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Resolving of discrete transformation chains and multisoliton solution of the 3-wave problem

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Abstract

The chain of discrete transformation equations is resolved in explicit form. The new found form of solution alow to solve the problem of interrupting of the chain in the most strigtforward way. More other this form of solution give a guess to its generalization on the case of arbitrary semisimple algebra in the case of n-wave problem. This technique is demonstraited on the example of construction multi-soliton solution of the 3-wave problem in explicit form.

1 Introduction

The problem of three waves in two dimensions arises in different form in many branches of the mathematical physics. For example it exists in radio and nonlinear optics applications, which can be found in [10].

Three-wave interactions in plasmas, stability criteria and asymptotic behavior for a general system of three interacting waves, the influence of mutually different linear damping coefficients in a system of three interacting waves of equal sign of energy are described in [13], where an accurate study of waves in a plasma has been performed.

Nonlinear water wave interactions and hydrodynamic turbulence including the three wave problem were investigated in [12, 9, 2]

The subject is also discussed in the study of nonlinear lattices [3] and photon crystals [11].

The present paper should be considered as a direct continuation of the previous one [4],[5]. For the convenience of the reader we first reproduce necessary results from [4] in a prologue.

2 Prologue

2.1 3-wave problem in dimensionless form and its discrete transformations

Arbitrary element of the A_2 algebra (without Cratan components) may be represented as

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3 The case of A_2 algebra

Algebra A_2 has the following Cartan matrix and basic commutation relations between two generators of the simple roots $X^{\pm}_{1,2}$ and its Cartan elements $h_{1,2}$

$$k = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} [h_1, X^{\pm}_1] = \pm 2X^{\pm}_1 & [h_1, X^{\pm}_2] = \mp X^{\pm}_2 \\ [h_2, X^{\pm}_1] = \mp X^{\pm}_2 & [h_2, X^{\pm}_2] = \pm 2X^{\pm}_2 \end{pmatrix}$$

Arbitrary element of the algebra may be represented as

$$f = f^+_{1.1} X^+_{\alpha_1+\alpha_2} + f^+_{0.1} 2X^+_{\alpha_2} + f^+_{1.9} X^+_{\alpha_1} + f^-_{1.0} X^-_{\alpha_1} + f^-_{0.1} X^-_{\alpha_2} + f^-_{1.1} X^-_{\alpha_1+\alpha_2}$$

$\alpha_{1,2}$ are the indexes of simple roots. $X^+_{\alpha_1+\alpha_2} \equiv [X^+_2, X^+_1]$.

In these notations the system of equations (**??**) looks as

$$\begin{gathered} D_{1,0} f^+_{1.0} = f^+_{1.1} f^-_{0.1}, \quad D_{1,0} f^-_{1.0} = f^-_{1.1} f^+_{0.1} \\ D_{0,1} f^+_{0.1} = f^+_{1.1} f^-_{1.0}, \quad D_{0,1} f^-_{0.1} = f^-_{1.1} f^+_{1.0} \\ D_{1,1} f^+_{1.1} = -f^+_{0.1} f^+_{1.0}, \quad D_{1,1} f^-_{1.1} = -f^-_{0.1} f^-_{1.0} \end{gathered} \tag{3.1}$$

where operatores of differentaion are the following ones $D_{1.0} = \frac{c_1}{\delta}\frac{\partial}{\partial t} + \frac{d_1}{\delta}\frac{\partial}{\partial x}, D_{0.1} = \frac{c_2}{\delta}\frac{\partial}{\partial t} + \frac{d_2}{\delta}\frac{\partial}{\partial x}, D_{1.1} = D_{1.0} + D_{0.1}, \delta \equiv (c_1 d_2 - c_2 d_1)$, t, x two independent arguments of 3-wave problem, c, d independent parameters. In what follows sometimes we use notation $D_{1.0} f = f_1, D_{0.1} f = f_2$.

The discrete transformation of this system are the following ones [4], in what reader can verified by direct notcombersom calculation.

3.0.1 T_3

The system (3.1) is invariant with respect to the following transformation T_3

$$\begin{gathered} \tilde{f}^+_{1.1} = \frac{1}{f^-_{1.1}}, \quad \tilde{f}^+_{1.0} = -\frac{f^-_{0.1}}{f^-_{1.1}}, \quad \tilde{f}^+_{0.1} = \frac{f^-_{1.0}}{f^-_{1.1}} \\ \tilde{f}^-_{0.1} = -f^-_{1.1} D_{1.0} \frac{f^-_{0.1}}{f^-_{1.1}}, \quad \tilde{f}^-_{1.0} = f^-_{1.1} D_{0.1} \frac{f^-_{1.0}}{f^-_{1.1}} \\ \frac{\tilde{f}^-_{1.1}}{f^-_{1.1}} = f^+_{1.1} f^-_{1.1} - D_{1.0} D_{0.1} \ln f^-_{1.1} \end{gathered} \tag{3.2}$$

where $D_{i,j} = \frac{(ic_1+jc_2)}{\delta}\frac{\partial}{\partial t} + \frac{(id_1+jd_2)}{\delta}\frac{\partial}{\partial x}$.

