Journal of Nonlinear Mathematical Physics Volume 14, Number 2 (2007), 164–173 LETTER

On a class of mappings between Riemannian manifolds

Thomas H OTWAY

Department of Mathematics, Yeshiva University, 500 W 185th Street, New York, NY 10033, U.S.A.
E-mail: otway@yu.edu

Received October 25, 2006; Accepted in Revised Form December 5, 2006

Abstract

Effects of geometric constraints on a steady flow potential are described by an elliptic-hyperbolic generalization of the harmonic map equations. Sufficient conditions are given for global triviality.

1 Introduction

In [12], local properties of maps which are critical points of a nonlinear Hodge energy were investigated (see also [10]). The target of the map has a physical interpretation as a geometric constraint on the potential of a steady flow. In this letter we illustrate these maps by deriving elementary but explicit physical examples, clarify the relation of the objects studied in [12] to other classes of maps which have been studied recently, and provide global conditions under which critical points reduce to the trivial map.

1.1 A column of tap water as a mapping

A simple motivating example for placing geometric constraints on a flow potential is provided by the steady flow of water from a faucet. If v_i is the velocity of the flow through a thin horizontal section of area A_i and if v_f and A_f are defined analogously, then the conservation of mass implies that

$$v_i A_i = v_f A_f. \tag{1.1}$$

But the particles accelerate under gravity, so if the cross section A_f is taken nearer to the drain than the cross section A_i, we conclude that $v_f > v_i$. Equation (1.1) then implies that $A_f < A_i$, which explains why the column of water is seen to taper. In this conventional approach the flow geometry is derived by imposing a physical law (acceleration under gravity) on a conservation law.

An alternative approach would be to observe that the column of water tapers, so that $A_f < A_i$. Conservation of mass implies eq. (1.1), so we conclude that $v_f > v_i$, that is, the particles accelerate under gravity. In this alternative approach the physical law is derived by imposing an observed flow geometry on a conservation law.

It is convenient to think of the geometry as a constraint applied to a flow potential u. In this example u maps a right circular cylinder into a tapered cylinder. Recall that every smooth curve has a dual representation as the envelope of its family of tangent lines. In a steady flow the velocity vectors appear as tangent lines to the potential surfaces, and so form an envelope of the cross sections represented in eq. (1.1). We see these cross sections taper by tracing the velocity vectors of the water droplets. Thus, as is often the case in fluid dynamics, the mathematical abstraction of a potential surface attains a visible representation in physical space.

While this simple example involves incompressible flow, the same alternatives exist in the more complicated case of compressible flow such as the flow of exhaust from a jet engine.

1.2 Shallow hydrodynamic flow

Now we consider a slightly more complicated case, that of steady, inviscid, hydrodynamic flow in a shallow channel. Write the flow velocity v in components (v_1, v_2, v_3), where v_1 is the horizontal component in the x-direction, v_2 is the horizontal component in the y-direction, and v_3 is the component in the (vertical) z-direction. Impose initial conditions under which v_3 is zero at time $t = 0$. Because we are assuming shallow depth, it is reasonable to suppose that the component of acceleration of water particles in the z-direction has negligible effect on pressure. The result of applying this hydrostatic law is that v_3 remains zero for all subsequent times and the horizontal velocity components v_1 and v_2 are independent of the z-coordinate. Because the flow is steady, the velocity components are also independent of t.

Generalizing eq. (1.1) to express the vanishing of an appropriate surface integral and applying the Divergence Theorem, we write the law of mass conservation in the form of a *continuity equation* (see, *e.g.*, [7], Sec. 1.1.1)

$$\frac{\partial}{\partial x}\left[h\left(x,y\right)v_1\left(x,y\right)\right]+\frac{\partial}{\partial y}\left[h\left(x,y\right)v_2\left(x,y\right)\right]=0, \tag{1.2}$$

where $h(x,y)$ represents the depth of the channel at the point (x,y). Bernoulli's formula expresses h as a function of $Q \equiv |v|^2$, that is, $h\left(Q\right) = \left(C - Q\right)/2g$, where C is a constant and g is the magnitude of gravitational acceleration. Substituting this relation into (1.2) and using the chain rule, we find that prior to the imposition of any geometric constraint the flow will satisfy

$$\left[\frac{C-Q}{2}-v_1^2\right]v_{1x}-v_1v_2\left(v_{1y}+v_{2x}\right)+\left[\frac{C-Q}{2}-v_2^2\right]v_{2y}=0. \tag{1.3}$$

