Weight Distributions of Divisible Codes Meeting the Griesmer Bound over F_5 with Dimension 3

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Abstract - Based on a relationship between generator matrix of given code and its weight distribution, all $[31s+t, 3]_5(t=7, 13, 19, 25)$ optimal divisible codes with divisor 5 are determined by solving systems of linear equations. Then their generator matrices and weight polynomials of these optimal divisible codes are also given.

Index Terms - Weight Distributions, divisible codes, Griesmer Bound

1. Introduction

Let F_q^n be the n-dimensional vector over the Galois field $GF(q)$, where q be a prime or power of prime. A q-ary linear code of length n, dimension k and minimum distance d is said to be an $[n, k, d]_q$ code.

Suppose C is an $[n, k, d]_q$ code. Any basis of C forms a k by n matrix G that is called a generator matrix of C, and C is uniquely determined by any of its generator matrices.

For a code C, let $A_i(C)$ be the number of words of Hamming weight i in C, the weight polynomial of C is given by

$$W_C(y)=\sum_{i=1}^{n}A_i y^i .$$

The famous Griesmer bound [1] asserts that the minimum value $n_q(k, d)$ of n satisfies

$$n\geq\sum_{i=0}^{k-1}\left\lceil\frac{d}{q^i}\right\rceil,$$

where $\lceil x\rceil$ denotes the smallest integer $\geq x$. A q-ary linear code is optimal in the sense that no shorter code exists for the same k and d. It is therefore of considerable interest in coding theory to study code meeting the Griesmer bound with equality. We shall abbreviate the right side of the bound by $g_q(k,d)$ and call a code meeting the bound a Griesmer code.

Divisible codes were introduced by Ward in 1981 [2]. A q-ary divisible code is a linear code over the F_q^n whose code-words all have weights divisible by some integer $\Delta\geq 1$, where Δ is called a divisor of the code. Ward proved in [2] that if a divisor Δ of a divisible code is relatively prime to the field characteristic, then the code is merely equivalent to a $\Delta-$folded replicated code. Thus for a q-ary divisible code C, one is most interested in the case where the greatest divisor of C equals p^e for some integer $e\geq 1$.

The study of divisible codes was motivated by a theorem of Gleason and pierce giving constraints on the divisor and field size for divisible codes that are formally self-dual. Later, people began to study the bounds for divisible codes, see [3] and [4] and the references therein.

Optimal codes are often divisible, and the purpose of this paper is to discuss the weight distributions of divisible codes meeting the Griesmer bound over F_5.

This paper is arranged as follows. In section 2, some preliminary materials are introduced. In section 3, we shall give all the weight polynomials of divisible codes with dimension $k=3$, which divisor $\Delta=5$ and meeting the Griesmer bound, and the main results of this paper are presented.

2. Preliminary Knowledge

Let $F_5=\{0,1,2,3,4\}$ be the Galois field with five elements, and let F_5^n be the n-dimensional row vector space over F_5. In the following, we always assume that all the matrices and classical codes are F_5.

Let $1_n=(1,1,\cdots,1)_{1\times n}$, $0_n=(0,0,\cdots,0)_{1\times n}$ and $0_{m\times n}$ be the all-ones vector, the zero vector of length n and the zero matrix of size $m\times n$, respectively.

A non-zero row (column) vector is *monic* if its first non zero coordinate is 1. Suppose $N_k=\frac{5^k-1}{4}$ for $k\geq 2$, then the total number of k-dimensional monic column vectors is N_k.

For k=2, let $\alpha_{2,1}=(1,0)^T$, $\alpha_{2,2}=(0,1)^T$, $\alpha_{2,3}=(1,2)^T$, $\alpha_{2,3}=(1,3)^T$, $\alpha_{2,4}=(1,3)^T$ and $\alpha_{2,6}=(1,4)^T$. Then $\alpha_{2,i}$ are the monic vectors of dimension 2. Let S_2 be the following matrix

$$S_2=(\alpha_{2,1},\alpha_{2,2},\cdots,\alpha_{2,6})=\begin{pmatrix}1&0&1&1&1&1\\0&1&1&2&3&4\end{pmatrix}.$$

Then S_2 is a generator matrix of [6, 2, 5] Simplex code C_2. From S_2, we can construct a generator matrix of $[N_k, k, 5^{k-1}]$ $(k\geq 3)$ Simplex code C_k by the following method:

$$S_k=\begin{pmatrix}S_{k-1}&0_{k-1}^T&S_{k-1}&\cdots&S_{k-1}\\0_{N_{k-1}}&1&1_{N_{k-1}}&\cdots&4\times 1_{N_{k-1}}\end{pmatrix}.$$

Let $M_k = S_k^T S_k$, then M_k is a matrix of size $N_k \times N_k$ for S_k is a $k \times N_k$ matrix.

