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$f(x)$ and $F(x)$ are connected by the relation

$$x^2 f(x) = [\beta + \alpha(-\ln F(x))]F(x). \tag{4.3}$$

Let $\{X_n,\ n \geqslant 1\}$ be a sequence of iid random variables as defined in Section 2. For a fixed $k \geqslant 1$, we define the sequence $\{L_m^{(k)},\ m \geqslant 1\}$ of k-th lower record times of $\{X_n,\ n \geqslant 1\}$ (as introduced by Pawlas and Szynal, 1998) as follows:

$$L_1^{(k)} = 1, \quad L_{m+1}^{(k)} = \min\left\{ j > L_m^{(k)} :\ X_{k:L_m^{(k)}+k-1} > X_{k:j+k-1} \right\}.$$

For $k = 1$, we write $L_m^{(1)} = L_m$ which are lower record times of $\{X_n,\ n \geqslant 1\}$.
The sequence $\{Z_m^{(k)},\ m \geqslant 1\}$ where $Z_m^{(k)} = X_{k:L_m^{(k)}+k-1}$ is called the sequence of generalized lower record values or k-th lower record values of $\{X_n,\ n \geqslant 1\}$. For convenience we shall also take $Z_0^{(k)} = 0$. For $k = 1$, we have $Z_m^{(1)} = X_{L_m}$, $m \geqslant 1$, which defines the usual sequence of lower record values of $\{X_n,\ n \geqslant 1\}$. We shall define,

$$\begin{aligned}
\nu_{m;k}^{(r)} &= E[(Z_m^{(k)})^r], && r, m = 1, 2, \ldots \\
\nu_{m,t;k}^{(r,s)} &= E[(Z_m^{(k)})^r (Z_t^{(k)})^s], && 1 \leqslant m \leqslant t-1,\ \ r, s = 1, 2, \ldots \\
\nu_{m,t;k}^{(r,0)} &= E[(Z_m^{(k)})^r] = \nu_{m;k}^{(r)}, && 1 \leqslant m \leqslant t-1,\ \ r = 1, 2, \ldots \\
\nu_{m,t;k}^{(0,s)} &= E[(Z_t^{(k)})^s] = \nu_{t;k}^{(s)}, && 1 \leqslant m \leqslant t-1,\ \ s = 1, 2, \ldots
\end{aligned}$$

The pdf of $Z_m^{(k)}$, $m \geqslant 1$ is given by

$$f_{Z_m^{(k)}}(x) = \frac{k^m}{(m-1)!}[-\ln F(x)]^{m-1}[F(x)]^{k-1} f(x) \tag{4.4}$$

and the joint density function of $Z_m^{(k)}$ and $Z_t^{(k)}$, $1 \leqslant m < t$, $t \geqslant 2$ is given by

$$\begin{aligned}
f_{Z_m^{(k)}, Z_t^{(k)}}(x,y) = {} & \frac{k^t}{(m-1)!(t-m-1)!}[\ln F(x) - \ln F(y)]^{t-m-1} \\
& \times [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [F(y)]^{k-1} f(y), \quad x > y.
\end{aligned} \tag{4.5}$$

Theorem 4.1. *Fix a positive integer $k \geqslant 1$. For $m \geqslant 1$ and $r = 1, 2, \ldots$*

$$\nu_{m+1;k}^{(r)} = \nu_{m;k}^{(r)} + \frac{\beta k}{m\alpha}(\nu_{m-1;k}^{(r)} - \nu_{m;k}^{(r)}) - \frac{r}{m\alpha}\nu_{m;k}^{(r+1)}. \tag{4.6}$$

Proof. For $m \geqslant 1$ and $r = 1, 2, \ldots$, we have from (4.3) and (4.4)

$$\begin{aligned}
\nu_{m;k}^{(r+1)} = {} & \frac{\beta k^m}{(m-1)!}\int_0^\infty x^{r-1}[-\ln F(x)]^{m-1}[F(x)]^{k-1} dx \\
& + \frac{\alpha k^m}{(m-1)!}\int_0^\infty x^{r-1}[-\ln F(x)]^m [F(x)]^k dx.
\end{aligned}$$