3.0.2 T_2

The system (3.1) is invariant with respect to the following transformation T_2

$$\begin{gathered} \tilde{f}^+_{0.1} = \frac{1}{f^-_{0.1}}, \quad \tilde{f}^-_{1.0} = -\frac{f^-_{1.1}}{f^-_{0.1}}, \quad \tilde{f}^+_{12} = \frac{f^+_{1.0}}{f^-_{0.1}} \\ \tilde{f}^+_{1.0} = -f^-_{0.1} D_{1.1} \frac{f^+_{1.0}}{f^-_{0.1}}, \quad \tilde{f}^-_{1.1} = -f^-_{0.1} D_{1.0} \frac{f^-_{1.1}}{f^-_{0.1}} \\ \frac{\tilde{f}^-_{0.1}}{f^-_{0.1}} = f^+_{0.1} f^-_{0.1} + D_{1.0} D_{1.1} \ln f^-_{0.1} \end{gathered} \tag{3.3}$$

3.0.3 T_1

The system (3.1) is invariant with respect to the following transformation T_1

$$\tilde{f}^{+}_{1.0} = \frac{1}{f^{-}_{1.0}}, \quad \tilde{f}^{-}_{0.1} = \frac{f^{-}_{1.1}}{f^{-}_{1.0}}, \quad \tilde{f}^{+}_{1.1} = -\frac{f^{+}_{0.1}}{f^{-}_{1.0}}$$

$$\tilde{f}^{+}_{0.1} = f^{-}_{1.0} D_{1.1} \frac{f^{+}_{0.1}}{f^{-}_{1.0}}, \quad \tilde{f}^{-}_{1.0} = f^{-}_{1.0} D_{0.1} \frac{f^{-}_{1.1}}{f^{-}_{1.0}} \tag{3.4}$$

$$\frac{\tilde{f}^{-}_{1.0}}{f^{-}_{1.0}} = f^{+}_{1.0} f^{-}_{1.0} + D_{0.1}(D_{1.1} \ln f^{-}_{1.0})$$

3.0.4 General properties of discrete transformations

Three above transformations are invertable. This means f may be expressed algebraicaly in terms of $\tilde{f}$. Exept of this $T_3 = T_1T_2 = T_2T_1$, what means that all discrete transformation are mutually commutative. This in its turn means that arbitrary discrete transformation may be represented in a form $T = T_1^{n_1} T_2^{n_2}$ [?]. Thus from each given initial solution $W_0 \equiv (f^{\pm}_{1.0}, f^{\pm}_{1.0}, f^{\pm}_{1.1})$ of the system (3.1) it is possible to obtain a chain of solutions labelled by two natural numbers (n_1, n_2) the number of applications of the discrete transformations (T^1, T^2) to it.

The chain of equations which occur with respect to the functions $(f^{-}_{1.0}, f^{-}_{1.0}, f^{-}_{1.1})$ correspondingly in the case T_1, T_2, T_3 discrete transformation are definitely two-dimensional Toda lattices. Their general solutions in the case of two fixed ends are well-known [6]. As the reader will see soon this fact allows constructing the many soliton solutions of the 3-wave problem in the most straightforward way.

3.1 Resolving discrete transformation chains

3.1.1 Two identities of Jacobi

The first Jacobi identity

$$D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^1 & \tau_{11} \end{pmatrix} D_n \begin{pmatrix} T_{n-1} & a^2 \\ b^2 & \tau_{22} \end{pmatrix} - D_n \begin{pmatrix} T_{n-1} & a^2 \\ b^1 & \tau_{12} \end{pmatrix} D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^2 & \tau_{21} \end{pmatrix} =$$

$$D_{n-1}(T_{n-1}) D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ b^1 & \tau_{11} & \tau_{12} \\ b^2 & \tau_{21} & \tau_{22} \end{pmatrix}$$

where a^i, b^i are $(n-2)$ dimensional columns (rows) vectors, $\tau_{i,j}$ components of 2-th dimensional matrix.

The second Jacobi identity

$$D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^1 & \tau \end{pmatrix} D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ d^1 & \nu & \mu \\ b^2 & \rho & \tau \end{pmatrix} - D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^2 & \rho \end{pmatrix} D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ d^1 & \nu & \mu \\ b^1 & \tau & \sigma \end{pmatrix} =$$

$$D_n \begin{pmatrix} T_{n-1} & a^1 \\ d^1 & \nu \end{pmatrix} D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ b^2 & \rho & \tau \\ b^1 & \tau & \sigma \end{pmatrix}$$

These identities can be generalized in the case of arbitrary semi-simple group. The reader can find these results in [7].