(In eq. (1.3) numerical subscripts denote vector components, whereas variable subscripts denote partial differentiation in the direction of the variable.) While Bernoulli's formula is valid in a broader context [4], when we applied it in deriving eq. (1.3) we tacitly assumed the flow to be irrotational. Thus its velocity vector has vanishing curl, which allows us to equate mixed partial derivatives and assume the local existence of a potential function $u(x,y)$ such that $\nabla u = v$.

Writing $c^2 = gh = (C-Q)/2$, we obtain (*c.f.* (10.12.5) of [19]) a second-order quasi-linear elliptic-hyperbolic equation for the potential function:

$$\left[c^2 - u_x^2\right] u_{xx} - 2u_x u_y u_{xy} + \left[c^2 - u_y^2\right] u_{yy} = 0. \tag{1.4}$$

The type of eq. (1.4) depends on whether or not the flow speed $\sqrt{Q}$ exceeds the propagation speed c. For *subcritical* flow speeds in which the *Froude number* $F = \sqrt{Q}/c$ is exceeded by 1, the continuity equation is of elliptic type and the flow is *tranquil*. For *supercritical* flow speeds in which F exceeds 1, eq. (1.4) is of hyperbolic type, which characterizes *shooting flow*.

We can prescribe flow geometry for this problem by means analogous to the simpler example of water flowing from a faucet. In this case the potential function can be considered as a map from the flow domain to a target manifold. In order for the map to have non-trivial geometry, we must assume that the potential function u is multi-valued. Its components can be imagined as local coordinates on the target manifold. However, unlike the usual representations of the flow potential in the complex plane (*e.g.,* Sec. 1.6 of [8]), our mappings are defined over the real field. A geometric variational problem for this model will be formulated in the next section. Note that multi-valued potentials arise naturally on flow domains having non-trivial topology.

Conjectures on the observable effects of geometric constraints on the flow of shallow water go back at least 400 years. Geometric arguments, to one degree or another, have been applied to explain tidal bores on the English rivers Severn and Trent, the French river Seine near Caudebec-en-Caux, and the Chinese river Tsien-Tang, as well non-tidal anomalies such as those involving the Agulhas Current. See [14] for a review. As our last physical example, we recall a model for those effects which has particularly simple geometry; the model is given in greater detail in [20].

Consider a steady current flowing in the positive-x direction. Suppose that an incline of magnitude δ occurring between the points x_0 and x_1 of an otherwise horizontal channel floor produces a surface elevation of height ε at $x = x_1$. Suppose that $x_0 < x_1$, that the velocity of the flow to the left of x_0 is v_1 and that the velocity of the flow to the right of x_1 is $\widetilde{v_1}$. An inviscid flow is a conservative system. Equating the kinetic and potential energies of the flow for an arbitrary surface particle of mass m, we have

$$\frac{mv_1^2}{2} - \frac{m\widetilde{v_1}^2}{2} = mg\varepsilon. \tag{1.5}$$

Write the continuity equation for this one-dimensional system in the form

$$(H+\delta)\, v_1 = (H+\varepsilon)\, \widetilde{v_1}. \tag{1.6}$$

Equations (1.5) and (1.6) can be combined into the single expression

$$\left(H^2 + 2H\varepsilon + \varepsilon^2\right) 2g\varepsilon = v_1^2\,(2H+\varepsilon+\delta)\,(\varepsilon-\delta)\,.$$

Approximating this expression to first order in ε and δ, we obtain

$$\varepsilon = \frac{\delta}{1 - gH/v_1^2} = \frac{\delta}{1 - (c/v_1)^2}.$$

Again we find that the character of the flow is determined by whether the Froude number F exceeds, equals, or is exceeded by the number 1, where in this case $\sqrt{Q} = |v_1|$. The blow-up singularity that develops as F tends to 1 is avoided by hypothesis, as ε is assumed to be small. In the case of tranquil flow, a positive elevation $|\varepsilon|$ occurs when δ is exceeded by zero and a negative elevation $-|\varepsilon|$ occurs when δ exceeds zero. The opposite relations hold for shooting flow. Of course the effects of turbulence are ignored.

