The Long Time Behavior of Solutions for a Generalized Boussinesq System

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Abstract

The initial-boundary value problem is investigated for a generalized Boussinesq equation with the quadratic nonlinearity. For small initial data and homogeneous boundary conditions, its solution is constructed in the form of a series which converges absolutely and uniformly. The long time asymptotic expansion of the solution is acquired to show the nonlinear effects of amplitude.

Keywords: Boussinesq equation, Well-posedness, Fourier transform, Exponential decay

1. Introduction

Boussinesq equation was first presented in [1] to research the propagation of small amplitude's long wave on the surface of shallow water. From then on, people have made a great deal of investigations for the equation (see[2]-[4]). One of the classical Boussinesq equations is written in the form

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1}$$

where constant coefficients α and β depend on the depth of fluid and the speed of waves. Clarkson [3] established a general method to determine the exact solutions of equation (1). Hirota deduced conservation laws and examined it's numerical solutions in literature [5]. Lai [9]-[10] studied the long time behavior of solutions for nonlinear wave equations. Nakamura [6] studied the exponential decay of the Boussinesq equation with spherical symmetry.

Varlamov [7] considered the following Boussinesq equation with initial-boundary assumptions

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{2}$$

where $\beta \in R^1$, α and b are positive constants and $\alpha > b^2$. Making use of the eigenfunction expansion method in a ball, Varlamov [8] investigated a long time asympotics of solution for a Boussinesq equation similar to equation (2). Polat [11] considered the blow up phenomena of solutions for the following Boussinesq equation with damping term

$$u_{tt} - u_{xx} + \delta u_{xxxx} - \lambda u_{xxtt} - ru_{xxt} = \beta(u^2)_{xx} - \gamma^2 u,$$

where $\delta, \lambda, r, \beta$ and γ are constants, which satisfy some assumptions.

The objective of this paper is to study the well-posedness of the following generalized Boussinesq equation with initial-boundary conditions

$$u_{tt} - au_{ttxx} - 2bu_{txx} + du = -cu_{xxxx} + u_{xx} + \beta(u^2)_{xx} + hu^2, \tag{3}$$

where $a > 0,\ b > 0,\ c > 0\ d \geqslant 0, h$ and β are constants and $a+c > b^2$. Under some assumptions, the existence and the uniqueness of solution for equation (3) are established. It will be shown that the long time behavior of the solution in equation shows the presence of damped oscillations decaying exponentially in time as $t \to \infty$. The methods for the proof of our main theorem in this paper are based on those of [7]. However, it should be emphasized that the technique for proving uniqueness of the solution is different from that presented in [7] in which the time extension method was used.

2. Main result

The task of this paper is to consider the following generalized Boussinesq system with initial-boundary conditions

$$\begin{cases} u_{tt} - au_{ttxx} - 2bu_{txx} + du = -cu_{xxxx} + \\ u_{xx} + \beta(u^2)_{xx} + hu^2, \\ u(0,t) = u(\pi,t) = 0, \\ u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \\ u(x,0) = \varepsilon^2\varphi(x), u_t(x,0) = \epsilon^2\psi(x), \end{cases} \tag{4}$$

where $a > 0, d \geqslant 0, b, c$ and ε are positive constants, $x \in (0,\pi)$, $t > 0$, h and $\beta \in R^1$.

Definition 2.1: If $u(0) = u(\pi) = u^{''}(0) = u^{''}(\pi) = \ldots = u^{2n-2}(0) = u^{2n-2}(\pi) = 0$ and

$u^{(2n)}(x) \in L^2(0,\pi)$, the function $u(x)$ is said to belong to the class $C^{2n}(0,\pi)$, $n \geqslant 1$.

Definition 2.2: The function $u(x,t)$ defined on $[0,\pi] \times [0,+\infty)$ is said to be the classical solution of the problem defined by system (4), if $u(x,t)$ and its derivatives included in (4) are bounded and continuous, and satisfy system (4).

