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A supersymmetric second modified KdV equation

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Abstract

In this paper, based on the Bäcklund transformation for the supersymmetric MKdV equation, we propose a supersymmetric analogy for the second modified KdV equation. We also calculate its one-, two- and three-soliton solutions.

1 Introduction

Bäcklund transformation (BT), originated from the classical differential geometry, has been one of most important ingredients in modern theory of integrable systems. On the one hand, one may use a BT to construct solutions such as multi-soliton solutions for a given system. On the other hand, as advocated by Hirota, BT may provide an effective way to supply nonlinear differential equations which are integrable. In the framework of Hirota's direct method, many systems have been interpreted as BT for the known systems and some new integrable systems have been found in this way. The first example is the modified Korteweg-de Vries (MKdV) equation, which can be rediscovered from the bilinear BT for the celebrated Korteweg-de Vries (KdV) equation. Later Nakamura and Hirota [12] found so called the second modified KdV equation. This idea has been further extended by Nakamura [11] and a third modified KdV equation was obtained in particular. The second modified KdV equation reads as

$$u_t + u_{xxx} - \frac{1}{2}u_x^3 + [\lambda + \frac{3}{2}(\mu + \mu')^2]u_x - \frac{3}{2}u_x(\mu e^{-u} + \mu' e^{u})^2 = 0. \tag{1.1}$$

It is interesting to note that the equation (1.1) contains an equation known as Calogero-Degaspris-Fokas equation as a particular example [1, 3].

Soliton equations have supersymmetric counterparts, which are relevant both physically and mathematically. We refer to [9] for physical motivations of studying supersymmetric soliton systems. From mathematical viewpoint, constructing a supersymmetric analogy for certain soliton equation is a nontrivial task. Since the seminal paper of Manin and Radul [7] supersymmetric integrable systems have been studied extensively. One of the most important supersymmetric integrable equations is the $N = 1$ supersymmetric KdV (sKdV) equation, for which many interesting results have been established [8, 13, 4]. In [10], this system was converted into a bilinear form and a general formula was conjectured for the whole hierarchy. Recently, Hirota's direct method is used to construct soliton solutions for the sKdV equation [2].

Very recently, the sKdV equation was reexamined within the framework of Hirota's direct method [5]. It was shown that, as in the KdV case, the supersymmetric MKdV (sMKdV) equation can be recovered from the bilinear BT for the sKdV equation. This allows us to obtain the proper bilinear form for the sMKdV equation and its bilinear BT and soliton solutions [6].

The purpose of this paper is to construct a candidate for the supersymmetric second modified KdV equation(ssMKdV). We will show that the bilinear BT of the sMKdV equation, proposed in [6], can be transformed into a supersymmetric second modified KdV equation via certain dependent variable transformation.

This paper is organized as follows: in the next section, we will derive the ssMKdV equation from the BT of the sMKdV equation. In section 3, using Hirota's bilinear method, we will calculate one-soliton, two-soliton and three-soliton solutions of the ssMKdV equation. Last section summarizes the results briefly.

2 Supersymmetric second mKdV equation

The supersymmetric modified KdV equation was proposed in [8, 15]. It reads as

$$\Psi_t + \mathcal{D}^6\Psi - 3(\mathcal{D}\Psi)\mathcal{D}^2(\Psi\mathcal{D}\Psi) = 0, \tag{2.1}$$

where $\Psi = \Psi(x, t, \theta)$ is a Grassmann odd variable depending on usual (even) spatial variables x, super (odd) spatial variables θ and usual (even) temporal variable t. $\mathcal{D}$ is the super derivative defined by $\mathcal{D} = \mathcal{D}_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$.