If $P = (a_{ij})$ is a matrix over F_5, its projection $P_p = (b_{ij})$ is a binary matrix, $b_{ij} = 0$ if $a_{ij} = 0$ and $b_{ij} = 1$ otherwise. We denote $P_k = P(M_k)$ as projective matrix of the matrix M_k.

Let $Q_k = J_{N_k} - P_k$, where J_{N_k} be the all-ones matrix of $N_k \times N_k$. The matrices P_k and P_k are introduced and their properties are discussed in [5].

Let $S_k = (\alpha_{k,1}, \alpha_{k,2}, \cdots, \alpha_{k,N_k})$ be the generator matrix of Simplex code S_k. Suppose C be a $[n, k, d]_5$ code without zero coordinate and its generator matrix G with columns are all monic vectors. If $\alpha_{k,i}$ appears l_i times in G, we denote $G = (l_1\alpha_{k,1}, l_2\alpha_{k,2}, \cdots, l_{N_k}\alpha_{k,N_k})$ and call $L = (l_1, l_2, \cdots, l_{N_k})$ as the defining vector of G.

Let $P_k L^T = W^T$, here $W = (w_1, w_2, \cdots, w_{N_k})$ is the projective weight vector of $[n, k, d]_5$. So we can change W into $W^T = (d, d, \cdots, d)^T + (\lambda_1, \lambda_2, \cdots, \lambda_{N_k})^T = d \cdot 1_{N_k} + \Lambda^T$. Figure the sum from two sides of this equation, left side is $q^{k-1}\sum_{i=1}^{N_k} l_i = nq^{k-1}$, and right side is $N_k d + \sum_{i=1}^{N_k} \lambda_i$. Then $nq^{k-1} = N_k d + \sum_{i=1}^{N_k} \lambda_i$, we denote $\sigma(\Lambda) = \sum_{i=1}^{N_k} \lambda_i$, so $\sigma(\Lambda) = nq^{k-1} - N_k d$.

Finally, we set up the connection between definition and projective weight distribution in the form of systems of equations as follows:

$$\begin{cases} P_k L^T = d \cdot 1_{N_k}^T + \Lambda^T \\ \sigma(\Lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_{N_k} \\ \lambda_i \geq 0 (1 \leq i \leq N_k) \\ n = l_1 + l_2 + \cdots + l_{N_k} \end{cases} \quad (1)$$

For a given $C = [n, k, d]_5$, it is easy to give all the possible projective weight distribution from $\sigma(\Lambda)$, thus $W^T = d \cdot 1_{N_k} + \Lambda^T$, $C = [n, k, d]_5$ exits if and only if (1) has non negative integer solution L. From L we can obtain $C = [n, k, d]_5$ generator matrix G and weight polynomial $W_C(y)$.

3. Main results

In this section, we will use the results of the above section to study the weight distributions of optimal $[n, 3]$ divisible codes for $n = 31s + t$, where $t \in \{7, 13, 19, 25\}$. To save space we only explain our process for case $[31s + 7, 3]$ optimal divisible codes, other cases can be deuced similarly.

Let $n = 31s + 7 (s \geq 1)$, using the Griesmer bound, we can deduce a $[31s + 7, 3]$ optimal divisible code has minimum distance $d = 25s + 5$ and $\sigma(\Lambda) = 20$, then

$$L = \frac{1}{5^2}(25 \cdot 1_{31} - 5Q_3\Lambda).$$

Since each code words of a divisible codes with divisor $\Delta = 5$ has weight $w \equiv (mod 5)$, we can assume $\Lambda = 5\Lambda'$, thus

$$L = 1_{31} - Q_3\Lambda', \quad (2)$$

where $\sigma(\Lambda') = 4$. Using MATLAB program, one can easily get all solutions satisfying (2) for $s \geq 1$.

In the following, we let $M^w(n, k)$ be the number of different weight polynomial of optimal $[n, k]$ divisible code, and use $sG = (G, G, \cdots, G)$ to denote the juxtaposition of s copies of G for given matrix G.

Case 1: If $s = 1, M^w(38, 3) = 1$, and this one weight polynomial is $1 + 108y^{30} + 16y^{35}$ for optimal $[38, 3, 30]$ divisible code, its generator matrix G_{38} with defining vector L_{38} is $L_{38} = (0011100112212112122111222002222)$.