Theorem 2.3: If $a > 0, c > 0, d \geqslant 0, a+c > b^2, \varphi(x) \in C^6(0,\pi), \psi(x) \in C^4(0,\pi)$, there is a ε_0, for any ε satisfying $0 < \varepsilon < \varepsilon_0$, problem (4) has a unique classical solution expressed in the form

$$u(x,t) = \sum_{N=1}^{\infty} \varepsilon^{N+1} u^{(N)}(x,t), \tag{5}$$

where $u^{(N)}(x,t)$ will be defined in the proof (see(23) or (24)). Series (5) and its derivatives, which appear in problem (4), converge absolutely and uniformly. For $x \in [0,\pi]$, $t \geqslant 0$, $0 < \varepsilon < \varepsilon_0$. The problem (5) has the following asymtotics as t is sufficiently large

$$u(x,t) = e^{-\frac{b}{1+a}t}[(A\cos\sigma t + B\sin\sigma t)\sin x + O(e^{[\frac{\eta b}{1+a}t]})], \tag{6}$$

where $\sigma = \frac{\sqrt{ac+a+c-b^2+1+ad+d}}{1+a}$ is a positive constant, $0 < \eta < \frac{3}{1+4a}$.

For simplicity, throughout the paper, we denote by C any possitive constants independent of t, which may depend on $\varphi(x)$, $\psi(x)$ and other constants appearing in system (4).

3. Proof of Theorem 2.3

3.1. Existence of solution

We make an odd extension for x on $[-\pi,0]$, and represent $u(x,t)$ in the form of Fourier series expressed by

$$u(x,t) = \sum_{n=-\infty,n\neq 0}^{\infty} \widehat{u}_n(t)e^{inx}, \tag{7}$$

where

$$\widehat{u}_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x,t)e^{inx}dx,$$

from which we have $\widehat{u}_{-n}(t) = -\widehat{u}_n(t), n \geqslant 1$.

In the sequel we shall denote the norm of the space of functions belonging to $L^2(-\pi,\pi)$ for each fixed $t > 0$,

$$\| u(t) \| = \| u(t) \|_{L^2(-\pi,\pi)} = (\int_{-\pi}^{\pi} | u(x,t) |^2 \, dx)^{\frac{1}{2}}.$$

It follows from (7) that

$$u(x,t) = 2i\sum_{n=1}^{\infty} \widehat{u}_n(t)\sin nx, \tag{8}$$

$$\widehat{u}_n(t) = \frac{1}{i\pi}\int_0^{\pi} u(x,t)\sin nx dx.$$

Noting the initial functions on $[-\pi, \pi]$, which have $\widehat{\varphi}_{-n} = -\widehat{\varphi}_n, \widehat{\psi}_{-n} = -\widehat{\psi}_n, n \geqslant 1$, we get

$$\varphi(x) = \sum_{n=-\infty,n\neq 0}^{\infty} \widehat{\varphi}_n e^{inx}, \tag{9}$$

$$\psi(x) = \sum_{n=-\infty,n\neq 0}^{\infty} \widehat{\psi}_n e^{inx}. \tag{10}$$

Furthermore on $[0,\pi]$, we obtain

$$\begin{cases} \varphi(x) = 2i\sum\limits_{n=1}^{\infty} \widehat{\varphi}_n \sin nx, \\ \widehat{\varphi}_n = \frac{1}{i\pi}\int_0^{\pi} \varphi(x)\sin nx dx, \\ \psi(x) = 2i\sum\limits_{n=1}^{\infty} \widehat{\psi}_n \sin nx, \\ \widehat{\psi}_n = \frac{1}{i\pi}\int_0^{\pi} \psi(x)\sin nx dx. \end{cases} \tag{11}$$

Integrating (11) and using the smoothness assumption of initial data yield the following inequalities

$$| \widehat{\varphi}_n | \leqslant C_1 n^{-6}, \quad | \widehat{\psi}_n | \leqslant C_1 n^{-4}, \quad n \geqslant 1, \tag{12}$$

where C_1 is a positive constant.