By means of the dependent variable transformation

$$\Psi = \mathcal{D}\ln\left(\frac{g}{f}\right),$$

the system (2.1) is bilinearized as [6]

$$(D_t + D_x^3)(g\cdot f) = 0, \tag{2.2a}$$
$$SD_x(g\cdot f) = 0, \tag{2.2b}$$

where f, g are Grassmann even functions and the Hirota derivative is defined as

$$SD_t^m D_x^n f\cdot g = (\mathcal{D}_{\theta_1} - \mathcal{D}_{\theta_2})\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)^m \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n f(x_1, t_1, \theta_1)g(x_2, t_2, \theta_2)\Big|_{\substack{x_1=x_2\\ t_1=t_2\\ \theta_1=\theta_2}}$$

this bilinerization proves to be ideal to construct soliton solutions for the sMKdV equation (2.1).

In [6], we further proved that the following system

$$D_x f \cdot g' - D_x g \cdot f' = \mu f g' - \mu f' g, \tag{2.3}$$

$$S f \cdot g' + S g \cdot f' = 0, \tag{2.4}$$

$$(D_t + D_x^3 - 3\mu D_x^2 + 3\mu^2 D_x) f \cdot f' = 0, \tag{2.5}$$

$$(D_t + D_x^3 - 3\mu D_x^2 + 3\mu^2 D_x) g \cdot g' = 0, \tag{2.6}$$

supplies us a BT for the sMKdV system (2.1). Where (f, g) and (f', g') are two solutions of the system (2.2a) and (2.2b), μ is a (Grassmann even) constant.

We first note that the following equations are the consequence of eqs. (2.3) and (2.4)

$$S D_x f \cdot f' = \mu S f \cdot f', \tag{2.7}$$

$$S D_x g \cdot g' = \mu S g \cdot g', \tag{2.8}$$

a detailed proof for this fact can be found in the Appendix A.

Next we consider the bilinear BT (2.3)-(2.8) of the sMKdV equation as a new system. To get the nonlinear evolution version of this new system, we define new variables $\varphi, \varphi', \rho, \rho'$ by

$$\varphi = \ln\left(\frac{f}{f'}\right), \quad \varphi' = \ln\left(\frac{g}{g'}\right), \quad \rho = \ln(f f'), \quad \rho' = \ln(g g'),$$

then our bilinear BT (2.3)-(2.8) can be reformulated as

$$(\varphi + \varphi' + \rho - \rho')_x - e^{\varphi' - \varphi}(\varphi + \varphi' + \rho' - \rho)_x - 2\mu + 2\mu e^{\varphi' - \varphi} = 0, \tag{2.9}$$

$$\mathcal{D}(\varphi + \varphi' + \rho - \rho') + e^{\varphi' - \varphi}\mathcal{D}(\varphi + \varphi' + \rho' - \rho) = 0, \tag{2.10}$$

$$\varphi_t + \varphi_{xxx} + 3\rho_{xx}\varphi_x + \varphi_x^3 - 3\mu(\rho_{xx} + \varphi_x^2) + 3\mu^2\varphi_x = 0, \tag{2.11}$$

$$\varphi'_t + \varphi'_{xxx} + 3\rho'_{xx}\varphi'_x + {\varphi'_x}^3 - 3\mu(\rho'_{xx} + {\varphi'_x}^2) + 3\mu^2\varphi'_x = 0, \tag{2.12}$$

$$\varphi_x \mathcal{D}\varphi + \mathcal{D}\rho_x - \mu\mathcal{D}\varphi = 0, \tag{2.13}$$

$$\varphi'_x \mathcal{D}\varphi' + \mathcal{D}\rho'_x - \mu\mathcal{D}\varphi' = 0. \tag{2.14}$$