3.1.2 Resolving the discrete lattice chains

Let us take an initial solution in the form

$$f^+_{1.0} = f^+_{1.0} = f^+_{1.1} = 0,$$

In this case solution of the system (3.1) has the form

$$\begin{aligned} f^-_{1.0} &= \int d\lambda p(\lambda) e^{\lambda(d_1 t - c_1 x)} \equiv \int d\lambda P(\lambda), \quad f^-_{0.1} = \int d\mu q(\mu) e^{\mu(d_2 t - c_2 x)} \\ &\equiv \int d\mu Q(\mu), \end{aligned} \tag{3.5}$$

$$f^-_{1.1} = \int d\lambda \int d\mu \frac{p(\lambda) q(\mu)}{\lambda - \mu} e^{\lambda(d_1 t - c_1 x) + \mu(d_2 t - c_2 x)} + \int d\nu r(\nu) e^{\nu((d_1 + d_2)t - (c_1 + c_2)x)}$$

Application to to this solution each of inverse transformations $(T^i)^{-1}$ is meaningless because of zeroes arising in denominators. The chain of equations under such boundary conditions we call the chain with the fixed end from the left (from one side).

The result of the application to such an initial solution by l_3 times T_3 transformation takes the following form (in order to check this fact only two Jacobi identities of the previous subsection are necessary)

$$(f^+_{1.1})^{(l_3)} = (-1)^{l_3 - 1} \frac{\Delta_{l_3 - 1}}{\Delta_{l_3}}, \quad (f^-_{1.1})^{(l_3)} = (-1)^{l_3} \frac{\Delta_{l_3 + 1}}{\Delta_{l_3}}, \quad \Delta_0 = 1$$

$$(f^+_{1.0})^{(l_3)} = (-1)^{l_3} \frac{\Delta_{l_3}^{(f^-_{1.0}}}{\Delta_{l_3}}, \quad (f^+_{0.1})^{(l_3)} = \frac{\Delta_{l_3}^{(f^-_{0.1})}}{\Delta_{l_3}}, \quad \Delta_0^{(f^-_{1.0}} = \Delta_0^{(f^-_{0.1}} = 0 \tag{3.6}$$

$$(f^-_{1.0})^{(l_3)} = \frac{\Delta_{l_3 + 1}^{(f^-_{1.0}}}{\Delta_{l_3}}, \quad (f^-_{0.1})^{(l_3)} = (-1)^{l_3} \frac{\Delta_{l_3 + 1}^{(f^-_{0.1}}}{\Delta_{l_3}}, \quad \Delta_{-1} = 0.$$

where Δ_n are minors of the n-th order of infinite dimensional matrix

$$\Delta = \begin{pmatrix} (f^-_{1.1}) & (f^-_{1.1})_2 & (f^-_{1.1})_{22} & ..... \\ (f^-_{1.1})_1 & (f^-_{1.1})_{12} & (f^-_{1.1})_{122} & ..... \\ (f^-_{1.1})_{11} & (f^-_{1.1})_{112} & (f^-_{1.1})_{1122} & ..... \\ ...... & .... & .... & ..... \end{pmatrix} \tag{3.7}$$

and $\Delta_{l_3}^{(f^-_{1.0}}, \Delta_{l_3}^{(f^-_{0.1}}$ are the minors of l_3 order in the matrices of which the last line (or column) is exchanged on the derivatives of the corresponding order on argument 2 of $f^-_{1.0}$ function (on argument 1 of the $f^-_{0.1}$ function in the second case.

In the next discussion the following notation will be used. W^{l_3, l_1} and W^{l_3, l_2}- denote the results of the application of discrete transformations $T^{l_3} T^{l_1}$ and $T^{l_3} T^{l_2}$ respectively to the corresponding component of the 3-wave field. Δ_{l_3,l_1} and Δ_{l_3,l_2} are determinants of orders $l_3 + l_1$ and $l_3 + l_2$, with the following structure of the determinant matrices. The first l_3 rows (columns) coincide with the matrix of (3.7) and last l_1, (l_2) rows (columns) are consructed from the derivatives of the functions $f^-_{0.1}$ and $f^-_{1.0}$ with respect to differetiation $D_{0.1}, D1.0$ respectivily.

The result of an additional application of the l_1 times T_1 transformation to the solution (3.6) is

$$(f^+_{1.0})^{l_3,l_1} = \frac{\Delta_{l_3,l_1-1}}{\Delta_{l_3,l_1}}, \quad (f^-_{1.0})^{l_3,l_1} = \frac{\Delta_{l_3,l_1+1}}{\Delta_{l_3,l_1}}, \quad \Delta_0 = 1, \quad \Delta^{l_3,-1} \equiv \Delta^{f^-_{0.1}}_{l_3}$$

$$(f^+_{1.1})^{l_3,l_1} = (-1)^{l_3+l_1-1}\frac{\Delta_{l_3-1,l_1}}{\Delta_{l_3,l_1}}, \quad (f^-_{1.1})^{l_3,l_1} = (-1)^{l_3+l_1}\frac{\Delta_{l_3+1,l_1}}{\Delta_{l_3,l_1}}, \tag{3.8}$$

$$((f^-_{1.0})^{l_3,l_1} = (-1)^{l_3+l_1}\frac{\Delta_{l_3+1,l_1-1}}{\Delta_{l_3,l_1}}, \quad (f^+_{0.1})^{l_3,l_1} = (-1)^{l_3+l_1}\frac{\Delta_{l_3-1,l_1+1}}{\Delta_{l_3,l_1}},$$

We do not present the explicit form for components $W^(l_3, l_2)$, which can be obtained without any difficulties from (3.8) by the exchange of the corresponding arguments and unknown functions.

