Fully Implicational Triple I Reasoning Method on Linguistic Truth-valued Lattice Implication Algebra

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Abstract

Fully implicational Triple I method is one of important fuzzy reasoning methods. In this paper, we discuss fully implicational triple I reasoning method on a linguistic truth-valued lattice implication algebra L_{18}, we define L_{18}-type α-triple I rule FMP and L_{18}-type α-triple I rule FMT, and give the corresponding algorithm formulas, respectively. Finally, the conclusions on the recovery properties of the two algorithms are obtained.

Keywords: Triple I reasoning method, Fuzzy logic, Linguistic truth-valued, Lattice implication algebra

1. Introduction

It is well known that fuzzy reasoning plays an important role in fuzzy control theory, and the methodology of its differs from the methodology of artificial intelligence. In the study of fuzzy control theory, many fuzzy reasoning methods have been proposed(see.e.g.[2]-[4]); however, there exists a gap between these methodology and artificial intelligence. Artificial intelligence emphasized symbolic manipulation and has rooted in logic, automated deduction using syntactic tools, and has very much neglected anything pertaining to "number crunching". On the contrary, most of proposed fuzzy reasoning methodology have been right away addressing methodology based on non-linear optimization.

To reduce the above mentioned gap,in [5], Wang proposed Triple I method for solving the problems of fuzzy modus ponens (briefly, FMP) and fuzzy modus tollens (briefly, FMT) based on the corresponding fuzzy logic. Since then this fuzzy reasoning methodology has been extensively investigated by several researchers (see e.g. [6]-[9]). In [10], Wu et.al. discussed triple I algorithm on complete residuated lattice. In [11], Wang discussed the formalized theory of general fuzzy reasoning through generalizing triple I method to multiple I method.

In recent years, linguistic variable(or terms) have been extensively investigated in approximate reasoning, fuzzy logic and decision-making theory and it will certainly increase in importance in the future. In 1970s, Zadeh [12] presented and developed the theory of approximate reasoning based on linguistic variable and fuzzy logic. In Zadeh's view of fuzzy logic, the truth-values are linguistic. In [13], Herrera et al. investigated aggregation of linguistic information, presented the linguistic aggregation operators LOWA and LWA, and applied them to decision-making and internet information processing. Herrera et al. also presented a processing model for 2-tuple linguistic information, which has been applied to fuzzy inference and fuzzy decision-making. Since 1990, in ([14]-[18]), Ho et al . has established the hedge algebra to process linguistic terms, and introduced a measure function to describe "if-then" inference rule. However, due to the absences of implication operator transferring truth values, and so based on hedge algebra, it is difficult to establish corresponding algebraic logic as well as fuzzy inference by computing with words.

In order to study uncertainty reasoning and automatic reasoning with linguistic terms, whose foundation is lattice-valued logic system based on lattice implication algebras, by defining the set of basic linguistic truth values and the set of modifiers, and also defining partially orderings on them according to common sense, Xu [19] established the notion of linguistic truth-valued lattice implication algebra L_{18} and discussed its properties in 2006, Xu considers that it is highly necessary to establish suitable logic foundation based on linguistic truth values for reasoning with words.

2. Preliminaries

In this section, for the purpose of reference, we present some basic information about linguistic truth-valued lattice implication algebra and fully implicational triple I method for fuzzy reasoning.

In [5], Wang proposed Triple I method for solving the problem of FMP based on the corresponding fuzzy logic. Consider the most fundamental form of fuzzy reasoning, i.e., FMP with the form

$$\begin{array}{lll} \textit{suppose that } A(x) \to B(y) & & -\textit{major premise} \\ \quad \textit{and given } A^*(x) & & -\textit{minor premise} \\ \hline \quad \textit{calculate} & B^*, & \cdots \textit{conclusion}, \end{array} \quad (1)$$