By differentiating eqs (2.13) and (2.14), we obtain

$$\rho_{xx} = \mu\varphi_x - \varphi_x^2 - (\mathcal{D}\varphi_x)\mathcal{D}\varphi, \tag{2.15}$$

$$\rho'_{xx} = \mu\varphi'_x - {\varphi'_x}^2 - (\mathcal{D}\varphi'_x)\mathcal{D}\varphi', \tag{2.16}$$

substituting eqs. (2.15-2.16) into eqs. (2.11-2.12), we eliminate ρ_{xx}, ρ'_{xx} and find

$$\begin{aligned}(\varphi - \varphi')_t + (\varphi - \varphi')_{xxx} + 3\mu(\varphi_x^2 - {\varphi'_x}^2) - \frac{3}{2}(\varphi - \varphi')_x(\varphi + \varphi')_x^2 - \frac{1}{2}(\varphi - \varphi')_x^3 \\ + 3\mu\big((\mathcal{D}\varphi_x)\mathcal{D}\varphi - (\mathcal{D}{\varphi'}_x)\mathcal{D}\varphi'\big) - 3\big(\varphi_x(\mathcal{D}\varphi_x)\mathcal{D}\varphi - {\varphi'}_x(\mathcal{D}{\varphi'}_x)\mathcal{D}\varphi'\big) = 0.\end{aligned} \tag{2.17}$$

Now we introduce a new variable $\Phi = \varphi - \varphi'$. Then eqs. (2.9) and (2.10) yield

$$(\rho - \rho')_x = \frac{1 - e^{-\Phi}}{1 + e^{-\Phi}}[2\mu - (\varphi + \varphi')_x], \tag{2.18}$$

$$(\rho - \rho')_x = \frac{2e^{-\Phi}}{(1 - e^{-\Phi})^2}(\mathcal{D}\Phi)\mathcal{D}(\varphi + \varphi') - \frac{1 + e^{-\Phi}}{1 - e^{-\Phi}}(\varphi + \varphi')_x. \tag{2.19}$$

Solving eqs. (2.18) and (2.19) with respect to $\varphi+\varphi'$, we have

$$\mathcal{D}(\varphi+\varphi')=-\frac{\mu}{2}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}})\mathcal{D}^{-1}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}}). \tag{2.20}$$

Differentiating above equation leads to

$$(\varphi+\varphi')_x=-\frac{\mu}{2}(e^{\Phi}+e^{-\Phi}-2)-\frac{\mu}{4}e^{\frac{\Phi}{2}}(1+e^{-\Phi})(\mathcal{D}\Phi)\mathcal{D}^{-1}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}}), \tag{2.21}$$

$$(\varphi+\varphi')_x^2=\frac{\mu^2}{4}(e^{\Phi}+e^{-\Phi}-2)^2+\frac{\mu^2}{4}(e^{\Phi}-e^{-\Phi})(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}})(\mathcal{D}\Phi)\mathcal{D}^{-1}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}}). \tag{2.22}$$

Now we can reformulate the equation (2.17) as a single equation for $\Phi=\varphi-\varphi'$. To this end, we note that the last two terms of the left hand side of the equation (2.17) can be rewritten as

$$\begin{aligned}(\mathcal{D}\varphi_x)\mathcal{D}\varphi-(\mathcal{D}{\varphi'}_x)\mathcal{D}\varphi' &\overset{(B.4)}{=} \frac{1}{2}[(\mathcal{D}(\varphi+\varphi')_x)\mathcal{D}\Phi+(\mathcal{D}\Phi_x)\mathcal{D}(\varphi+\varphi')]\\ &=\frac{\mu}{8}\left[(e^{\frac{\Phi}{2}}+e^{-\frac{\Phi}{2}})\Phi_x\mathcal{D}\Phi-2(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}})\mathcal{D}\Phi_x\right]\mathcal{D}^{-1}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}}),\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\varphi_x(\mathcal{D}\varphi_x)\mathcal{D}\varphi-\varphi'_x(\mathcal{D}\varphi'_x)\mathcal{D}\varphi' &\overset{(B.5)}{=} \frac{1}{4}(\varphi+\varphi')_x\big(\mathcal{D}(\varphi+\varphi')_x\big)\mathcal{D}\Phi+\frac{1}{4}\Phi_x(\mathcal{D}\Phi_x)\mathcal{D}\Phi\\ &+\frac{1}{4}(\varphi+\varphi')_x(\mathcal{D}\Phi_x)\mathcal{D}(\varphi+\varphi')+\frac{1}{4}\Phi_x\big(\mathcal{D}(\varphi+\varphi')_x\big)\mathcal{D}(\varphi+\varphi')\\ &=\frac{\mu^2}{16}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}})^3(\mathcal{D}\Phi_x)\mathcal{D}^{-1}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}})+\frac{1}{4}\Phi_x(\mathcal{D}\Phi_x)\mathcal{D}\Phi.\end{aligned} \tag{2.24}$$