4 Multi-soliton solution of the 3-wave problem

4.1 General consideration

The system (**??**) allows a reduction (under additional asumption that all operators of differentiation are real, i.e. $\partial_\alpha = \partial^*_\alpha$)

$$f^+_{1.0} = (f^-_{1.0})^*, \quad f^+_{0.1} = (f^-_{1.0})^*, \quad f^+_{1.1} = (f^-_{1.1})^* \tag{4.1}$$

In this case the system (3.1) is reduced to three equations

$$(f^-_{1.0})_1 = f^-_{1.1}(f^-_{0.1})^*, \quad (f^-_{0.1})_2 = f^-_{1.1}(f^-_{1.0})^*, \quad (f^-_{1.1})_3 = -f^-_{1.0}f^-_{0.1} \tag{4.2}$$

for three complex unknown functions. This system of equations is the standard 3-wave problem aplicated in many branches of mathematical physics (see introduction).

In [4] it was shown that begining with initial solution (3.5) after the $(l_3, l_1, (l_2))$ steps of the discrete transformation we arrive at a solution of the reduced system (4.2) if the following conditions are satisfied

$$\Delta_{2l_3+1,2l_1} = \Delta_{2l_3+1,2l_1-1} = \Delta_{2l_3,2l_1+1} = 0$$

$$\frac{\Delta_{2l_3-1,2l_1+1}}{\Delta_{2l_3,2l_1}} = (-1)^{l_3+l_1}E^* \quad \frac{\Delta_{2l_3-1,2l_1}}{\Delta_{2l_3,2l_1}} = (-1)^{l_1-1}D^* \quad \frac{\Delta_{2l_3,2l_1-1}}{\Delta_{2l_3,2l_1}} = (-1)^{l_3}B^* \tag{4.3}$$

Thus the problem is representation of the typical determinant Δ_{l_3,l_1} in more observable form.

4.2 Representation determinantes in the form of multidementional integrals

The first l_3 columnes of the determinant matrix $\Delta_(l_3, l_2)$ of the degree $l_3 + l_2 \times l_3 + l_2$ are constructed from the function $f^-_{1.1}$ and its derivatives with respect to differenciation on $D_{1.0}, D_{0.1}$ by the rules of (3.7). The function $f^-_{1.1}$ is parametrised by (3.5) in which we will consider at first (for simplicity) the case with $r(\nu) = 0$. How include in consideration general case with this function different from zero will be explained something later. From its definition (3.5) it follows rules of defferentiation

$$D_{0.1}f^-_{1.1} = \int d\lambda \int d\mu(-\lambda)\frac{p(\lambda)q(\mu)}{\lambda-\mu}e^{\lambda((d_1+d_2)t-(c_1+c_2)x)} \equiv \int d\lambda \int d\mu(-\lambda)\frac{P(\lambda)Q(\mu)}{\lambda-\mu}$$

$$D_{1.0}f^-_{1.1} = \int d\lambda \int d\mu\mu\frac{p(\lambda)q(\mu)}{\lambda-\mu}e^{\lambda((d_1+d_2)t-(c_1+c_2)x} \equiv \int d\lambda \int d\mu\mu\frac{P(\lambda)Q(\mu)}{\lambda-\mu}$$

in other words integrant function multipllcates on $-\lambda$ under $D_{0.1}$ differetiation and on μ under $D_{1.0}$ differetiation. By the same reasons integrant function of $f^-_{1.0}$ multiplicats on μ on the first differetiation and on zero the second one. Result of differentiation of $f^-_{0.1}$ is zero and $-\lambda$.

Let us parametrisied first column of matrix under consideration by parameters of integration λ_1, μ_1. Then integration it is possible take out of sign of determinant and we obtain deteminant with the first column constructed from consequent degree of the parameters μ_1 from zero up to $l_3 + l_2$. All terms contain the same factor $\frac{1}{\lambda_1-\mu_1}$. Let us represent second column in integral form by the parametres λ_2, μ_2. Each term of column contains the same degrees of μ_2 parameter and commomon factor $\frac{-\lambda_2}{\lambda_2-\mu_2}$. This procedure may continued up to l_3 column with the obvious result. Next columns are constructed with the help of derivation of $f^-_{0.1}$ with respect to $D_{1.0}$. The terms in the first after l_3 column are connected with integral representation for $f^-_{0.1}$ function with the parameter of intgration μ_{l_3+1} . Elemeents of this column are consequent degrees of μ_{l_3+1}. The next column exept of the common factor μ_{l_3+2} are consequent degrees of this parameter. Combining all results above together we obtain for deteminant under the sign of integration the following expression

$$(-1)^{l_3}W_{l_3+l_2}(\mu)\prod_{i=1}^{l_3}\frac{1}{\lambda_i-\mu_i}\prod_{i=1}^{l_3}\lambda_i^{i-1}\prod_{k=1}^{l_2}\mu_{l_3+k}^{k-1} \tag{4.4}$$

where $W_{l_3+l_2}(\mu_1, \mu_2....\mu_{l_3+l_1})$ Vandermond determinant constructed from all l_3+l_2 parameters μ.