and FMT with the form

$$\begin{array}{lll} \textit{suppose that } A(x) \to B(y) & & -\textit{major premise} \\ \quad \textit{and given} \quad B^*(x) & & -\textit{minor premise} \\ \hline \quad \textit{calculate } A^* & , & \cdots \textit{conclusion}, \end{array} \quad (2)$$

where $A(x), A^*(x)$ and $B(y), B^*(y)$ are subsets of X and Y, respectively. According to triple I rule FMP in [5], B^* is the smallest fuzzy subset of Y such that the major premise $A(x) \to B(y)$ fully sustains the fact that the minor premise $A^*(x)$ sustains the conclusion B^*, i.e., $(A(x) \to B(y)) \to (A^*(x) \to B^*(y))$ is a "tautology" with respect to X and Y. Especially, if we assume that $\to$ satisfies that $a \to b = 1$ if and only if $a \leqslant b$, then the following condition holds

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) = 1, \quad (3)$$

for any $x \in X, y \in Y$, and B^* is the smallest fuzzy subset of Y satisfying (3). Similarly, according to triple I rule FMT in [5], A^* is the largest fuzzy subset of X satisfying (4),

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) = 1, \quad (4)$$

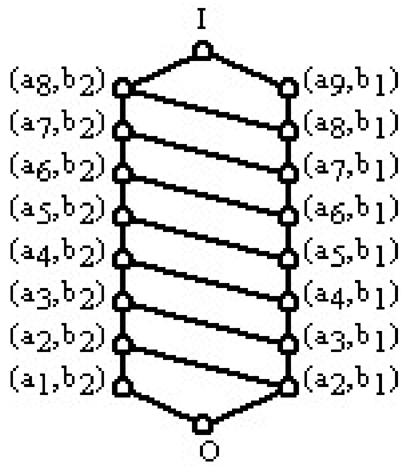
for any $x \in X, y \in Y$,

Definition 2.1. *[1] A bounded lattice $(L, \vee, \wedge, O, I)$ with order-reversing involution $'$ and a binary operation $\to$ is called a lattice implication algebra if it satisfies the following axioms:*
(I_1) $x \to (y \to z) = y \to (x \to z)$,
(I_2) $x \to x = I$,
(I_3) $x \to y = y' \to x'$,
(I_4) $x \to y = y \to x = I \Rightarrow x = y$,
(I_5) $(x \to y) \to y = (y \to x) \to x$,
(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,
for all $x, y, z \in L$.

In a lattice implication algebra L, for all $x, y, z \in L$, the following hold (see [1]):

(i) $I \to x = x$ and $x \otimes O = x'$;
(ii) $x \leqslant y \Leftrightarrow x \to y = I$;
(iii) $x \vee y = (x \to y) \to y$;
(iv) $x \to y \geqslant x' \vee y$;
(v) $(L, \vee, \wedge)$ is a distributive lattice;
(vi) $x \leqslant y$ implies $y \to z \leqslant x \to z$ and $z \to x \leqslant z \to y$.

Let {false, true} be the basic linguistic truth-valued set with ordering relation {false < true} in common sense, and {slightly, little, somewhat, closely, basically, more, most, very, utterly} be the basic modifier set with ordering relation {slightly < little < somewhat < closely < basically < more < most < very < utterly} in common sense, by using basic modifiers to modify the basic linguistic truth-valued, then we get a basic linguistic truth-valued set L_{18} = {O = (a_1, b_1) = utterly false, I = (a_9, b_2) = utterly true, (a_1, b_2) = slightly true, (a_2, b_2) = little true, (a_3, b_2) = somewhat true, (a_4, b_2) = closely true, (a_5, b_2) = basically true, (a_6, b_2) = more true, (a_7, b_2) = most true, (a_8, b_2) = very true, (a_9, b_1) = slightly false, (a_8, b_1) = little false, (a_7, b_1) = somewhat false, (a_6, b_1) = closely false, (a_5, b_1) = basically false, (a_4, b_1) = more false, (a_3, b_1) = most false, (a_2, b_1) = very false}. According to the intuitive meaning of synthesized linguistic truth-valued terms, further, we define a semantically partial ordering on L_{18} as follows,