Finally substituting the above expressions (2.21)-(2.24) into equation (2.17), and using new variable $\Phi=\varphi-\varphi'$, we have

$$\begin{aligned}\Phi_t+\Phi_{xxx}-\frac{1}{2}\Phi_x^3-\frac{3}{8}\mu^2\Phi_x(e^{\Phi}-e^{-\Phi})^2-\frac{3}{4}\Phi_x(\mathcal{D}\Phi_x)\mathcal{D}\Phi&\\ -\frac{3}{8}\mu^2(e^{\frac{3\Phi}{2}}+e^{-\frac{3\Phi}{2}})\Phi_x(\mathcal{D}\Phi)\mathcal{D}^{-1}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}})&\\ -\frac{3}{16}\mu^2(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}}+e^{\frac{3\Phi}{2}}-e^{-\frac{3\Phi}{2}})(\mathcal{D}\Phi_x)\mathcal{D}^{-1}(e^{\frac{\Phi}{2}}-e^{-\frac{\Phi}{2}})&=0.\end{aligned} \tag{2.25}$$

which is our supersymmetric second modified KdV equation. To see the relationship with the known second modified KdV system, we rewrite it in terms of components. By assuming $\Phi=u+\theta\xi,\mathcal{D}^{-1}(e^{\Phi/2}-e^{-\Phi/2})=a+\theta b$, where ξ,a are Grassmann odd variables, equation (2.25) in components reads as

$$u_t + u_{xxx} - \frac{1}{2}u_x^3 - \frac{3}{8}\mu^2 u_x(e^u - e^{-u})^2 + \frac{3}{4}u_x\xi\xi_x$$
$$- \frac{3}{8}\mu^2 u_x\xi a(e^{\frac{3u}{2}} + e^{-\frac{3u}{2}}) - \frac{3}{16}\mu^2\xi_x a(e^{\frac{u}{2}} - e^{-\frac{u}{2}} + e^{\frac{3u}{2}} - e^{-\frac{3u}{2}}) = 0, \tag{2.26a}$$

$$\xi_t + \xi_{xxx} - \frac{3}{4}u_x^2\xi_x - \frac{3}{4}\mu^2 u_x(e^{2u} - e^{-2u})\xi - \frac{3}{8}\mu^2(e^u - e^{-u})^2\xi_x - \frac{3}{4}u_x u_{xx}\xi$$
$$- \frac{3}{8}\mu^2 u_x(e^{\frac{3u}{2}} + e^{-\frac{3u}{2}})(u_x a - b\xi) - \frac{3}{16}\mu^2(u_{xx}a - \xi_x b)(e^{\frac{u}{2}} - e^{-\frac{u}{2}} + e^{\frac{3u}{2}} - e^{-\frac{3u}{2}})$$
$$- \frac{3}{32}\mu^2\xi\xi_x a(e^{\frac{u}{2}} + e^{-\frac{u}{2}} - e^{\frac{3u}{2}} - e^{-\frac{3u}{2}}) = 0. \tag{2.26b}$$

It is easy to see that when the fermionic fields vanish, equation (2.26b) holds identically and equation (2.26a) reduces to

$$u_t + u_{xxx} - \frac{1}{2}u_x^3 - \frac{3}{8}\mu^2 u_x(e^u - e^{-u})^2 = 0. \tag{2.27}$$

which may be compared with the equation (1.1).