The next step in calculation is connected with the fact that the domain of integration is simmetrical with respect to arbirary permutation the parameters λ and μ indepedently. Let us realise all $l_3!$ permutations of parameters λ_i simultaneosly with the same permutations of first l_3 parameters μ. Under such kind of transformation in (4.4) only terms from the product $\prod_{i=1}^{l_3}\lambda_i^{i-1}$ change. Keeping in mind properties of Vandermond determinant we conclude that these terms regrouping into Vandermond deteminant of l_3 order constructed from l_3 parameters λ $\frac{1}{l_3!}W_{l_3}(\lambda_{l_3}, ..\lambda_1)$.

Now we continue equality (4.4)

$$(-1)^{l_3}\frac{1}{l_3!}W_{l_3+l_2}(\mu)W_{l_3}(\lambda)\prod_{i=1}^{l_3}\frac{1}{\lambda_i-\mu_i}\prod_{k=1}^{l_2}\mu_{l_3+k}^{k-1} \tag{4.5}$$

The following comment allows realize further calculation. The symmetrzaition with respect to all permutations in the expresion $W_n(x_1, x_2, ..x_n)P_N(x)$ (P_N some polinomial function of x degree N) may be different from zero only in the case $\frac{n(n-1)}{2} \leq N$ and result is the following one $W_n^2(x_1, x_2, x_n)U_{N-\frac{n(n-1)}{2}}(x)$, where $U_{N-\frac{n(n-1)}{2}}(x)$ symmetrical function of the degree $N - \frac{n(n-1)}{2}$.

In (4.5) natural numeros l_3, l_2 connected with with n_1, n_2 (number of T_1, T_2 transformation) obviously as $n_2 = l_3 + l_2, n_1 = l_3$.

Now let us multiply numerator and denominator (4.5) on common factor in order to separate antisymmetrical function with respect all $l_3!$ permutation λ_i and $(l_3 + l_2)!$ parameters μ_a out of the sign of symmetrization . Result is the following one

$$(-1)^{l_3} \frac{1}{l_3!} \prod_{i=1}^{l_3} \prod_{a=1}^{l_2+l_3} \frac{1}{\lambda_i - \mu_a} W_{l_3+l_2}(\mu) W_{l_3}(\lambda) \times$$

$$\prod_{k=1}^{l_2} \mu_{l_3+k}^{k-1} \prod_{i=1}^{l_3} \prod_{a=1, a\neq i}^{l_2+l_3} (\lambda_i - \mu_a) \tag{4.6}$$

Terms in the first line of the last equality (4.6) are antisymmetrical with respect to all permutatuions of λ and μ indexes. In the second row the degree of polinomial equal $\frac{l_2(l_2-1)}{2} + l_3(l_3 + l_2 - 1) \equiv \frac{l_3(l_3-1)}{2} + \frac{(l_3+l_2)(l_3+l_2-1)}{2}$. The last expression is exactly summa demsiones of Vandermond determinantes constructed from parameters λ and μ. As it was noted above (comments after (4.5)) in the process of antisymmetrsation different from zero may give only terms with degree no less then $\frac{(l_3+l_2)(l_3+l_2-1)}{2}$ in calculation with respect to μ parameters and no less then $\frac{l_3(l_3-1)}{2}$ with respect to λ parameters. The form of these terms are also known. They are (elements of Vandermond determinant)

$$\prod_{a=1}^{l_2+l_3} \mu_a^{a-1} \prod_{i=1}^{l_3} \lambda_i^{i-1}$$

or which may be obtained from them by some permutation of paramters. This is pure combinatorical problem. Let us rewrite product in(**??**) in equivalent form

$$\prod_{k=1}^{l_2} \mu_{l_3+k}^{k-1} \prod_{i=1}^{l_3} \prod_{k=1, k\neq i}^{l_3} (\lambda_i - \mu_k) \prod_{j=1}^{l_2} (\lambda_i - \mu_{l_3+j}) \tag{4.7}$$

The terms containing parameters μ with indexes more then l_3 are only in the last product and to have necessary (Vandermond) degree together (with the first product) from these multiplicators it is necessary to conserve only products of μ terms with the result

$$(-1)^{l_3 l_2} \prod_{k=1}^{l_2} \mu_{l_3+k}^{l_3+k-1} \prod_{i=1}^{l_3} \prod_{k=1, k\neq i}^{l_3} (\lambda_i - \mu_k) \tag{4.8}$$

