p-adic interpolating function associated with Euler numbers

Taekyun KIM [*], *Daeyeoul KIM* [†] *and Ja Kyung KOO* [‡]

[*] *EECS, Kyungpook National University Taegu 702-701, S. Korea*
 E-mail: tkim@knu.ac.kr

[†] *National Institute for Mathematical Sciences Tajeon 305-340, Korea,*
 E-mail: daeyeoul@nims.re.kr

[‡] *KAIST, Department of Mathematics, Tajeon 305-701, Korea*
 E-mail: jkkoo@math.kaist.ac.kr

Abstract

In this paper, we investigate some relations between Bernoulli numbers and Frobenius-Euler numbers, and we study the values for p-adic l-function.

1 Introduction

Recently, many mathematician have studied the Bernoulli polynomials and the Bernoulli numbers. The Bernoulli polynomials and Bernoulli numbers are of significant importance in physics and mathematics. q-Bernoulli polynomials and q-Bernoulli numbers possess many surprising properties and arise in many areas of applied mathematics and physics(see [2,7,8,9,10,17,18,19]). These numbers and polynomials are used not only number theory, complex analysis, and the other branch of mathematics but also in other parts of the p-adic analysis and mathematical physics. Hensel invented the so-called p-adic numbers around the end the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within scientific community[11]. Although they have penetrated several mathematical fields, number theory, algebraic geometry, analysis, stochastic differential equations on the real Banach spaces and manifolds and other parts of the natural sciences which are turbulence theory, dynamical systems, statistical physics, biology, etc ([1,20]). The motivation of this paper is to give some relations and formulae of the L-function.

Let $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively denote the field of rational numbers, the complex number field, the field of $p-$adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. It is well known that Bernoulli numbers B_k can be determined inductively by

$$B_0 = 1, (B+1)^k - B_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing B^i by B_i. The Bernoulli polynomials are defined by

$$B_k(x) = (B+x)^k = \sum_{l=0}^{k} \binom{k}{l} B_l x^{k-l}, \quad \text{cf. } [3,5,6,7].$$

Let $F(t)$ be the generating function of B_k. Then we see that

$$F(t) = \frac{t}{e^t - 1} = e^{Bt} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

This is a consequence of the following difference equation

$$F(t) = e^t F(t) - t.$$

Let u be a complex number with $|u| > 1$. Then the Frobenius-Euler numbers and polynomials are defined inductively by

$$H_0(u) = 1, \quad (H+1)^k - uH_k(u) = 0 \quad \text{for } k \geq 1,$$

with the usual convention of replacing H^k by $H_k(u)$ for $k \geq 0$, see [6,7,8,9].

In [7, 10], Kim studied the $p-$adic $L-$function $L_p(s,\chi)$ which interpolates Bernoulli numbers at negative integers. Kim [4] constructed a new q-extension of generalized Bernoulli polynomials attached to χ and proved the existence of a specific p-adic interpolation function that interpolates the q-extension of generalized Bernoulli polynomials at negative integers. The p-adic q-integral (or q-Volkenborn integral) was originally constructed by T. Kim [7, 10], who indicated a connection between the q-integral and non-archimedean combinatorial analysis. The q-integral is used in mathematics physics, for example, in the derivation of the functional equation for the q-L-function, in the theory of q-stirling numbers, and q-Mahler theory of integration with respect to a ring $\mathbb{Z}_p$ together with Iwasawa p-adic q-L-function. In this paper, we shall investigate some relations between Bernoulli numbers and Frobenius-Euler numbers. Carlitz defined q-extension of Frobenius-Euler numbers and polynomials and proved properties analogous to those satisfies $H_n(u)$ and $H_n(u,x)$. Satoh [12,13] used these properties, especially the so-called distribution relation for q-Frobenius-Euler polynomials, in order to construct the corresponding q-extension of the p-adic measure and to define a q-extension of p-adic l-function $l_{p,q}(s,u)$. By using a method similar to that used in [10], we give the values for p-adic l-function.