The above process may be reversed and we conclude that by the dependent variable transformation

$$\Phi = \varphi - \varphi' = \ln\left(\frac{fg'}{f'g}\right).$$

equation (2.25) is reduced to its bilinear form eqs. (2.3-2.6).

3 Soliton solutions

It is well known that Hirota's bilinear method is a very effective approach to construct soliton solutions. In this section, we calculate soliton solutions for the ssMKdV equation.

The calculation involved is standard, so we will omit most detail and present results.

One-soliton:

$$\begin{aligned} f &= 1 + e^{\eta+\theta\xi}, \\ g &= 1 - e^{\eta+\theta\xi}, \\ f' &= 1 + \frac{\mu - k}{\mu + k}e^{\eta+\theta\xi}, \\ g' &= 1 - \frac{\mu - k}{\mu + k}e^{\eta+\theta\xi}, \end{aligned}$$

where $\eta = kx - k^3t + c_0$.

Two-soliton:

$$\begin{aligned} f &= 1 + e^{\eta_1+\theta\xi_1} + e^{\eta_2+\theta\xi_2} + A_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)}, \\ g &= 1 - e^{\eta_1+\theta\xi_1} - e^{\eta_2+\theta\xi_2} + A_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)}, \\ f' &= 1 + \frac{\mu - k_1}{\mu + k_1}e^{\eta_1+\theta\xi_1} + \frac{\mu - k_2}{\mu + k_2}e^{\eta_2+\theta\xi_2} + B_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)}, \\ g' &= 1 - \frac{\mu - k_1}{\mu + k_1}e^{\eta_1+\theta\xi_1} - \frac{\mu - k_2}{\mu + k_2}e^{\eta_2+\theta\xi_2} + B_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)}. \end{aligned}$$

Three-soliton:

$$
\begin{aligned}
f &= 1 + e^{\eta_1+\theta\xi_1} + e^{\eta_2+\theta\xi_2} + e^{\eta_3+\theta\xi_3} \\
&\quad + A_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)} + A_{13}e^{\eta_1+\eta_3+\theta(\xi_1+\xi_3)} + A_{23}e^{\eta_2+\eta_3+\theta(\xi_2+\xi_3)} \\
&\quad + (\alpha_{13}\alpha_{23}A_{12} + \alpha_{12}\alpha_{32}A_{13} + \alpha_{12}\alpha_{13}A_{23})e^{\eta_1+\eta_2+\eta_3+\theta(\xi_1+\xi_2+\xi_3)}, \\
g &= 1 - e^{\eta_1+\theta\xi_1} - e^{\eta_2+\theta\xi_2} - e^{\eta_3+\theta\xi_3} \\
&\quad + A_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)} + A_{13}e^{\eta_1+\eta_3+\theta(\xi_1+\xi_3)} + A_{23}e^{\eta_2+\eta_3+\theta(\xi_2+\xi_3)} \\
&\quad - (\alpha_{13}\alpha_{23}A_{12} + \alpha_{12}\alpha_{32}A_{13} + \alpha_{12}\alpha_{13}A_{23})e^{\eta_1+\eta_2+\eta_3+\theta(\xi_1+\xi_2+\xi_3)}, \\
f' &= 1 + \frac{\mu-k_1}{\mu+k_1}e^{\eta_1+\theta\xi_1} + \frac{\mu-k_2}{\mu+k_2}e^{\eta_2+\theta\xi_2} + \frac{\mu-k_3}{\mu+k_3}e^{\eta_3+\theta\xi_3} \\
&\quad + B_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)} + B_{13}e^{\eta_1+\eta_3+\theta(\xi_1+\xi_3)} + B_{23}e^{\eta_2+\eta_3+\theta(\xi_2+\xi_3)} \\
&\quad + \frac{\mu-k_1}{\mu+k_1}\frac{\mu-k_2}{\mu+k_2}\frac{\mu-k_3}{\mu+k_3}(\alpha_{13}\alpha_{23}A_{12} + \alpha_{12}\alpha_{32}A_{13} + \alpha_{12}\alpha_{13}A_{23}) \times \\
&\quad \times e^{\eta_1+\eta_2+\eta_3+\theta(\xi_1+\xi_2+\xi_3)}, \\
g' &= 1 - \frac{\mu-k_1}{\mu+k_1}e^{\eta_1+\theta\xi_1} - \frac{\mu-k_2}{\mu+k_2}e^{\eta_2+\theta\xi_2} - \frac{\mu-k_3}{\mu+k_3}e^{\eta_3+\theta\xi_3} \\
&\quad + B_{12}e^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)} + B_{13}e^{\eta_1+\eta_3+\theta(\xi_1+\xi_3)} + B_{23}e^{\eta_2+\eta_3+\theta(\xi_2+\xi_3)} \\
&\quad - \frac{\mu-k_1}{\mu+k_1}\frac{\mu-k_2}{\mu+k_2}\frac{\mu-k_3}{\mu+k_3}(\alpha_{13}\alpha_{23}A_{12} + \alpha_{12}\alpha_{32}A_{13} + \alpha_{12}\alpha_{13}A_{23}) \times \\
&\quad \times e^{\eta_1+\eta_2+\eta_3+\theta(\xi_1+\xi_2+\xi_3)},
\end{aligned}
$$