The structure of the terms of last product which give different from zero input is explained above. Let consider procedure of obtaing typical necessary term $\lambda_1^{l_3-1} \lambda_2^{l_3-2} .. \lambda_{l_3-1}$. From multiplicatores with $i = 1$ let us conserve only λ_1 terms The number of multiplcators exactly $l_3 - 1$. From muliplicatotors with $i = 2$ we conserve λ_2 in first $l_3 - 2$ factors and $-\mu_{l_3}$ in the last one. Among multiplicators with $i = 3$ we conserve λ_3 in first $l_3 - 3$ factors and $\mu_{l_3-1}\mu_{l_3}$ in two last ones. Continuing this procedur as result we obtain the term of the form $(-1)^{l_3-1}\lambda_1^{l_3-1}\lambda_2^{l_3-2}..\lambda_{l_3-1}\mu_{l_3}^{l_3-1}\mu_{l_3-1}^{l_3-2}..\mu_2$. All other terms obtained from the above by symmetrization of this expession with respect to all $l_3!$ permutations of λ indexes simultaneously with the same one of μ indexes. Now we remind the reader that up to now we have transformed only the second line of the (4.5). In connection of the first line the result must be antisymmetrised on λ and μ indexes independently. Antisymmetrization on μ with indexes $1 \leq i \leq l_3$ leads to $(-1)^{l_3} W_{l_3}(\lambda_1, ...\lambda_{l_3}) \prod_{a=1}^{a=l_1+l_3} \mu_a^{a-1}$. Further antisymmetrization with respect $(l_1 + l_3)$ indexes μ leads to finally result for integrant function of Δ_{l_3,l_1}

$$\frac{1}{l_3!}\frac{1}{(l_3+l_2)!}\frac{W_{l_3}^2(\lambda)W_{l_3+l_2}^2(\mu)}{\prod(\lambda_i-\mu_a)} \equiv \frac{1}{n_1!}\frac{1}{(n_2)!}\frac{W_{n_1}^2(\lambda)W_{n_2}^2(\mu)}{\prod(\lambda_i-\mu_a)} \tag{4.9}$$

4.3 Explicite form of solution after n_1 times transformation T_1 and n_2 times transformation T_2

Let us introduce in consideration the basis function $V(n_1, n_2)$

$$V(n_2, n_1) = \frac{1}{n_1!n_2!}\int \prod_{i=1}^{n_1} P(\lambda_i)d\lambda_i \prod_{k=1}^{n_2} Q(\mu_k)d\mu_k \frac{W_{n_1}^2(\lambda)W_{n_2}^2(\mu)}{\prod_{i,k}(\lambda_i-\mu_k)} \tag{4.10}$$

From the results of the previous subsection ($T_3 = T_1T_2!$)it follows that solution of the 3-wave problem looks as

$$f_{1.0}^+ = \frac{V(n_1-1,n_2)}{V(n_1,n_2)} \quad ,f_{0.1}^+ = \frac{V(n_1,n_2-1)}{V(n_1,n_2)} \quad ,f_{1.1}^+ = \frac{V(n_1-1,n_2-1)}{V(n_1,n_2)}, \tag{4.11}$$

$$f_{1.0}^- = \frac{V(n_1+1,n_2)}{V(n_1,n_2)} \quad ,f_{0.1}^- = \frac{V(n_1,n_2+1)}{V(n_1,n_2)} \quad ,f_{1.1}^- = \frac{V(n_1+1,n_2+1)}{V(n_1,n_2)}$$

4.4 Conditions of interupting of the chain equations

Let us consider the initial function $f_{1.0}^-$ in a form

$$f_{1.0}^- = \sum_{i=1}^{n_1} p_i e^{L_i(d_1t-c_1x)} = \int d\lambda \sum_{L=1}^{n_1} \delta(\lambda - L_i)p(\lambda)e^{\lambda(d_1t-c_1x)}$$

$$f_{0.1}^- = \sum_{k=1}^{n_2} q_k e^{M_k(d_2t-c_2x)} = \int d\mu \sum_{M=1}^{n_2} \delta(\mu - M_k)q(\mu)e^{\mu(d_2t-c_2x)}$$

where $\delta(x)$ ussual Dirac function. Under such choise of innitial conditions for $V(n_1, n_2)$ from (4.10) we obtain

$$V(n_1, n_2) = \prod_{i=1}^{n_1} p_i e^{L_i(d_1t-c_1x)} \prod_{k=1}^{n_2} q_k e^{M_k(d_2t-c_2x)} \frac{W_{n_1}^2(L)W_{n_2}^2(M)}{\prod_{i,k}(L_i-M_k)} \tag{4.12}$$

Also obviously, that

$$V(n_1+1,n_2) = V(n_1,n_2+1) = V(n_1+1,n_2+1) = 0$$

because in this case in arguments of Vandermond determinant arised two equal parameters L or M.

In connection with results of the previouse subsection (4.11)

$$f^-_{1.0} = f^-_{0.1} = f^-_{1.1} = 0$$

and to this solution it is not possible applicate no one of discrete transformation T_i. Thus in the case under consideration it is possible to say about the chain restrickted from both ends. On "left" end the inverse discrete transformation are mingless; on "right" one direct descrete transfomations are mingless. This is the caine closed from both sides.