Substituting (7), (9) and (10) into (4) gives rise to the following Cauchy problem for $\widehat{u}_n(t)$

$$\begin{cases} (1+an^2)\widehat{u}_n''(t) + 2bn^2\widehat{u}_n'(t) + \\ (cn^4+n^2+d)\widehat{u}_n(t) \\ = (-\beta n^2 + h)p(\widehat{u}_n(t)), \\ \widehat{u}_n(0) = \varepsilon^2\widehat{\varphi}_n, \quad \widehat{u}_n'(t) = \varepsilon^2\widehat{\psi}_n, \end{cases} \tag{13}$$

where

$$p(\widehat{u}_n(t)) = \sum_{g=-\infty,g\neq 0,n}^{\infty} \widehat{u}_{n-g}(t)\widehat{u}_g(t),$$

$$\widehat{u}_{-n}(t) = -\widehat{u}_n(t), \quad n \geqslant 1.$$

If $n = 1$, it has

$$p(\widehat{u}_1(t)) = 2\sum_{g=1}^{\infty} \widehat{u}_{-g}(t)\widehat{u}_{1+g}(t) = -2\sum_{g=1}^{\infty} \widehat{u}_g(t)\widehat{u}_{1+g}(t). \tag{14}$$

If $n \geqslant 2$, we have

$$p(\widehat{u}_n(t)) = \sum_{g=1}^{n-1} \widehat{u}_{n-g}(t)\widehat{u}_g(t) - 2\sum_{g=1}^{\infty} \widehat{u}_g(t)\widehat{u}_{n+g}(t).$$

Setting $\Phi_n = \varepsilon\widehat{\varphi}_n$ and $\Psi_n = \varepsilon\widehat{\psi}_n$, we get the solution formula for problem (13) in the form

$$\begin{aligned}\widehat{u}_n(t) &= \varepsilon e^{\frac{-bn^2 t}{1+an^2}}\{[\cos(\sigma_n t) \qquad (15)\\ &+ \frac{bn^2}{1+an^2}\frac{\sin(\sigma_n t)}{\sigma_n}]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n\}\\ &- \frac{\beta n^2 - h}{\sigma_n(1+an^2)}K,\end{aligned} \qquad (16)$$

where

$$K = \int_0^t \exp[\frac{-bn^2}{1+an^2}(t-\tau)]\sin[\sigma_n(t-\tau)]p(\widehat{u}_n(t))d\tau$$

$$\sigma_n = \frac{\sqrt{acn^6 + (a+c-b^2)n^4 + (1+ad)n^2 + d}}{1+an^2},$$

in which we require $a+c > b^2, d \geqslant 0$.

Now, we consider integral equation (16) by using perturbation technique. Firstly, we express $\widehat{u}_n(t), (n \geqslant 1)$ as a form of series about ε

$$\widehat{u}_n(t) = \sum_{N=1}^{\infty} \varepsilon^{N+1}\widehat{\xi}_n^{(N)}(t). \qquad (17)$$

Substituting (17) into (16) and equating the coefficients of like powers of ε result in the following formula.

When $N = 0$, we have

$$\widehat{\xi}_n^{(0)}(t) = \varepsilon e^{\frac{-bn^2 t}{1+an^2}}\{[\cos(\sigma_n t) + \frac{bn^2}{1+an^2}\frac{\sin(\sigma_n t)}{\sigma_n}]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n\}. \qquad (18)$$

