A Modified Duffing System with a Sawtooth-wave Excitation and its Synchronization

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Abstract

A duffing system under approximate sawtooth wave excitation obtained by a relaxation oscillator is proposed, and its dynamic properties (including phase trajectory, time evolution, bifurcation diagram) are analyzed, also the impact of excitation magnitude change on phase trajectory is discussed. The synchronization of two duffing systems under various excitations is achieved by nonlinear integral feedback method based on the sum of squared errors. Simulation results show that excitation amplitude affects the state of the system. Finally, it has been explained the effectiveness of this scheme on the generation of chaotic source in secure communication and synchronization control.

Keywords: Sawtooth wave; modified duffing equation; chaotic synchronization; feedback control

1. Introduction

There have been many studies on chaos which exhibits complicated dynamic phenomena in nonlinear systems. Famous chaotic systems such as Lorenz, Chua, Chen systems, and discrete Logistic, Henon maps [1-4] have been addressed extensively. These studies mainly concentrate on the following aspects: dynamical behaviors analyses of chaos and complex chaos (see for example fractional-order chaos and multi-scroll chaotic system) in theory; Chaos control and synchronization, chaotic weak signal detection, and chaotic secure communication (stream cipher, chaotic modulation, chaotic encoding, chaotic masking and chaotic shift keying etc) in application. Oscillator source has played an important role in communication. Nonlinear oscillator usually includes complex continuous chaotic signal [5-6]. Duffing oscillator is a typical nonlinear system, and it exhibits different dynamic properties under the periodic force excitation. In most papers on the research of duffing systems, the case of definite harmonic excitation is taken into account [7]. In some papers this problem is further extended to multi-frequency excitation [8-9]. Reference [10] studies the chaotic control of duffing system with random phase. In our paper we have analyzed various state characteristics based on the sawtooth-wave excitation obtained by relaxation oscillator.

From the perspective of chaotic communication, useful information can be demodulated and detected usually under chaotic carrier synchronization. In such cases we need to utilize synchronization. Sometimes we need to suppress chaos and desynchronize in order to avoid the destruction of chaos resonance [11]. Hence, by studying the mechanics of chaos and synchronization, and the factors of affecting synchronization, we can utilize and control chaos freely. Different methods about chaos synchronization have been presented in prior papers, such as PC synchronization, linear or nonlinear feedback synchronization, adaptive synchronization, delay feedback or delay coupling synchronization, state observation synchronization, impulsive synchronization, active control synchronization and so on. The synchronization of two duffing systems under various excitations and various states is investigated based on nonlinear integral feedback technique in this paper.

2. Duffing system based on sawtooth wave excitation obtained by relaxation oscillator

Many researchers have obtained the sawtooth-wave from the relaxation oscillator based on different methods and different circuits. Actually, a relaxation oscillator has been widely used in science and engineering. Toshimichi has implemented the coupling relaxation oscillator by utilizing the piecewise linearity of operational amplifiers combined with RC circuits [12]. Ref. [13] has designed the relaxation oscillator by coupling a simple S-shaped current controlled nonlinear resistor to a timing RC network, and directly modified it to be a low-power consumption chaotic oscillator. In this paper, the sawtooth wave by relaxation oscillator (see [12]) is applied as the excitation of duffing system, in which various dynamical features are shown and synchronization of two duffing systems is achieved.

The equation of duffing system with a sawtooth wave excitation is as follows:

$$\begin{cases} \dot{x} = hy \\ \dot{y} = -ax - by - cx^3 + \gamma z \\ \dot{z} = -p - g(z - w) \\ \dot{w} = -d * q - g(w - z) \\ \dot{p} = \left(z - f(p)\right)/e \\ \dot{q} = \left(w - f(q)\right)/e \end{cases} \tag{1}$$

where

$$f(p) = \begin{cases} p - 2 & p \geq 1 \\ -p & |p| < 1 \\ p + 2 & p \leq -1 \end{cases} \tag{2}$$

In Eq.(1), x, y are the state variables of duffing system, b is the damping ratio, $ax + cx^3$ is the nonlinear restoring force, γ is the excitation amplitude coefficient, z is the excitation term of duffing system provided by the sawtooth wave of the relaxation oscillator, w, p, q are the other state variables, and d denotes the detuning.