where $\eta_i = k_i x - k_i^3 t + c_i, \alpha_{ij} = \frac{k_i-k_j}{k_i+k_j}$ and

$$
A_{ij} = \left(\frac{k_i-k_j}{k_i+k_j}\right)\left(\frac{k_i-k_j-2\xi_i\xi_j}{k_i+k_j} + 2\theta\frac{k_j\xi_i-k_i\xi_j}{k_i+k_j}\right), \quad B_{ij} = \frac{\mu-k_i}{\mu+k_i}\frac{\mu-k_j}{\mu+k_j}A_{ij}.
$$

4 Discussions

In this paper, we have presented a supersymmetric second modified KdV equation which was obtained from the BT of the supersymmetric MKdV equation. We have also constructed one-, two- and three-soliton solutions. Since the existence of three-soliton solution often implies integrability, we therefore have a new candidate of integrable supersymmetric equation. It would be interesting to find out other integrability properties of this system, such as BT, Lax representation and Painlevé property.

We followed Hirota's idea to construct the second modification for the supersymmetric KdV equation. In literature, there do exist other approaches to the same problem, notably the method based on symmetry and nontrivial conservation laws [3, 14]. An interesting problem is to extend symmetry approach to the supersymmetric case to see what sort of equations would be produced.

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Appendix

A Proof of eqs. (2.7-2.8)

Equations (2.7) and (2.8) are obtained as follow:

$$
\begin{aligned}
&(SD_xf\cdot f')gg' - ff'(SD_xg\cdot g')\\
&\overset{(B.1)}{=} \frac{1}{2}\Big\{D_x[(Sf\cdot g')\cdot gf' + fg'\cdot(Sg\cdot f')] + S[(D_xf\cdot g')\cdot gf' + fg'\cdot(D_xg\cdot f')]\Big\}\\
&\overset{(2.3-2.4)}{=} \frac{1}{2}\Big\{-D_x[(Sg\cdot f')\cdot gf'] - D_x[fg'\cdot(Sf\cdot g')] + \mu S(fg'\cdot gf')\\
&\qquad +S[(D_xg\cdot f')\cdot gf'] + S[fg'\cdot(D_xf\cdot g')] + \mu S(fg'\cdot f'g)\Big\}\\
&\overset{(B.2)}{=} \mu S(fg'\cdot f'g),
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
& (SD_xf\cdot f')gg' + ff'(SD_xg\cdot g')\\
\overset{(B.3)}{=}\; & (SD_xg\cdot f)f'g' + gf(SD_xg'\cdot f') + (D_xf'\cdot g)(Sg'\cdot f) + (D_xg'\cdot f)(Sf'\cdot g)\\
\overset{(2.2b)}{=}\; & (D_xf'\cdot g)(Sg'\cdot f) + (D_xg'\cdot f)(Sf'\cdot g)\\
\overset{(2.4)}{=}\; & (D_xf'\cdot g)(-Sf'\cdot g) + (D_xg'\cdot f)(Sf'\cdot g)\\
=\; & (D_xg'\cdot f - D_xf'\cdot g)Sf'\cdot g\\
\overset{(2.3)}{=}\; & \mu(f'g - fg')Sf'\cdot g.
\end{aligned}
\tag{A.2}
$$