When $N \geqslant 1$, we get

$$\widehat{\xi}_n^{(N)}(t) = \frac{\beta n^2 + h}{\sigma_n(1+an^2)}\int_0^t \exp[\frac{-bn^2}{1+an^2}(t-\tau)] \sin[\sigma_n(t-\tau)]Q_N(\widehat{\xi}_n^{(j)}(\tau))d\tau, \qquad (19)$$

where $n \geqslant 1$ and

$$Q_N(\widehat{\xi}_n^{(j)}(\tau)) = \varepsilon_n \sum_{g=1}^{n-1}\sum_{j=1}^{N}\widehat{\xi}_{n-g}^{(j-1)}(\tau)\widehat{\xi}_g^{(N-j)}(\tau) -2\sum_{g=1}^{\infty}\sum_{j=1}^{N}\widehat{\xi}_{n+g}^{(j-1)}(\tau)\widehat{\xi}_g^{(N-j)}(\tau),$$

$\varepsilon_1 = 0; \varepsilon_n = 1$, if $n \geqslant 2$.

In order to state that formula (7) represents the solution of problem (4), we need to prove that the series

$$\begin{aligned}u(x,t) &= \sum_{n=-\infty, n\neq 0}^{\infty} e^{inx}\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{\xi}_n^{(N)}(t)\\ &= 2i\sum_{n=1}^{\infty}\sin nx\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{\xi}_n^{(N)}(t)\end{aligned}$$

converges absolutely and uniformly. To do this, we construct the following estimate for $n \geqslant 1,\ t > 0,\ N > 0$,

$$|\ \widehat{\xi}_n^{(N)}(t)\ | \leqslant C^N(N+1)^{-2}n^{-6}e^{-\frac{b}{1+a}t}, \qquad (20)$$

where C is a positive constant independent of N, n, ε and t. We will use the induction method to prove inequality (20).

When $N = 0$, form (18), we have

$$\begin{aligned}|\ \widehat{\xi}_n^{(0)}(t)\ | &\leqslant \varepsilon e^{\frac{-bn^2 t}{1+an^2}}[(1 + \frac{bn^2}{\sigma_n(1+an^2)})\ |\ \Phi_n\ | \\ &+ \frac{1}{\sigma_n}\ |\ \Psi_n\ |]\\ &\leqslant \varepsilon n^{-6}e^{-\frac{b}{1+a}t}.\end{aligned}$$

Assuming that (20) is valid for all $\widehat{\xi}_n^{(s)}(t)$ with $0 \leqslant s \leqslant N-1$, we shall prove that (20) also holds for $s = N$. According to [7], for any integer $n \geqslant 1, g \geqslant 1$, and $g \neq n$, we have

$$|\ n-g\ |^{-6}\ g^{-6} \leqslant 2^6 n^{-6}[g^{-6} + |\ n-g\ |^{-6}]$$

$$j^{-2}(N+1-j)^{-2} \leqslant 2^2(N+1)^{-2}[j^{-2} + (N+1-j)^{-2}].$$

Using $\frac{bn^2}{1+an^2} \geqslant \frac{b}{1+a}(n \neq 0)$ and (19) leads to

$$\begin{aligned}&|\widehat{\xi}_n^{(N)}(t)| \leqslant C|\beta|(N+1)^{-2}n^{-6}\\ &\times\sum_{g=1}^{\infty}(g^{-6} + |n-g|^{-6})\\ &\times\sum_{j=1}^{N}C^{j-1}C^{N-j}[j^{-2} + (N+1-j)^{-2}]\\ &\times |\ S_N(n,t)\ |,\end{aligned}$$

where

$$\begin{aligned}&|S_N(n,t)|\\ \leqslant\ & Ce^{\frac{-bn^2}{1+an^2}t}\int_0^t \exp(\frac{bn^2}{1+an^2} - \frac{2b}{1+a}\tau)d\tau\\ =\ & Ce^{-\frac{bn^2}{1+an^2}t} \times |\frac{e^{(\frac{bn^2}{1+an^2} - \frac{2b}{1+a})\tau} - 1}{\frac{bn^2}{1+an^2} - \frac{2b}{1+a}}|\\ \leqslant\ & Ce^{-\frac{b}{1+a}t}.\end{aligned} \qquad (21)$$

Therefore (20) holds.