2 A geometric variational problem

Equation (1.4) can be derived by a variational principle from an energy functional having the form

$$E_\rho(u; \Sigma, \mathbb{R}^n) = \int_\Sigma \int_0^Q \rho(s) ds d\Sigma, \tag{2.1}$$

where Σ is a surface and $\rho(Q) = c^2$. In generalizing this problem we consider an energy functional of the form

$$E_\rho(u; M, N) = \int_M \int_0^{Q(du)} \rho(s)\, ds dM, \tag{2.2}$$

where M is a Riemannian manifold of dimension n; N is a Riemannian manifold of dimension m; $u : M \to N$ is a bounded map;

$$Q(du) = \langle du, du \rangle_{|T^*M \otimes u^{-1}TN};\tag{2.3}$$

$\rho : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ is a $C^{1,\alpha}$ function of Q satisfying the differential inequality

$$0 < \frac{\frac{d}{dQ}\left[Q\rho^2(Q)\right]}{\rho(Q)} < \infty \tag{2.4}$$

for $Q \in [0, Q_{crit})$; Q_{crit} is the square of the sonic flow speed. Inequality (2.4) is a condition for tranquil channel flow.

We are interested in maps u which are *stationary* with respect to $E_\rho(u; M, N)$ in the sense that

$$\delta E = \frac{d}{dt} E(u_t; M, N)|_{t=0} = 0, \tag{2.5}$$

where $u_t : M \times (-\epsilon < t < \epsilon) \to N$, $u_0 = u$, is a smooth, compactly supported one-parameter family of variations (to be further specified below).

The variational equations of $E_\rho(u; M, N)$ are satisfied by maps which are extremal within a competing homotopy class of finite-energy maps from M to N. However, there are solutions of (2.5) which are not extremal with respect to any class of maps.

Condition (2.4) is a condition for ellipticity of the variational equation for E,

$$trace\nabla_{cov}(\rho(Q)du) = 0, \tag{2.6}$$

where ∇_{cov} denotes the covariant derivative in the bundle $T^*M \otimes u^{-1}TN$. Note that this equation introduces geometry into both the domain and range of eq. (1.4).

As an alternative to the hydrodynamic interpretation, the manifold N may be chosen to represent a geometric constraint placed on the flow potential of a steady, irrotational, polytropic, perfect, compressible fluid, which is adiabatic and isentropic and for which the closed 1-form $du \in \Gamma(T^*M)$ is dual to the flow velocity. In that case we choose

$$\rho(Q) = \left(1 - \frac{\gamma-1}{2}Q\right)^{1/(\gamma-1)}, \tag{2.7}$$

where $\gamma > 1$ is the adiabatic constant of the medium [3]. Those choices transform (2.4) into a condition for subsonic compressible flow of mass density ρ. The *sonic transition* as Q tends to Q_{crit} is a gas-dynamic analogy for the change in the aspect of tap water from clear to white at a sufficiently high velocity or for the transition from tranquil to shooting channel flow in hydraulics. We recover harmonic maps as the incompressible limit $\rho(Q) \equiv 1$.

Note that two distinct terms are both referred to as *density* in the mathematical/fluid dynamics literature. The physical, or mass density of the flow is $\rho(Q)$, but the density of the variational integral E_ρ is given by

$$e(u) = \int_0^Q \rho(s)ds.$$

In particular, the physical density given by (2.7) is a decreasing function of Q, whereas the corresponding variational density is an increasing function of Q provided $Q < 2/(\gamma - 1)$. The variational density $e(u)$ corresponding to the physical density (2.7) vanishes (or *cavitates*) at the flow speed $Q = 0$, but the physical density itself does not. The physical density (2.7) cavitates at the flow speed $Q = 2/(\gamma - 1)$, but the variational density does not.