3. Characteristic simulation and analysis

The dynamic characteristics of duffing system is addressed in this section. The approximate periodic sawtooth wave (the variable z in equation (1)) is shown in Fig. 1. By the aid of MATLAB simulation, phase portraits and time histories are shown in Figs. 2(a)-(h). With the variation of excitation magnitude, different phenomena exhibit including equilibrium point, period, chaos to large scale period are obtained.

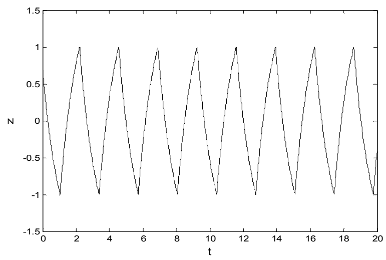

Fig. 1: The approximately periodic relaxation oscillator

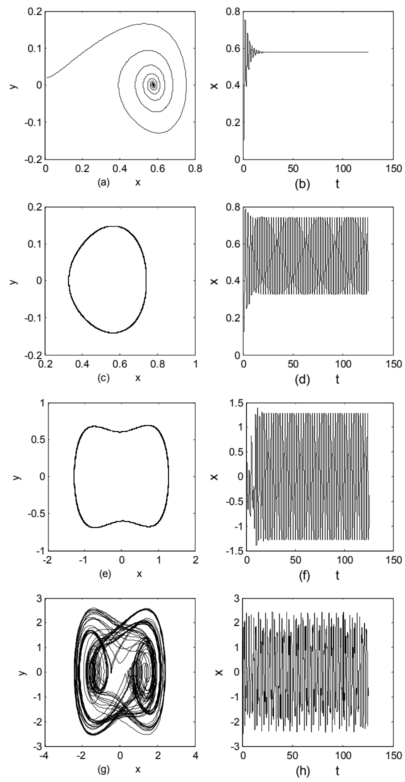

Fig. 2: Duffing system with different excitation magnitudes, initial value x_0=(0.01, 0.02, 0.685, 1.1, 0, 0)

Figs. 2(a), 2(c), 2(e), 2(g) denote the phase trajectories; Figs. 2(b), 2(d), 2(f), 2(h) depict the time evolution waveforms. It corresponds to equilibrium point with $\gamma = 0$ (see Figs. 2(a) and 2(b)), periodic orbit with $\gamma = 0.1$ (see Figs. 2(c) and 2(d)), large scale period with $\gamma = 1$ (see Figs. 2(e) and 2(f)), and chaos status with $\gamma = 10$ (see Figs. 2(g) and 2(h)), respectively.

For clarity, the previous 5000 instant points are ignored in phase trajectory. Other parameters are taken as:

$$g = 0.4, \quad e = 1/10^4, \quad d = 0.9,$$
$$a = -1, \quad b = 0.5, \quad c = 3, \quad h = 4$$

where the duffing system is under a sawtooth wave excitation derived from differential equation (1), γ impacts the magnitude of sawtooth wave excitation. Simulation results show that the variation of periodic oscillation amplitude impacts the dynamic characteristics of the system.

Besides the phase trajectories and time evolution waveforms shown in Fig. (2), the bifurcation diagram obtained by Poincare section is analyzed in Fig.3.