Now, integrating by parts treating x^{r-1} for integration and the rest of the integrand for differentiation and simplifying we get relation (4.6). □

Theorem 4.2. *For $1 \leqslant m \leqslant t-2$ and $r, s = 1, 2, \ldots$*

$$r v_{m,t;k}^{(r+1,s)} = \beta k \left(v_{m-1,t-1;k}^{(r,s)} - v_{m,t-1;k}^{(r,s)} \right) + m\alpha \left(v_{m,t;k}^{(r,s)} - v_{m+1,t;k}^{(r,s)} \right) \tag{4.7}$$

and for $m \geqslant 1$, $r, s = 1, 2, \ldots$

$$r v_{m,m+1;k}^{(r+1,s)} = \beta k \left(v_{m-1,m;k}^{(r,s)} - v_{m;k}^{(s+r)} \right) + m\alpha \left(v_{m,m+1;k}^{(r,s)} - v_{m+1;k}^{(s+r)} \right). \tag{4.8}$$

Proof. The proof follows exactly in the same manner as in Theorem 2.2 and hence omitted. □

5. Characterization result for inverted Gompertz distribution

Here we obtain a characterization property of inverted Gompertz distribution based on conditional expectation of function of generalized lower record values. First we consider the conditional distribution of a generalized lower record value $Z_t^{(k)}$ given $Z_m^{(k)} = x$ for $m < t$. The joint pdf of $Z_m^{(k)}$ and $Z_t^{(k)}$ is given in (4.5) and the marginal distribution of $Z_m^{(k)}$ is given by (4.4). If we write $g(y \mid x)$ to denote this conditional density, then the conditional pdf of $Z_t^{(k)}$ given $Z_m^{(k)} = x$, $1 \leqslant m < t$ is given by

$$g(y \mid x) = \frac{k^{t-m}}{(t-m-1)![F(x)]^k} [\ln F(x) - \ln F(y)]^{t-m-1} [F(y)]^{k-1} f(y). \tag{5.1}$$

Clearly $g(y \mid x)$ is distributed as the pdf of the $(t-m)th$ generalized lower record value arising from the distribution truncated on the right at x. Thus we conclude that the conditional pdf of generalized lower record values also shows a property similar to that of conditional pdf of an order statistic arising from an absolutely continuous distribution given the value of a higher order statistic.
In particular, the conditional pdf of $Z_{m+1}^{(k)}$ given $Z_m^{(k)} = x$, is given by

$$g(y \mid x) = \frac{k[F(y)]^{k-1} f(y)}{[F(x)]^k}. \tag{5.2}$$

Theorem 5.1. *Let X be an absolutely continuous r.v. with pdf $f(x)$ and cdf $F(x)$. Then X follows an inverted Gompertz distribution with $F(x) = e^{-\frac{\beta}{\alpha}(e^{\frac{\alpha}{x}}-1)}$ if and only if*

$$E\left[e^{-e^{\alpha/Z_{m+1}^{(k)}}} \mid Z_m^{(k)} = x \right] = \frac{\beta k}{(\alpha+\beta k)} e^{-e^{\frac{\alpha}{x}}}. \tag{5.3}$$

Proof. Using (5.2), the necessary part follows by direct computation.
Conversely, assume that (5.3) holds. Then

$$\int_0^x e^{-e^{\frac{\alpha}{y}}} [F(y)]^{k-1} f(y) dy = \frac{\beta}{(\alpha+\beta k)} e^{-e^{\frac{\alpha}{x}}} [F(x)]^k. \tag{5.4}$$

Differentiating both sides of (5.4) and simplifying we get,

$$\frac{d}{dx} \ln F(x) = \frac{\beta e^{\frac{\alpha}{x}}}{x^2}$$

which on further simplification leads to, $F(x) = e^{-\frac{\beta}{\alpha}(e^{\frac{\alpha}{x}}-1)}$. □

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