4.5 Conditions of reality

At first we repeat here some result from [4]. Let we have some hermitian solution $f = f^H$ and let us applicate to it direct and inverse one of transformation T_i from section 2. Of course condition of hermition will be forbidden in both cases but (up to sign factors see [4]) solutions after direct f_d and inverse transformation f_i are connected by the condition $f_d = f_i^H$. After m-steps of such procedure we have $f_{d;m} = f^H_{i;m}$. After this comment consideration below is obvious.

As it was shown in [4] solution will be hermitian on n_1, n_2 step of discrete transformation solution after $2n_1, 2n_2$ steps of discrete transformation connected with the initial solution by conditions of "reality" (up to signs)

$$(f^-_{1.0})_{0.0} = (\bar{f}^+_{1.0})_{2n_1,2n_2}, \quad (f^-_{0.1})_{0.0} = (\bar{f}^+_{0.1})_{2n_1,2n_2}, \quad (f^-_{1.1})_{0.0} = (\bar{f}^+_{1.1})_{2n_1,2n_2} \tag{4.13}$$

where $\bar{f} = f^*$. For finding the positive components of the final solution is necessary in connection with (4.11) calculate $V(n_1-1,n_2), V(n_1,n_2-1), V(n_1-1,n_2-1)$ functions. We have

$$V(n_1,n_2-1) = \sum_{i=1}^{n_1} W^2_{n_1-1}(L_1..L_{i-1}L_{i+1}...L_{n_1}) \prod_{j=1,j\neq i}^{n_1} p_j e^{L_j(d_1t-c_1x)}$$

$$\prod_{k=1}^{n_2} q_k e^{M_k(d_2t-c_2x)} \frac{W^2_{n_2}(M)}{\prod_{j,j\neq i,k}(L_j - M_k)}$$

After deviding the last expression on $V(n_1,n_2)$ from (4.12) we obtain in connection of (4.11)

$$(f^+_{1.0})_{n_1,n_2} = \sum_{i=1}^{n_1} \frac{1}{p_i} e^{-L_i(d_1t-c_1x)} \prod_{j=1,j\neq i}^{n_1} \frac{1}{(L_i-L_j)^2} \prod_{k=1}^{n_2}(L_i - M_k) \tag{4.14}$$

Absolutely by the same way we obtain

$$(f^+_{0.1})_{n_1,n_2} = \sum_{k=1}^{n_2} \frac{1}{q_k} e^{-M_i(d_2t-c_2x)} \prod_{j=1,j\neq k}^{n_2} \frac{1}{(M_k-M_j)^2} \prod_{i=1}^{n_1}(L_i - M_k) \tag{4.15}$$

The functions $f^+_{1.0}, f^+_{0.1}$ repite the structure of the initial functions $f^-_{1.0}, f^+_{0.1}$ (3.5) and thus conditions of reality (4.1) after comparising (4.14),(4.15) with (3.5) means that $2n_1$ parameteres L_i are connected by the conditions $L^*_i = -(PL)_i$ and the same for $2n_2$ parameters M_a $M^*_a = -(QM)_a$, where P, Q elements from the group of permutations of $2n_1, 2n_2$ indexes such as $P^2 = 1, Q^2 = 1$ Of course all this true if all parameters of theory c, d equaly as arguments of the problem t, x are real numbers. Parametres p_i, q_a are conected by the obvious relations typical for multi-soliton solutions in one component theoris. The arising products $\prod_{k=1}^{n_2}(L_i - M_k), \prod_{i=1}^{n_1}(L_i - M_k)$ may be considered as results of interuction.

4.6 Comments about calculation of in the general case $r(\nu) \neq 0$

All calculations in the general case it is possible to do using the following trik. We remind reader one formula from the theory of disributions (generalized functions)

$$\frac{1}{x - y + \epsilon z} = \frac{1}{x - y} + \frac{i\pi z}{2}\delta(x - y)$$

in the zero limit of ϵ.

On anlogy we introduce by definition the function

$$\frac{1}{\lambda - \mu + \epsilon r(\lambda\mu)} \equiv \frac{1}{\lambda - \mu} + \frac{r(\lambda, \mu)}{p(\lambda)q(\mu)}\delta(\lambda - \mu) \tag{4.16}$$

with the obvious propertie

$$\frac{1}{\lambda - \mu + \epsilon r(\lambda\mu)}(\lambda - \mu) = 1$$

as a consequence of the equality $x\delta(x) = 0$

With the help of definition (4.16) the initial function $f^-_{1.1}$ may be represented in a form

$$f^-_{1.1} = \int d\lambda \int d\mu \frac{p(\lambda)q(\mu)}{\lambda - \mu + \epsilon\frac{r(\lambda,\mu)}{p(\lambda)q(\mu)}} e^{\lambda(d_1 t - c_1)x + \mu(d_2 t - c_2)}$$

Indeed using the formulae above we come to two correct terms in (3.5).