For $n \geqslant 2$, we derive

$$\frac{bn^2}{1+an^2} \geqslant \frac{(1+\eta)b}{1+a},$$

where $0<\eta<\frac{3}{1+4a}$. Furthermore, we get

$$|\widehat{\xi_n}^{(N)}(t)| \leqslant C^N(N+1)^{-2}n^{-6}e^{-\frac{1+\eta b}{1+a}t}. \tag{22}$$

Substituting (17) into (8) and interchanging the order of summation in the series, for $x \in [0,\pi], t \geqslant 0, \varepsilon \in [0,\varepsilon_0]$, we get

$$\begin{aligned} u(x,t) &= 2i\sum_{n=1}^{\infty}\widehat{u}_n(t)\sin nx \\ &= 2i\sum_{n=1}^{\infty}\sin nx\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{\xi_n}^{(N)}(t) \\ &= \sum_{N=0}^{\infty}\varepsilon^{N+1}u^{(N)}(x,t), \end{aligned} \tag{23}$$

where

$$u^{(N)}(x,t) = 2i\sum_{n=1}^{\infty}\widehat{\xi}_n^{(N)}(t)\sin nx. \tag{24}$$

Differentiating (18)-(19), for $k=1,2$, we have

$$\partial_t^k\widehat{\xi}_n^{(0)}(t) = \sum_{l=0}^{k}c_k^l(-1)^l(\frac{bn^2}{1+an^2})^l e^{\frac{-bn^2t}{1+an^2}} \times$$
$$\partial_t^{k-l}\{[\cos(\sigma_n t) + \frac{bn^2}{1+an^2}\frac{\sin(\sigma_n t)}{\sigma_n}]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n\},$$

$$\partial_t^k\widehat{\xi}_n^{(N)}(t) = -\frac{\beta n^2-h}{\sigma_n(1+an^2)} \times \int_0^t g_k(n,t-\tau)Q_N(\widehat{\xi}_n^{(j)}(\tau))d\tau + R_k(n,t),$$

where

$$g_k(n,t) = \sum_{l=0}^{k}c_k^l(-1)^l(\frac{bn^2}{1+an^2})^l e^{\frac{-bn^2}{1+an^2}t}\sigma_n^{k-l} \times \sin[\sigma_n t + \frac{(k-l)\pi}{2}],$$

$Q_N(\widehat{\xi_n}^{(j)}(\tau))$ is defined by (19) and c_k^l are binomial coefficients. $R_k(n,t)$ are defined as follows

$$R_1(n,t)=0, \qquad R_2(n,t) = -\frac{\beta n^2-h}{1+an^2}Q_N(\widehat{\xi}_n^{(j)}(\tau)).$$

By the bounded properties of $\frac{bn^2}{1+an^2}$ and $\frac{\beta n^2}{\sigma_n(1+an^2)}$ and making the estimates similar to (20) for $n \geqslant 1, N \geqslant 0, t>0, k \geqslant 0,1,2$, we know that the following inequalities hold

$$|\partial_t^k\widehat{\xi}_n^{(N)}(t)| \leqslant C^N(N+1)^{-2}n^{k-6}e^{-\frac{b}{1+a}t}, \tag{25}$$

$$|\partial_t^k\widehat{u}_n^{(N)}(t)| \leqslant Cn^{k-6}e^{-\frac{b}{1+a}t}. \tag{26}$$

The u_{xxxx} and u_{ttxx} are the highest order of derivative terms appearing in system (4). Inequalities (25) and (26) show that the derivatives of $u(x,t)$ in problem (4) are absolutely and uniformly convergent. Thus, we know $u(x,t)$ is a classical solution of problem (4).

3.2. Uniqueness of the solution

Assuming that problem (4) has two classical solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$, we shall prove that $u^{(1)}(x,t)$ is equal to $u^{(2)}(x,t)$. Making an odd extension for the two solutions on $(-\pi,0]$, we notice that the two solutions belong to space $L^2(-\pi,\pi)$. According to Definition 2.2, for each fixed $t>0$, we have

$$\max_{x\in[-\pi,\pi]}|u^{(1)}(x,t)| < c_t, \max_{x\in[-\pi,\pi]}|u^{(2)}(x,t)| < c_t,$$

where c_t is a constant depending on t.