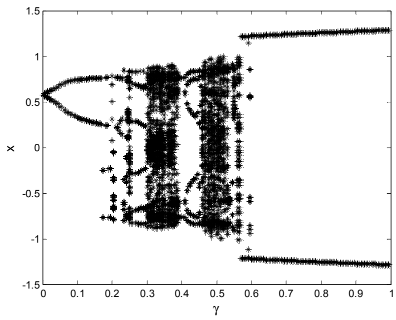

Fig. 3: Bifurcation diagram of the system. (x vs γ)

Fig.3 shows the situations from equilibrium point to period, and from period to chaos, then to large scale period. Simultaneously, Fig.3 shows that period may exist in the regimes of chaos, like γ=0.4~0.45, period 3 exists among large region of chaos. Moreover, chaos and period may coexist in one system with same parameters. Under γ=0.25, single scroll chaos is observed. Until t =280, it degenerates to period status under the excitation as shown in Fig. 4. Here the excitation is equivalent to the role of interference.

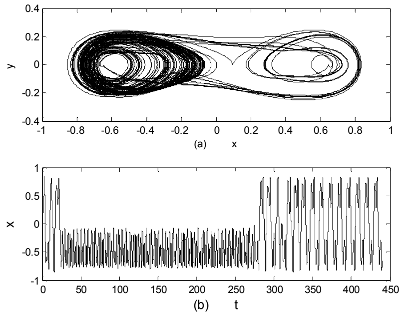

Fig. 4: The Duffing system with the excitation amplitude coefficient γ=0.25, and the initial value x_0=(0.01, 0.02, 0.685, 1.1, 0, 0)

4. Synchronization

Synchronization plays an important role in communication as mentioned in Section 1. It has been applied in many fields such as chemistry reaction, power transfer and so on [14]. Synchronization phenomena also exist in biological system like cardiovascular, neural system etc [15]. Synchronization is a broad concept in science, many literatures have studied different aspects of synchronization. The importance and mechanics of oscillator synchronization are analyzed in detail [16]. In the above sections of this paper, a relaxation oscillator based duffing system with a sawtooth wave excitation is constructed, and the synchronization of duffing systems will be addressed in this section.

The synchronization of oscillator system will be realized with different controlling methods, but it is usually caused by external disturbance and the coupling of systems [16]. The phase trajectories of two independent duffing systems are shown in Fig. 5. Fig. 5(a) is system (1) with a large scale periodic behavior where h=1, c=3, γ=10, a=-1, g=0.4, d=0.9. Fig. 5(b) shows a chaotic duffing system with a common harmonic function (0.425 $\cos t$) excitation as in Eq. (3).

$$\begin{cases} \dot{x}_1 = y_1 + u_1(t) \\ \dot{y}_1 = x_1 - 0.5y_1 - x_1^3 + 0.425\cos t + u_2(t) \end{cases} \tag{1}$$

where $u_1(t)$, $u_2(t)$ are nonlinear controlling variables.

$$\begin{cases} u_1(t) = k(x_1 - x) \\ u_2(t) = k(y_1 - y) \end{cases} \tag{4}$$

and

$$\dot{k} = m(x - x_1)^2 + m(y - y_1)^2, \quad m = -10 \tag{5}$$

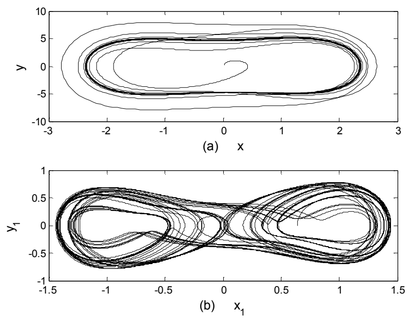

Fig. 5: Two independent duffing systems with different excitations and parameters. m=0; h=1; c=3; γ=10; a=-1; g=0.4; d=0.9.

Combing Eqs.(3), (4) and (5), the controlling variables $u_1(t)$, $u_2(t)$ are functions of integral for k. Figs.(6) and (7) show that the synchronization of two systems has been achieved ($e_{x-x_1} = x - x_1$).