The purpose of this paper is to study the values of the $p-$adic $l-$function $l_{p,u}(s,\chi)$ which interpolates Frobenius-Euler numbers at negative integers. Finally, we derive the values of $L_p(s,\chi)$ at positive integers from the relation between $l_{p,u}(s,\chi)$ and $L_p(s,\chi)$. We list the following typical results in the paper.

2   On Dirichlet's L−function

Let χ be a primitive Dirichlet character with conductor $f \in \mathbb{N}$. Then the generalized Bernoulli numbers $B_{k,\chi}$ are defined by

$$B_{k,\chi} = f^{k-1} \sum_{a=1}^{f} \chi(a) B_k \left(\frac{a}{f} \right), \quad \text{cf. } [8,9,14,15,16].$$

It is well known that the p−adic L−function is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, s \in \mathbb{C}, \quad \text{cf. } [3,5,6,8,9].$$

Thus, we have $L(1-k,\chi) = -\dfrac{B_{k,\chi}}{k}$, for any positive integer k.

Let u a be complex number with $|u| > 1$. Then the Frobenius-Euler numbers $H_k(u)$ and polynomials $H_k(u,x)$ are defined by $H_0(u) = 1$, $(H+1)^k - uH_k(u) = 0$ if $k \geq 1$, with the usual convention of replacing H^k by $H_k(u)$ and $H_k(u,x) = (H+x)^k$ for $k \geq 0$, cf. [5,14]. The generalized Frobenius-Euler numbers $H_{k,\chi}(u)$ are defined by

$$H_{k,\chi}(u) = f^k \sum_{a=1}^{f} u^{f-a} \chi(a) H_k(u^f, \frac{a}{f}),$$

for $k \geq 0$. In [5,14], the complex function $l_u(s,\chi)$ is considered by

$$l_u(s,\chi) = \sum_{n=1}^{\infty} u^{-n} \chi(n) n^{-s}.$$

This function interpolates the Frobenius-Euler numbers at negative integers as follows:

$$l_u(-k,\chi) = \frac{1}{u^f - 1} H_{k,\chi}(u), \quad \text{cf. } [5,14].$$

Now we shall investigate the relation between Frobenius-Euler numbers and Bernoulli numbers. We take d as an element of positive integer with $(d, fp) = 1$ and $\chi(d) \neq 1$. Let ζ_d be a primitive d−th root of unity. Then we have

$$\sum_{j=1}^{d-1} l_{\zeta_d^{-j}}(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \sum_{j=1}^{d-1} \zeta_d^{jn}$$

$$= -\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} + \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \sum_{j=0}^{d-1} \zeta_d^{jn}$$

$$= -\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} + d^{-s+1} \chi(d) \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Thus, we have

$$-\sum_{j=1}^{d-1} l_{\zeta_d^{-j}}(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} - d^{1-s}\chi(d)\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

$$= (1 - d^{1-s}\chi(d))\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

$$= (1 - d^{1-s}\chi(d))L(s,\chi).$$

We set $s = 1 - k$ for $k > 1$,

$$(1 - d^k\chi(d))L(1-k,\chi) = -\sum_{j=1}^{d-1} l_{\zeta_d^{-j}}(1-k,\chi).$$

Thus, we obtain the following:

Proposition 1. For $k \in \mathbb{Z}$ with $k > 1$, we have

$$(1 - d^k\chi(d))\frac{B_{k,\chi}}{k} = \sum_{j=1}^{d-1} \frac{1}{\zeta_d^{-jf} - 1} H_{k-1,\chi}(\zeta_d^{-j}).$$