Although in realistic physical contexts the ratio γ of specific heats must be taken to exceed 1, many of the analytic properties of formula (2.7) extend to the limiting case in which γ tends to unity. In this limiting case the variational density is given by the function

$$\frac{1}{2}e(u) = 1 - \exp\left[-Q/2\right].$$

(We can easily see this by writing $y = \log\rho$ and applying L'Hôpital's rule to (2.7).) Regularity arguments for weak subsonic solutions of (2.5) presented for the unconstrained case in, *e.g.*, [17] can be extended to the limiting value of γ by the Arzelá-Ascoli Theorem. One thus obtains the existence of continuous subsonic solutions in that limit by semicontinuity, using the convexity of the limiting energy on the subsonic range (*c.f.* Proposition 1 of [10]). Some comments on the geometry of the limiting case are given in Sec. 2.1 of [11].

Reference [12] reviews the isometric embedding of the target manifold for critical points of (2.2), (2.3) into a higher-dimensional Euclidean space $\mathbb{R}^k$ and the use of nearest-point projection to obtain a form of the geometric constraint which is convenient for variational analysis in a Sobolev space. This is a familiar trick in the theory of harmonic maps [16]. We obtain variations of the form $u_t = \pi_N \circ (u + t\psi)$, where ψ is a smooth map from M into $\mathbb{R}^k$ and π_N is the *nearest-point projection,* assigning to every y in a Euclidean neighborhood of N the point on N that minimizes the distance to y. The embedding of the flow geometry in Euclidean space does not necessarily embed the corresponding physics in an ambient Euclidean space. To illustrate the distinction, compare water poured over a small sphere on the surface of Earth with water flowing on the surface of a spherical planet. In the former case we would take the gravitational acceleration vector to point in the vertical direction of the Euclidean coordinate system in which the sphere sits; in the latter case we would take the gravitational acceleration to point in the radial direction of the sphere itself. In the former case the gravitational potential comes from the Euclidean space in which the sphere is embedded, whereas in the latter case the intrinsic geometry of the surface affects the gravitational potential.

A brief review of the literature relevant to eq. (2.6) − in particular, its relation to the nonlinear Hodge theory introduced in [18] − is given in Sec. 1 of [12]. To those remarks we add that the extension of geometric variational problems for the Dirichlet energy to more general classes of energies was already outlined in [6], with a suggested application of the harmonic map energy to the theory of elasticity and that, whereas nonlinear Hodge theory generally considers the elliptic case of the variational equations, eq. (2.6) will be allowed to change from elliptic to hyperbolic type as Q_{crit} is exceeded. While the present letter addresses the case of irrotational flow, it is possible that similar considerations might apply to flows with vorticity; see, for example, the recent paper [5].

3 Conditions for trivial flow

We will say that a flow is *trivial* if its flow potential u is a constant function. Equivalently, the velocity field of a trivial flow associates the zero vector to every point of the flow domain.

We expect that a relative minimum for a smooth function of a single variable will occur at a point for which the second derivative is non-negative. The analogue of the second derivative for variational integrals is the *second variation.* Let $u_{s,t} : M \to N$, $-\epsilon < s, T < \epsilon$, be a compactly supported two-parameter variation such that $u_{0,0} = u$. Define

$$V = \frac{\partial u_{s,t}}{\partial t}|_{s,t=0}$$

and

$$W = \frac{\partial u_{s,t}}{\partial s}|_{s,t=0}.$$

The *second variation* is the quantity

$$I(V,W) = \frac{\partial^2}{\partial s \partial t} E(u_{s,t})|_{s,t=0}.$$

A map is said to be *stable* if $I(V,V)$ is non-negative for any compactly supported vector field V along u.