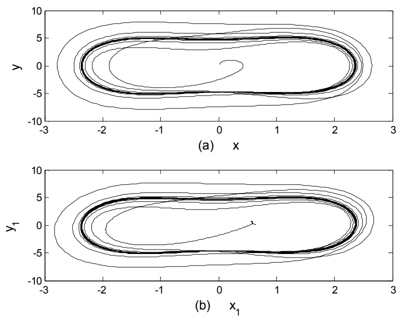

Fig. 6: Synchronization of two duffing systems with different excitations and parameters. *m*=-10; *h*=1; *c*=3; γ=10; *a*=-1; *g*=0.4; *d*=0.9

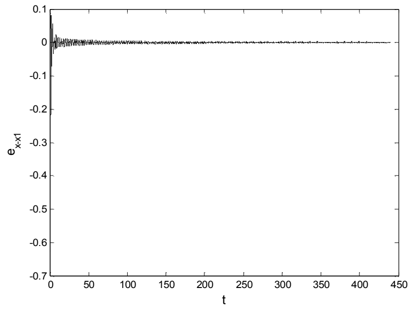

Fig.7: Time evolution of errors. *m*=-10; *h*=1; *c*=3; γ=10; *a*=-1; *g*=0.4; *d*=0.9

When drive and response systems are in different chaotic states (Fig. 8), how is the controlling result?

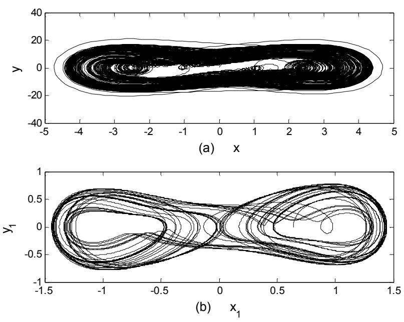

Fig. 8: Two independent duffing systems with different excitations and parameters. *m*=0; *h*=1; *c*=3; γ=80; *a*=1; *d*=0.6; *g*=0.1

Simulation results show that the synchronization has also been achieved through integral feedback control, as shown in Figs. (9) and (10), respectively.

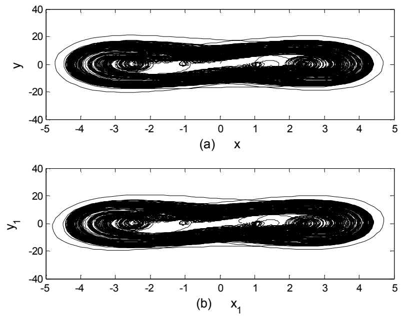

Fig. 9: Synchronization of two duffing systems with different excitations and parameters. *m*=-10; *h*=1; *c*=3; γ=80; *a*=1; *d*=0.6; *g*=0.1

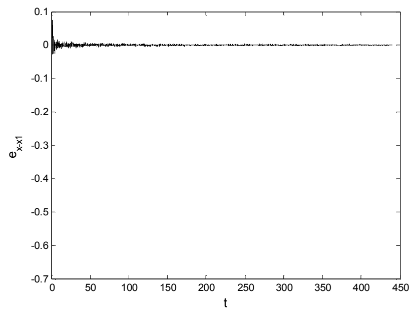

Fig. 10: Time evolution of errors. *m*=-10; *h*=1; *c*=3; γ=80; *a*=1; *d*=0.6; *g*=0.1

5. Conclusion

Duffing system which exhibits special chaotic characteristics under sawtooth wave excitation by relaxation oscillator is presented. The effect of excitation magnitude on dynamical properties is described through the plots of time evolution, phase portrait, bifurcation diagram etc. And it shows dynamical behaviors transit from equilibrium to cycle, then to chaos, large scale period. Finally, synchronization of two duffing systems by nonlinear integral feedback control of sum of squared errors. It is useful for secure communication by selecting period carrier or chaotic carrier.

6. Acknowledgement

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7. References

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