3   On the $p-$adic $L-$function

Let u be an element of $\mathbb{C}_p$ with $|1 - u|_p \geq 1$. Then the $p-$adic Euler measure is defined on $\mathbb{Z}_p$ by

$$E_u(a + p^N\mathbb{Z}_p) = \frac{u^{p^N - a}}{1 - u^{p^N}}, \quad \text{cf. } [5,6,14],$$

for $a \in \mathbb{Z}$ with $0 \leq a \leq p^N - 1$ and $N \geq 0$. Let f be a positive integer. We denote

$$X = \varprojlim_{N} \mathbb{Z}/dp^N,$$

$$X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.
Note that the natural map

$$\mathbb{Z}/fp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}$$

induces

$$\pi : X \to \mathbb{Z}_p.$$

If g is a function on $\mathbb{Z}_p$, we denote by the same g the function $g \circ \pi$ on X. Namely we consider g as a function on X. We can express the Frobenius-Euler numbers as an integral on X, by using the measure E_u, that is,

$$\int_X \chi(x)x^k dE_u(x) = \begin{cases} \dfrac{1}{1-\dfrac{1}{u}u^f} H_{k,\chi}(u) & \text{if } x \neq 1, \\[2mm] \dfrac{\dfrac{1}{u}}{1-u} H_k(u) & \text{if } x = 1. \end{cases}$$

Let w denote the Teichmüller character mod p (if $p = 2$, mod 4). For $x \in X^*$, we set $< x >= \dfrac{x}{w(x)}$. Note that $| < x >^{-1} |_p < p^{-1/p-1}$, $< x >^s$ is defined by $\exp(s \log_p < x >)$ for $|s|_p \leq 1$. Now we consider an interpolation function $l_{p,u}(s, \chi)$ for the Frobenius-Euler numbers as follows

$$l_{p,u}(s, \chi) = \int_{X^*} < x >^{-s} \chi(x) dE_u(x),$$

for $s \in \mathbb{Z}_p$. Note that

$$l_{p,u}(-k, \chi w^k) = \frac{1}{1-u^f} H_{k,\chi}(u) - \frac{p^k \chi(p)}{1-u^{fp}} H_{k,\chi}(u^p) \quad \text{if } \chi \neq 1.$$

For a fixed $\alpha \in X^*$, it is well known that

$$L_p(1-k, \chi) = -\frac{1}{k}\left(1 - p^{k-1}\chi_k(p)\right)\left(1 - (\frac{1}{\alpha})^k \chi_k(\frac{1}{\alpha})\right) B_{k,\chi_k}, \quad \text{see [13, 14]},$$

where $\chi_k = \chi w^{-k}$. Now we refine the above result

$$L_p(1-k, \chi w^k) = -\frac{1}{k}(1 - p^{k-1}\chi(\frac{1}{\alpha})) B_{k,\chi_k}.$$

Here we shall investigate the relation between $l_{p,u}(s, \chi)$ and $L_p(s, \chi)$. We set $\dfrac{1}{\alpha} = d$.

$$L_p(1-k, \chi w^k) = -\frac{1}{k}\left(1 - p^{k-1}\chi(p)\right)\left(1 - d^k \chi(d)\right) B_{k,\chi}.$$

By using Proposition 1, we see that

$$\begin{aligned} L_p(1-k, \chi w^k) &= -(1 - p^{k-1}\chi(p)) \sum_{j=1}^{d-1} \frac{H_{k-1,\chi}(\zeta_d^{-j})}{(\zeta_d^{-jf} - 1)} \\ &= \sum_{j=1}^{d-1} \int_{X^*} \chi(x)x^{k-1} dE_{\zeta_d^{-j}}(x). \end{aligned}$$

Therefore we obtain the following :

Proposition 2. For any positive integer k, we have

$$L_p(1-k, \chi w^k) = \sum_{j=1}^{d-1} \int_{X^*} \chi(x)x^{k-1} dE_{\zeta_d^{-j}}(x).$$