Setting $w(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$ and making an even extension for $w(x,t)$ on $\ldots(-3\pi,-2\pi)$, $(-2\pi,-\pi),(\pi,2\pi),(2\pi,3\pi),\ldots$, from (4), we have

$$\begin{aligned} &w_{tt} - aw_{ttxx} - 2bw_{txx} + dw = -cw_{xxxx} + w_{xx} \\ &+\beta[w(x,t)(u^{(1)}(x,t)+u^{(2)}(x,t))]_{xx} \\ &+hw(x,t)[u^{(1)}(x,t)+u^{(2)}(x,t)], \\ &w(x,0) = w_t(x,0) = 0. \end{aligned}$$

Taking the Fourier transform of w on $(-\infty,+\infty)$, namely,

$$\widehat{w}(\xi,t) = \int_{-\infty}^{+\infty}w(x,t)e^{-i\xi x}dx,$$

we obtain

$$\begin{aligned} &(1+a\xi^2)\widehat{w}''(\xi,t) + 2b\xi^2\widehat{w}'(\xi,t) + \\ &(c\xi^4+\xi^2+d)\widehat{w}(\xi,t) = (-\beta\xi^2+h)\widehat{f}(\xi,t), \end{aligned} \tag{27}$$

where $\widehat{f}(\xi,t) = w(u^{(1)}+u^{(2)})(x,t)$. It follows from (27) that

$$\begin{aligned} \widehat{w}(\xi,t) = &-\tfrac{\beta\xi^2-h}{\sigma_\xi(1+a\xi^2)}\int_0^t \exp[-\tfrac{b\xi^2}{1+a\xi^2}(t-\tau)] \\ &\times\sin[\sigma_\xi(t-\tau)]\widehat{f}(\xi,t)d\tau, \end{aligned} \tag{28}$$

where

$$\sigma_\xi = \frac{\sqrt{ac\xi^6 + (a+c-b^2)\xi^4 + (1+ad)\xi^2 + d}}{1+a\xi^2}.$$

Hence, we have

$$\begin{aligned} |\widehat{w}(\xi,t)| &\leqslant C\int_0^t |\exp[\frac{-b\xi^2}{1+a\xi^2}(t-\tau)]\widehat{f}(\xi,t)|\,d\tau \\ &\leqslant C[\int_0^t |\widehat{f}(\xi,t)|^2 d\tau]^{\frac{1}{2}}. \end{aligned} \quad (29)$$

Using inequality (29) and the Parseval inequality leads to

$$\begin{aligned} &\int_{-\infty}^{+\infty} |\widehat{w}(\xi,t)|^2\,d\xi \leqslant \int_{-\infty}^{+\infty}\int_0^t |\widehat{f}(\xi,t)|^2\,d\tau d\xi \\ &\leqslant C\int_0^t \|\widehat{f}(\xi,t)\|_{L^2}^2\,d\tau \\ &\leqslant C\int_0^t \|w(x,t)(u^{(1)}(x,t)+u^{(2)}(x,t))\|_{L^2}^2\,d\tau \\ &\leqslant C\int_0^t c_\tau \|w(x,t)\|_{L^2}^2\,d\tau. \end{aligned}$$

By Growall's inequality, we get $w(x,t)=0$ (in L^2). Using the continuity of functions $u^1(x,t)$ and $u^2(x,t)$ results in $u^1(x,t)=u^2(x,t)$. It completes the proof of uniqueness.