Now it is possible to repeate the process of calculations $V(n_1, n_2)$ function absolutely by the same steps as it was done in corresponding subsection above. Result is the following

$$V(n_1, n_2) = \frac{1}{n_1!}\frac{1}{(n_2)!}\int \prod_{i=1}^{n_1} P(\lambda_i)d\lambda_i \prod_{k=1}^{n_2} Q(\mu_k)d\mu_k \frac{W^2_{n_1}(\lambda)W^2_{n_2}(\mu)}{\prod(\lambda_i - \mu_k + \epsilon r(\lambda_i, \mu_k))}$$

The last expression allow further transformation, which include in $V(n_1, n_2)$ second part of initial function $f^-_{1.1}$ in explicit form (we assume that $n_2 \neq n_1$:

$$V(n_1, n_2) = \frac{1}{n_1!}\frac{1}{(n_2)!}\sum_{k=0}^{n_2}\int \prod_{i=1}^{n_1-k} P(\lambda_i)d\lambda_i \prod_{a=1}^{n_2-k} Q(\mu_a)d\mu_a \prod_{\theta=1}^{k} R(\nu_\theta)d\mu_\theta \times$$

$$\frac{W^2_{n_1-k}(\lambda)W^2_{n_2-k}(\mu)}{\prod(\lambda_j - \mu_b)} \prod(\nu_\alpha - \mu_c)\prod(\lambda_s - \nu_\beta)W^2_k(\nu)$$

where $R(\nu_\theta) = r(\nu_\theta)e^{\nu_\theta((d_1+d_2)t - (c_1+c_2)x)}$.

At this moment we have no idea what are conditions of interrupting of these chain and by this reason we can't say how extract soliton solutions from the general case of the initial conditions. From the mathematical point of view the fact that it is possible to find solution in explicite form is very intrigued.

5 Outlook

None of the authors is a specialist in the field of applications of 3-wave problem to certain physical phenomenon. For this reason we cannot discuss the usefullness of such applications.

From a computational point of view seems that here is the first paper having systematical investigation of a multicomponent integrable system. It is interesting to compare the results in the case of a multicomponent (the simplest one) integrable system connected with the algebra A_2 and numerous integrable systems connected with A_1 algebra. The strategy of computations in both cases are the same. Beginning with some simple solution connected with the upper triangular nilpotent subalgebra (($P = A = Q = 0$) in the case of the present paper; after taking the corresponding $2n$ steps of the discrete transformation we reach a solution connected with the lower triangular nilpotent subalgebra. Connection with Hermiticity of these solutions leads to a limitation of numerical parameters of the problem which were arbitrary up to now. In the case of a two component integrable system for instance in nonlinear Schrödinger equation this dependence looks as [8]

$$c_s^* = \prod_{k=1,k\neq s}^{2l_3} (\lambda_k - \lambda_s)^{-2} \frac{1}{c_{(Ps)}}, \quad \lambda_s^* = -\lambda_{Ps}$$

In the case under consideration this dependence is modified by the formulas of subsections "conditionofreality". This comparison gives the possibility of assuming that each simple root of semisimple algebra spawns its own systems of pairs of parameters; amplitude-phase and the numbers of these pairs are arbitrary. Only by then multi-soliton solutions are defined in the case of multicomponent integrable systems.

In our considerations we were unable to the solutions constructed with the Lax pair formalism have no idea, how this might be done, if indeed it is possible. From the result we see that we have deel with two spectral parameters λ, μ. Thus it is possible assume existence of two possibilities: or it is necessary to generalise $L - A$ formalism on the case of two independent spectral parameters or in some way have consequent calculations with two independent $L-A$ paires. Up to now we where not able realize anyone of these (known for us) possibilities.

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References

[1] Berger K and Milevski U, *SIAM J. Appl. Math.* **63** (2003), 1121–1140.

[2] Katou K, *J. Phys. Soc. Japan* **51** (1982), 996–1000.

[3] Konotop V V, Cunha M D, Christiansen P L and Clausen C B, *Phys Rev E* **160** (1999), 6104–6110.

[4] Leznov A N and Torres-Cordoba R, *J. Math. Phys.* **44** (2003), 2342–2352; Leznov A N, Escobedo-Alatrore J and Torres-Cordoba R, *J. Nonlinear Math. Phys.* **10** (2003), 243–251.

[5] Leznov A N, Discrete symmetries of the n-wave problem, *Theor. Math. Phys.* **132** (2002), 955–969.

[6] Leznov A N and Saveliev M V, Group theretical methods for integration of nonlinear dynamic systems, Birchoiser, 1992.

[7] Leznov A N, *Theor. Math. Phys.* **122** (1998), 211–228.

[8] Leznov A N, Shabat A B and Yamilov R I, *Phys. Lett. A* **199** (1993).

[9] Mahrenholtz O and Markiewicz M, Advances in Fluid Mechanics **24**, 1999, 272.

[10] Sukhorukov A P, Nonlinear interactions in optics and radiophysics, Nauka, Moskow, 1988 [Russian].

[11] Tkeshelashvili L and Busch K, *Appl. Phys. B* **81** (2005), 225–229.

[12] Vlahos L, Sharma R R and Papadopoulos K, *Astrophys. J.* **275** (1983), 374–390.

[13] Weiland J and Wilhelmsson H, Coherent non-linear interaction of waves in plasmas Oxford and New York, Pergamon Press, 1977, 366.