This definition implies the triviality of a certain class of flows:

Theorem 1. *Let the domain of a shallow, steady, irrotational hydrodynamic flow satisfying the hydrostatic law be represented by a compact Riemannian manifold M and let the flow potential u take M into the m-sphere $\mathbb{S}^m$ for $m \geq 2$. Let the flow speed Q be given by (2.3) for $0 \leq Q < 2$. Then any stable flow potential takes every point of M to a single point on $\mathbb{S}^m$.*

Proof. We impose a geometric constraint on the variational problem (1.4), (2.1), declaring that the image of the flow potential u must lie on a smooth, compact, Riemannian manifold N, where N is a submanifold of $\mathbb{R}^k$ for some sufficiently large number k. This transforms (2.1) into (2.2) and the Euler-Lagrange equation (1.4) into the equation

$$\delta_g \left[\left(1 - \frac{Q}{2}\right) du \right] = \left(1 - \frac{Q}{2}\right) A\left(du, du\right), \tag{3.1}$$

where δ_g is the formal adjoint of the exterior derivative; g is the Riemannian metric on M; Q satisfies (2.3); A is the second fundamental form of N, where N is expressed as a submanifold of some higher-dimensional Euclidean space as noted earlier. The system (3.1) is identical to the variational equations for $E_\rho\left(u; M, N\right)$ with ρ given by (2.7) in the special case $C = n = \gamma = 2$, where C is the constant of Bernoulli's formula.

Computing the second variation of $E_\rho\left(u; M, \mathbb{S}^m\right)$, we obtain

$$I(V,V) = \int_M |du|^2 \left\{ |du|^2 \frac{d^2}{dQ^2} \int_0^Q \rho(s)\, ds + (2-m) \frac{d}{dQ} \int_0^Q \rho(s)\, ds \right\} dM. \tag{3.2}$$

Because $\rho(s) = 1 - s/2$, the right-hand side of (3.2) is negative under the hypotheses of the theorem unless u is trivial almost everywhere. But u is continuous because $|du|$ is bounded, so the conclusion holds everywhere on M. This completes the proof. ■

Remarks. *i)* The conditions on the domain M will only correspond locally to the physical model of Sec. 1.2.

ii) Computing (2.4) for ρ given by (2.7) with $C = n = \gamma = 2$, we find that Theorem 1 holds for flow speeds that extend well into the range of shooting flow. Moreover, solving $Q = c^2 = 1 - Q/2$ for Q, we verify that the Froude number attains the value 1 at $Q = 2/3 = 2/(\gamma + 1) = Q_{crit}$.

iii) A map $u : M \to N$ is said to be *F-harmonic* if it is a critical point of the F-energy functional

$$E_F(u; M, N) = \int_M F\left(\frac{|du|^2}{2}\right) dM,$$

where $F : [0, \infty) \to [0, \infty)$ is a strictly increasing, twice-differentiable function of its argument. F-harmonic maps were introduced in [1] as a unification of p-harmonic and exponentially harmonic maps. A version of inequality (3.2) holds for any map which is F-harmonic; *c.f.* Theorem 7.1 of [1], taking

$$F\left(\frac{|du|^2}{2}\right) = e(u) = \int_0^{2\left(|du|^2/2\right)} \rho(s)\, ds.$$

In fact, the extension of Theorem 1 to a larger class of target manifolds is easily obtained by adapting the ideas of [2]. However, Theorem 1 is false for the best known special cases of F-harmonic maps: p-harmonic, exponentially harmonic, and α-harmonic maps; *c.f.* the remark following Corollary 7.2 of [1]. Moreover, critical points of (2.2), (2.3) will not be F-harmonic whenever the variational density $e(u)$ is a decreasing function of Q. That possibility is allowed by condition (2.4) and by our initial definition of ρ.

iv) Theorem 1 recalls the famous statement that the wind can never be blowing simultaneously in the same direction everywhere on Earth. But of course the two assertions are mathematically different and have completely different proofs. They each result from the combination of a harsh global hypothesis − maximal symmetry in the range of a map − with the global imposition of what would be a reasonable local hypothesis − in one case, continuity and in the other, stability.

v) Adding to the mass density $\rho(s)$ a generalized "surface tension" of the form

$$\tau(s) = \mu\,(1 + s)^{-1/2},$$

where μ is a positive constant, does not affect the result of Theorem 1. But this extension is dependent on the sign of μ.

vi) Because the proof of Theorem 1 relies on applying a compressible model to shallow hydrodynamic flow, it can be viewed as a corollary of a theorem about compressible flow:

Theorem 2. *Let the compact Riemannian manifold M be the domain of a steady, polytropic, irrotational, perfect flow and let the flow potential u take M into the m-sphere $\mathbb{S}^m$ for $m \geq 2$. Let the flow speed Q be given by (2.3) for $0 < Q < 2/(\gamma - 1)$, where γ is the adiabatic constant of the fluid. Then any stable flow takes every point of M into a single point of $\mathbb{S}^m$.*

Proof. Follow the proof of Theorem 1 based on eq. (1.4) (*c.f.* (2.14) of [3]), defining ρ as in (2.7) for any $\gamma > 1$ and any $n \geq 2$. ■

The correspondence between gas dynamics and shallow hydrodynamic flow, illustrated in (1.4) and leading to the similarity of Theorems 1 and 2, was apparently first reported in [15] for time-dependent compressible flow in 1 space dimension. Elliptic-hyperbolic systems similar − or dual − to (1.3) with a side condition of vanishing curl arise in projective geometry and optics, as well as in hydrodynamics and gas dynamics; *c.f.* [13].

This discussion provides a context for reinterpreting an earlier result. The variational arguments in [9] were given in terms of a function $w\left(|du|^p\right)$. That function can be interpreted as the variational density of the map u, taking $p = 2$ and

$$w(t) = \int_0^t \rho(s)ds.$$

Thus the results of [9] can be applied to hydrodynamic and compressible flow, which gives another set of conditions for triviality of the flow potential.

Acknowledgments. *I am grateful to an anonymous referee for comments on flow with vorticity.*

References

[1] Ara M, Geometry of F-harmonic maps, *Kodai Math. J.* **22** (1999), 243–263.

[2] Ara M, Stability of F-harmonic maps into pinched manifolds, *Hiroshima Math. J.* **31**(2000), 171–181.

[3] Bers L, Mathematical Aspects of Transonic Gas Dynamics, Wiley, New York, 1958.

[4] Constantin A and Strauss W, Exact steady periodic water waves with vorticity, *Comm. Pure Appl. Math.* **57** (2004), 481–527.

[5] Constantin A, Sattinger D and Strauss W, Variational formulations of steady water waves with vorticity, *J. Fluid Mech.* **548** (2006), 151–163.

[6] Eells J and Sampson J H, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.

[7] Johnson R S, A Modern Introduction to the Mathematical Theory of Water Waves, Cambridge, 1997.

[8] Meyer R E, Introduction to Mathematical Fluid Dynamics, Dover, New York, 1971.

[9] Otway T H, An asymptotic condition for variational points of nonquadratic functionals, *Annales Fac. Sci. Toulouse* **11**, No. 2 (1990), 187–195.

[10] Otway T H, Nonlinear Hodge maps, *J. Math. Phys.* **41**, No. 8 (2000), 5745–5766.

[11] Otway T H, Hodge equations with change of type, *Annali di Matematica, pura ed applicata* **181** (2002), 437–452.

[12] Otway T H, Maps and fields with compressible density, *Rendiconti del Seminario Matematico dell' Università di Padova* **111** (2004), 133–159.

[13] Otway T H, Harmonic fields on the extended projective disc and a problem in optics, *J. Math. Phys.* **46** (2005), 113501.

[14] Otway T H, Anomalies of the tides, in Galileo and the Cultural Universe of the Renaissance, Editors: Cwilich G and Felsenstein F, `arXiv:math-ph/0312047`.

[15] Riabouchinsky D, Sur l'analogie hydraulique des mouvements d'un fluide compressible, *Comptes Rendus Academie des Sciences, Paris* **195** (1932), 998.

[16] Schoen R, Analytic aspects of the harmonic map problem, in Seminar on Nonlinear Partial Differential Equations, Editor: Chern S S, Springer-Verlag, New York, 1985, 321–358.

[17] Shiffman M, On the existence of subsonic flows of a compressible fluid, *J. Rat. Mech. Anal.* **1** (1952), 605–652.

[18] Sibner L M and Sibner R J, A nonlinear Hodge-de Rham theorem, *Acta Math.* **125** (1970), 57–73.

[19] Stoker J J, Water Waves, Interscience, New York, 1987.

[20] Tricker R A R, Bores, Breakers, Waves, and Wakes, American Elsevier, New York, 1965.