3.3. Long time asymptotics

In order to find the long time behavior of the constructed solution, we firstly determine a subtle asymptotic $\widehat{u}_1(t)$. Since $\widehat{u}_1(t) = \sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{\xi}_1^{(N)}(t)$, from (18) and (19), we get

$$\widehat{\xi}_1^{(0)}(t) = e^{-\frac{b}{1+a}t}[A^0\cos(\sigma t) + B^0\sin(\sigma t)],$$

$$\widehat{\xi}_1^{(N)}(t) = e^{-\frac{b}{1+a}t}\,[A^N\cos(\sigma t)]+[B^N\sin(\sigma t)], \quad (30)$$

where

$$\begin{aligned} A^{(0)} &= \varepsilon\widehat{\varphi}_1, \\ B^{(0)} &= \frac{\varepsilon}{\sigma}(\frac{b}{1+a}\widehat{\varphi}_1 + \widehat{\psi}_1), \\ \sigma &= \frac{\sqrt{ac+a+c-b^2+1+ad+d}}{1+a}, \\ A^{(N)} &= \frac{\beta-h}{\sigma(1+a)}\int_0^t e^{\frac{b}{1+a}\tau}\sin(\sigma\tau)S_{u_n}(\tau)d\tau, \\ B^{(N)} &= \frac{\beta-h}{\sigma(1+a)}\int_0^t e^{\frac{b}{1+a}\tau}\cos(\sigma\tau)S_{u_n}(\tau)d\tau, \\ S_{u_n}(t) &= \sum_{g=1}^{\infty}\sum_{j=1}^{N}\widehat{\xi}_{1+g}^{j-1}(t)\widehat{\xi}_g^{(N-j)}(t) N \geqslant 1, \end{aligned}$$

where $\widehat{\xi}^{(j)}(t)$, $j=0,1,\dots N-1$ are defined by (18) and (19).

For $n\geqslant 2, N\geqslant 1$, using a method similar to that used in [8], it follows from (22) that there exists a positive number η $(0<\eta<\frac{3}{1+4a})$ such that

$$|R_A^{(N)}(t)| \leqslant Ce^{\frac{-\eta bt}{1+a}}, \qquad |R_B^{(N)}(t)| \leqslant Ce^{\frac{-\eta bt}{1+a}}. \quad (31)$$

Hence, as t is sufficiently large, we have proved that

$$2i\widehat{u}_1(t) = e^{\frac{-bt}{1+a}}[A\cos(\sigma t) + B\sin(\sigma t)], \quad (32)$$

where

$$A = 2i\sum_{N=0}^{\infty}\varepsilon^{N+1}A^{(N)},$$
$$B = 2i\sum_{N=0}^{\infty}\varepsilon^{N+1}B^{(N)},$$

in which $A^{(N)}$ and $B^{(N)}$ are defined by (30), and the series A and B converge absolutely and uniformly for $\varepsilon\in[0,\varepsilon_0]$. Now, we have

$$u(x,t) = 2i\widehat{u}_1(t)\sin x + R_u(x,t), \quad (33)$$

where

$$R_u(x,t) = 2i\sum_{n=2}^{\infty}\sin nx\sum_{N=1}^{\infty}\varepsilon^{N+1}\widehat{\xi}_n^{(N)}(t).$$

Using inequality (22) results in

$$\begin{aligned} &|R_u(x,t)| \leqslant \exp[\frac{-(1+\eta)bt}{1+a}] \\ &\times\sum_{N=1}^{\infty}C^N\varepsilon^{N+1}(N+1)^{-2}\sum_{n=2}^{\infty}n^{-6}, \\ &\leqslant C\exp[\frac{-(1+\eta)bt}{1+a}]. \end{aligned} \quad (34)$$

It derives from (32), (33) and (34) that formula (6) is valid. Thus, the proof of the theorem is complete.

4. Conclusions

In conclusion, we would like to return the issue of the smallness of initial data associated in problem (4). Although this assumption is needed to show the global-in-time existence for the problem, some equations admit global-in-time solutions for large initial data. Such solutions may have time decay due to dissipation or dispersion. It is the time decay that solutions will become small beginning from some $t=T$ and the asymptotics will be valid for the large initial data as t tends to infinite.

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