For χ a primitive Dirichlet character mod f, ζ_f a fixed primitive $f-$th root of unity, $\tau(\chi) = \sum \chi(j)\zeta_f^j$, $\epsilon \neq 1$ a $d-$th root of unity, and $(d, fp) = 1$, we have for any continuous $f : X^* \to \mathbb{C}_p$:

$$\int_{X^*} \chi(x)f(x)dE_\epsilon(x) = \frac{\tau(\chi)}{f} \sum_{0 \leq a < f} \overline{\chi}(a) \int_{X^*} f(x)dE_{\zeta_f^{-a}\epsilon}(x).$$

Let $f(x) = x^{-k}$ and $\epsilon = \zeta_d^{-1}$. Then we see that

$$
\begin{aligned}
L_p(1-k, \chi w^k) &= \sum_{j=1}^{d-1} \int_{X^*} \chi(x)x^{k-1}dE_{\zeta_d^{-j}}(x) \\
&= \frac{\tau(\chi)}{f} \sum_{0 \leq a < f} \overline{\chi}(a) \sum_{j=1}^{d-1} \int_{X^*} x^{k-1}dE_{\zeta_f^{-a}\zeta_d^{-j}}(x).
\end{aligned}
$$

Therefore we obtain the following :

Proposition 3. For any positive integer k, we have

$$L_p(1-k, \chi w^k) = \frac{\tau(\chi)}{f} \sum_{0 \leq a < f} \overline{\chi}(a) \sum_{j=1}^{d-1} \int_{X^*} x^{k-1}dE_{\zeta_f^{-a}\zeta_d^{-j}}(x).$$

4   The value of $l_{p,\zeta_d^{-1}}(s, \chi)$ at $s = 1$

For $u = \zeta_d^{-1}$, we note that

$$
\begin{aligned}
l_{p,\zeta_d^{-1}}(1-k, \chi w^k) &= \int_{X^*} x^{k-1}\chi(x)dE_{\zeta_d^{-1}}(x) \\
&= \frac{\tau(\chi)}{f} \sum_{0 \leq a < f} \overline{\chi}(a) \int_{X^*} x^{k-1}dE_{\zeta_d^{-1}\zeta_f^{-a}}(x).
\end{aligned}
$$

Thus, we see that

$$l_{p,\zeta_d^{-1}}(1, \chi) = \frac{\tau(\chi)}{f} \sum_{0 \leq a < f} \overline{\chi}(a) \int_{X^*} \frac{1}{x}dE_{\zeta_d^{-1}\zeta_f^{-a}}(x).$$

It is well known that the $p-$adic logarithm was defined by

$$\log_p(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

From this, we derive

$$\int_{X^*} \frac{1}{x} dE_{\zeta_d^{-1}\zeta_f^{-a}}(x) = -\log_p(1 - \zeta_d^{-1}\zeta_f^{-a}) + \frac{1}{p}\log_p(1 - \zeta_d^{-p}\zeta_f^{-pa})$$

$$= \log_p \frac{(1 - \zeta_d^{-p}\zeta_f^{-pa})^{\frac{1}{p}}}{(1 - \zeta_d^{-1}\zeta_f^{-a})}.$$

Therefore we obtain the following.

Theorem 4. For a primitive Dirichlet character $(\mod f)$, ζ_f a fixed primitive $f-$th root of unity, $\tau(\chi) = \sum \chi(j)\zeta_f^j$, $\zeta_d \neq 1$ a $d-$th root of unity, and $(d, fp) = 1$, we have

$$l_{p,\zeta_d^{-1}}(1, \chi) = \frac{\tau(\chi)}{f} \sum_{0 \leq a < f} \overline{\chi}(a) \log_p \left(\frac{(1 - \zeta_d^{-p}\zeta_f^{-pa})^{\frac{1}{p}}}{1 - \zeta_d^{-1}\zeta_f^{-a}} \right).$$

Remark. Carlitz has introduced an interesting q-analogue of Frobenius-Euler numbers in [L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15(1948) 987-1000; L. Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc. 76 (1954) 332-350]. We constructed $L_p(1 - k, \chi w^k)$ which are found useful in number theory and p-adic analysis. As applications, we obtain the values of $l_{p,\zeta_d^{-1}}(1, \chi)$. For powers of the Teichmuller character, T. Kim used the integral representation to extend the L-function to the large domain, in which it is a meromorphic function in the first variable and an analytic element in the second. These integral representations imply systems of congruences for the generalized Bernoulli polynomials. The problem of constructing multiple p-adic L-functions in the p-adic number field is still open.

