Riccati hierarchy have been known for many years to be quite exceptional. Nevertheless certain patterns are evident. The first is that only (3.3), *the* Ermakov–Pinney Equation, is the only member of the hierarchy (as revealed, but a certain extrapolation is not unreasonable) to possess the Painlevé Property. For one value of the leading order coefficient equations (3.3) and (3.4) pass the weak form of the test. As in the case of the Riccati hierarchy one would expect the pattern to persist. For other values of the leading order coefficient we obtain mixed resonances so that incomplete Left Painlevé Series and Right Painlevé Series can be found for these leading order coefficients. As we noted above, the existence of a subsiduary solution is not incompatible with the possession of the (weak) Painlevé Property. By way of contrast to the Riccati hierarchy there is no suggestion of the equations passing the Painlevé Test with a complete Left Painlevé Series as in the case of the Riccati hierarchy. This is already found in the Ermakov–Pinney equation for which there is effectively just one leading order coefficient since in the analysis it always appears as a square. The same is repeated for the higher members of the Ermakov–Pinney hierarchy. From the three equations considered there is evidently a pattern of resonances associated with the sequence of leading order coefficients.

Our study cannot be considered complete. Nevertheless we have demonstrated that the concept of recursion operators, applied so fruitfully to evolution equations, is equally applicable to ordinary differential equations. We further see that subclasses of hierarchies, witness the Riccati hierarchy, can have richer properties than the general class. We confined our attention to just one of the eight classes of linearisable hierarchies and one of the two quasilinear equations reported by Petersson *et al.* even with these restrictions we have unfolded a rich set of results. Exploration of the other classes is thereby encouraged.

Acknowledgments. *We thank the Institut Mittag-Leffler and The Royal Swedish Academy of Sciences for their kind hospitality while this work was performed. PGLL also thanks the University of KwaZula-Natal for its continued support.*

References

[1] Abraham-Shrauner B, Leach PGL, Govinder KS and Ratcliff G, Hidden and contact symmetries of ordinary differential equations, *J. Phys. A* **23** (1995), 6707–6716.

[2] Cavalcante J A and Tenenblat K, Conservation Laws for Nonlinear Evolution Equations, *J. Math. Phys.* **29** (1988), 1044–1049.

[3] Conte R, Singularities of differential equations and integrability, in Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Nonlinear Waves, Editors: Benest D and Frœschlé, Éditions Frontières, Gif-sur-Yvette, 1994, 49–143.

[4] Ermakov V, Second-order differential equations. Conditions of complete integrability, *Universita Izvestia Kiev Series III* **9** (1880), 1-25 (trans. Harin A O).

[5] Euler M, Euler N and Petersson N, Linearizable hierarchies of evolution equations in $(1+1)$ dimensions, *Stud. Appl. Math.* **111** (2003), 315–337.

[6] Feix M R, Géronimi C, Cairó, Leach P G L, Lemmer R L and Bouquet S É, On the singularity analysis of ordinary differential equations invariant under time translation and rescaling *J. Phys. A* **30** (1997), 7437–7461.

[7] Fokas A S, A symmetry approach to exactly solvable evolution equations, *J. Math. Phys.* **21** (1980), 1318–1325.

[8] Fokas A S, Symmetries and integrability, *Stud. Appl. Math.* **77** (1987), 253–299.

[9] Fuchssteiner B, Application of hereditary symmetries to nonlinear evolution equations, *Nonlinear Anal.-Theory Methods Appl.* **3** (1979), 849–862.

[10] Géronimi C, Leach P G L and Feix M R, Singularity analysis and a function unifying the Painlevé and the Psi series, *J. Nonlinear Math. Phys.* **9** Sup. 2 (2002), 36–48.

[11] Gorringe V M and Leach P G L, The first integrals and their Lie algebra of the most general autonomous Hamltonian of the form $H = T + V$ possessing a Laplace-Runge-Lenz vector, *J. Aust. Math. Soc. B* **34** (1993), 511–522.

[12] Hall L S, A theory of exact and configurational invariants, *Physica D* **8** (1982), 90–113.

[13] Ince E L, Ordinary Differential Equations, Longmans, Green & Co, London, 1927.

[14] Leach P G L, Cotsakis S and Flessas G P, Symmetry, singularities and integrability in complex dynamics I: The reduction problem, *J. Nonlinear Math. Phys.* **7** (2000), 445–479.

[15] Mahomed F M and Leach P G L, Symmetry Lie algebras of nth order ordinary differential equations, *J. Math. Anal. Appl.* **151** (1990), 80–107.

[16] Morozov V V, Classification of six-dimensional nilpotent Lie algebras, *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **5** (1958), 161–171.

[17] Mubarakzyanov G M, On solvable Lie algebras, *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **32** (1963), 114–123.

[18] Mubarakzyanov G M, Classification of real structures of five-dimensional Lie algebras, *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **34** (1963), 99–106.

[19] Mubarakzyanov G M, Classification of solvable six-dimensional Lie algebras with one nilpotent base element, *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **35** (1963), 104–116.

[20] Olver P J, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.* **18** (1977), 1212–1215.

[21] Painlevé P, Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, *Acta Math.* **25** (1902), 1–86.

[22] Petersson N, Euler N and Euler M, Recursion operators for a class of integrable third-order evolution equations, *Stud. Appl. Math.* **112** (2004), 201–225.

[23] Pinney E, The nonlinear differential equation $y''(x) + p(x)y + cy^{-3} = 0$, *Proc. Amer. Math. Soc.* **1** (1950), 681.

[24] Rajasekar S, Painlevé analysis and integrability of two coupled nonlinear oscillators, *Pramana – J. Phys.* **62** (2004), 1–12.

[25] Ramani A, Grammaticos B and Bountis T, The Painlevé Property and singularity analysis of integrable and nonintegrable systems, *Phys. Rep.* **180** (1989), 159–245.

[26] Riccati J F, *Acta Eruditorum quæ Lipsiae publicantur Supplementa* **VIII** (1724), 74

[27] Sarlet W, Leach P G L and Cantrijn F, Exact versus configurational invariants and a weak form of complete integrability, *Physcia D* **17** (1985), 331–338.

[28] Sen T, A class of integrable potentials, *J. Math. Phys.* **28** (1987), 2841–2850.

[29] Tabor M, Chaos and Integrability in Nonlinear Dynamics, John Wiley, New York, 1989.