From equations (A.1) and (A.2), we have:

$$
\begin{aligned}
SD_xf\cdot f' &= \frac{\mu}{2gg'}[S(fg'\cdot f'g) + (f'g - fg')Sf'\cdot g]\\
&= \frac{\mu}{2}[2Sf\cdot f' - \frac{f'}{g'}(Sf\cdot g' + Sg\cdot f')]\\
&\overset{(2.4)}{=} \mu Sf\cdot f',\\
SD_xg\cdot g' &= \frac{\mu}{2ff'}[-S(fg'\cdot f'g) + (f'g - fg')Sf'\cdot g]\\
&= \frac{\mu}{2}[2Sg\cdot g' - \frac{g}{f}(Sf\cdot g' + Sg\cdot f')]\\
&\overset{(2.4)}{=} \mu Sg\cdot g'.
\end{aligned}
$$

B Some Bilinear Identities

In this Appendix, we list the relevant bilinear identities, which can be proved directly. Here a, b, c and d are arbitrary Grassmann even functions of the independent variable x, t and θ.

$$
\begin{aligned}
(SD_xa\cdot b)cd - ab(SD_xc\cdot d) = \frac{1}{2}&D_x[(Sa\cdot d)\cdot cb + ad\cdot(Sc\cdot b)]\\
&+\frac{1}{2}S[(D_xa\cdot d)\cdot cb + ad\cdot(D_xc\cdot b)],
\end{aligned}
\tag{B.1}
$$

$$S[(D_x a \cdot b) \cdot ab] = D_x[(Sa \cdot b) \cdot ab], \tag{B.2}$$

$$\begin{aligned}(SD_x a \cdot b)cd + ab(SD_x c \cdot d) \quad &= \quad (SD_x a \cdot c)bd + ac(SD_x b \cdot d) \\ &\quad +(D_x b \cdot c)(Sd \cdot a) + (D_x d \cdot a)(Sb \cdot c),\end{aligned} \tag{B.3}$$

$$(\mathcal{D}a_x)\mathcal{D}a - (\mathcal{D}b_x)\mathcal{D}b = \frac{1}{2}[(\mathcal{D}(a+b)_x)\mathcal{D}(a-b) + (\mathcal{D}(a-b)_x)\mathcal{D}(a+b)], \tag{B.4}$$

$$\begin{aligned}a_x(\mathcal{D}a_x)\mathcal{D}a - b_x(\mathcal{D}b_x)\mathcal{D}b \quad &= \quad \frac{1}{4}(a+b)_x\big(\mathcal{D}(a+b)_x\big)\mathcal{D}(a-b) \\ &\quad +\frac{1}{4}(a-b)_x\big(\mathcal{D}(a-b)_x\big)\mathcal{D}(a-b) \\ &\quad +\frac{1}{4}(a+b)_x\big(\mathcal{D}(a-b)_x\big)\mathcal{D}(a+b) \\ &\quad +\frac{1}{4}(a-b)_x\big(\mathcal{D}(a+b)_x\big)\mathcal{D}(a+b).\end{aligned} \tag{B.5}$$

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