References

[1] S. Albeverio, M Gundalch, A. Yu. Khrennikiv and K.-O. Lindahl, On the Markovian behavior of p-adic random dynamical systems, Russ. J. Math. Phys. 8 (2001), pp.135-152.

[2] A. S. Hegazi and M. Mansour, A note on q-Bernoulli numbers and polynomials, J. Nonlinear Math. Phys. 13 (2006) pp.9–18.

[3] K. Iwasawa, Lectures on $p-$adic $L-$function, Princeton Univ. 1972.

[4] T. Kim, On the analogs of Euler numbers and polynomials associated with *p*-adic *q*-integrals on $\mathbb{Z}_p$ at $q = -1$, J. Math. Anal. Appl. 331(2007), pp.779-792.

[5] T. Kim, An analogue of Bernoulli numbers and their congruences, Rep. Fac. Sci. Engrg. Saga Univ. Math. 22 (1994), pp.21–26.

[6] T. Kim, On explicit of $p-$adic $q-L-$function, Kyushu J. Math. 48(1994), pp.73–86.

[7] T. Kim, *q*-Volkenborn Integration, Russian J. Math. Phys. 9(2005), pp.288-299.

[8] T. Kim, Power series and asymptotic series associated with the $q-$analog of two-variable $p-$adic $L-$function, Russ. J. Math. Phys. 12(2005), pp.186–196.

[9] T. Kim, Multiple *p*-adic *L*-functions, Russian J. Math. Phys. 13(2006), pp.151-157.

[10] T. Kim, On certain values of *p*-adic *q*-*L*-functions, Rep. Fac. Sci. Engrg. Saga Univ. Math. 23(1995), pp.1–7.

[11] T. Kim, A new Approach to q-zeta function, Journal of Computational Analysis and Applications 9(2007), pp.395-400.

[12] B. A. Kupershmidt, Reflection symmetries of *q*-Bernoulli polynomials, J. Nonlinear Math. Phys. 12(2005), pp.412–422.

[13] A. M. Robert, A course in *p*-adic Analysis, Springer-Verlag, 2000.

[14] J. Satho, *q*-analogue of Riemann's ζ-function and *q*-Euler numbers, J. Number Theory 31(1989), pp.346–362.

[15] K. Shiratani, S. Yamamoto, On $p-$adic interpolating function for the Euler numbers and its derivatives, Mem. Fac. Sci. Kyushu Univ. ser. A 39(1985), pp.113–125.

[16] Y. Simsek, D. Kim, S.-H. Rim, On the two-variable Dirichlet $q-L-$series, Adv. Sdud. Contemp. Math. 10(2005), pp.131–142.

[17] H. S. Cho, E.-S. Kim, Translation-invariant $p-$adic integral on $\mathbb{Z}_p$, Notes Number Theory Discrete Math. 7(2001), pp.70–77.

[18] L. C. Jang, H. K. Pak, S. H. Rim, A note on analogue of Euler and Bernoulli numbers, JP J. Algebra Number Theory Appl. 3(2003), pp.461–469.

[19] H. M. Srivastava, T. Kim, Y. Simsek, *q*-Bernoulli numbers and polynomials associated with multiple $q-$zeta functions and Basic $L-$series, Russ. J. Math. Phys. 12(2005), pp.241–268.

[20] V. S. Viladimirov, I. V. Volovich and E. I. Zelenov, *p*-adic Analysis and Mathematical Physics, Cambridge Univ. Press, London and New York, 1990.