Case 2: If $s = 2$, $M^w(69, 3) = 2$. These two weight polynomials for optimal $[69, 3, 55]$ divisible code are

$$1 + 108y^{55} + 16y^{60} (\textbf{two cdoes}),$$
$$1 + 112y^{55} + 8y^{60} + 4y^{65}.$$

There are two optimal divisible codes have weight polynomial $1 + 108y^{55} + 16y^{60}$, their corresponding generator matrices are $G_{69,1} = (G_3, G_{38})$ and $G_{69,2}$ with defining vectors $L_{69,1} = 1_{31}^T + L_{38}$ and $L_{69,2}$ are

$$L_{69,1} = (1122211223323223233222333113333),$$

and $L_{69,2} = (11222102233332223332223332223332)$,

respectively.

There is one optimal divisible codes has weight polynomial $1 + 112y^{55} + 8y^{60} + 4y^{65}$, its generator matrices $G_{69,3}$ with defining vector $L_{69,3}$ is

$$L_{69,3} = (0111101233332233332233332233332).$$

Case 3: If $s \geq 3$, $M^w(69, 3) = 5$. These five weight polynomials for optimal $[31s + 7, 3, 25s + 5]$ divisible code as follows:

$$1 + 108y^{25s+5} + 16y^{25s+10} (\textbf{three cdoes}),$$
$$1 + 112y^{25s+5} + 8y^{25s+10} + 4y^{25s+15} (\textbf{two cdoes}),$$
$$1 + 116y^{25s+5} + 8y^{25s+15},$$
$$1 + 116y^{25s+5} + 4y^{25s+10} + 4y^{25s+20},$$
$$1 + 120y^{25s+5} + 4y^{25s+25}.$$

Let $G_{100,i} (1 \leq i \leq 5)$ are generator matrices of optimal $[100, 3, 80]$ divisible code, their defining vectors $L_{100,i}$ are

$L_{100,1} = 2 \cdot 1_{31}^T + L_{38}$,

$L_{100,2} = 1_{31}^T + L_{69,2}$,

$L_{100,3} = (3343430334343334343334343334343)$,

$L_{100,4} = 1_{31}^T + L_{69,3}$

$L_{100,5} = (3442430344243344243344243344223)$,

$L_{100,6} = (4442420444242444242444242444242)$,

$L_{100,7} = (4441430444143444143444143444143)$,

$L_{100,8} = (4440440444044444044444044444044)$.

Then, there exit three different generator matrices for weight polynomial for $1+108y^{25s+5}+16y^{25s+10}$, we denote its by $G_{31s+7,i}(1 \le i \le 3)$.

There exit two different generator matrices for weight polynomial for $1+112y^{25s+5}+8y^{25s+10}+4y^{25s+15}$, we let its be $G_{31s+7,i}(4 \le i \le 5)$.

For weight polynomials $1+116y^{25s+5}+8y^{25s+15}$,

$1+116y^{25s+5}+4y^{25s+10}+4y^{25s+20}$ and

$1+120y^{25s+5}+4y^{25s+25}$, their corresponding generator matrices denote by $G_{31s+7,i}(6 \le i \le 8)$.

To above generator matrices $G_{31s+7,i}(1 \le i \le 8)$, their corresponding defining vectors are

$L_{31s+7,i} = (s-3) \cdot 1_{31}^T + L_{100,i}$.

In sum up all statements above, then we could get:

Theorem 3.1 Let $n = 31s + 7(s \ge 1)$, if $s = 1$, then $M^w(38,3) = 1$; if $s = 2$, then $M^w(38,3) = 2$; if $s \ge 3$, then $M^w(38,3) = 5$. All weight polynomials see above.

Similar to the above discussion, we have the following:

Theorem 3.2 Let $n = 31s + 13(s \ge 1)$,

(1) If $s = 1$, then $M^w(44,3) = 1$, and this weight polynomial is $1+112y^{35}+12y^{40}$.

(2) If $s \ge 2$, then $M^w(31s+13,3) = 3$, and these three weight polynomials are

$1+120y^{25s+10}+4y^{25s+25}$;
$1+116y^{25s+10}+4y^{25s+15}+4y^{25s+20}$;
$1+112y^{25s+10}+12y^{25s+15}(\textbf{two cdoes})$.

Theorem 3.3 Let $n = 31s + 19(s \ge 1)$, then $M^w(31s+19,3) = 2$, and these two weight polynomials are

$1+120y^{25s+15}+4y^{25s+25}$,
$1+116y^{25s+15}+8y^{25s+20}$.

Theorem 3.4 Let $n = 31s + 25(s \ge 0)$, then $M^w(31s+25,3) = 1$, and this one weight polynomial is

$1+120y^{25s+20}+4y^{25s+25}$.

4. Conclusions

In this paper, we have given the complete weight distribution of $[31s+t,3]_5(t=7,13,19,25)$ optimal divisible codes, our results of these codes have lengths one above the Griesmer bound. Our method given can also be used to $[n,k]_5$ optimal divisible codes for dimension$k \ge 4$.

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