and so we obtain an algebraic lattice $(L_{18}, \vee_{18}, \wedge_{18})$. In [19], the order-reversing involution ' and the implication operation $\rightarrow_{18}$ on $(L_{18}, \vee_{18}, \wedge_{18})$ have been defined as follows,

$$(a_i, b_j)'^{18} = (a_{10-i}, b_{3-j}),$$

$$(a_{i_1}, b_{j_1}) \rightarrow_{18} (a_{i_2}, b_{j_2}) =$$

$$\begin{cases} I & i_1 \leqslant i_2 \text{ and } j_1 \leqslant j_2 \\ (a_9, b_1) & i_1 \leqslant i_2 \text{ and } j_1 > j_2 \\ (a_{9-i_1+i_2}, b_2) & i_1 > i_2 \text{ and } j_1 \leqslant j_2 \\ (a_{9-i_1+i_2}, b_1) & i_1 > i_2 \text{ and } j_1 > j_2 \end{cases}$$

where $i = 1, 2, \cdots, 9$ and $j = 1, 2$. It is easy to validate that $(L_{18}, \vee_{18}, \wedge_{18}, '^{18}, \rightarrow_{18})$ is a lattice implication algebra, and is called as an linguistic truth-valued lattice implication algebra.

3. Algorithm of triple I reasoning on linguistic truth-valued lattice implication algebra

In this section, we define L_{18}-type α-triple I rule fuzzy modus ponens (briefly, FMP) and L_{18}-type α-triple I rule fuzzy modus tollens (briefly, FMT) and give the algorithm formulas of this two rules. We also discuss the recovery properties of L_{18}-type α-triple I rule FMP and L_{18}-type α-triple I rule FMT.

Definition 3.1. *Given a nonempty set X, an arbitrary mapping $f : X \rightarrow L_{18}$ is called a L_{18}-fuzzy subset of X (or a L_{18}-fuzzy set in X). The collection of all L_{18}-fuzzy subset of X denotes by $\mathscr{F}(X)$.*

Definition 3.2. *(L_{18}-type α-triple I rule FMP)Let X, Y be two nonempty set, $A, A^* \in \mathscr{F}(X)$, $B, B^* \in \mathscr{F}(Y)$, then B^* in (1) is the smallest L_{18}-fuzzy subset in $\mathscr{F}(Y)$ such that for any $x \in X, y \in Y$*

$$(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow B^*(y)) \geqslant \alpha. \quad (5)$$

Remark 3.1. *If $\alpha = I$, we call L_{18}-type α-triple I rule FMP as L_{18}-type triple I rule FMP.*

Theorem 3.1. *(algorithm formula of L_{18}-type α-triple I rule FMP)Let X, Y be two nonempty set, $A, A^* \in \mathscr{F}(X)$, $B, B^* \in \mathscr{F}(Y)$, then (1) can be computed as follows, for $y \in Y$,*

$$B^*(y) = \bigvee_{x \in X} ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))'. \quad (6)$$

Proof. we first prove that (6) satisfies the condition (5). In fact, for any $x \in X, y \in Y$,

$$\begin{aligned}
&\alpha \rightarrow ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \bigvee_{x \in X} ((A(x) \\
&\quad \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))')) = \\
&(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow (\alpha \rightarrow \bigvee_{x \in X} ((A(x) \\
&\quad \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))')) = \\
&(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \bigvee_{x \in X} (\alpha \rightarrow ((A(x) \\
&\quad \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))')) = \\
&(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \bigvee_{x \in X} (((A(x) \rightarrow \\
&\quad B(y)) \rightarrow (A^*(x) \rightarrow \alpha')) \rightarrow \alpha')) = \\
&(A(x) \rightarrow B(y)) \rightarrow \bigvee_{x \in X} (A^*(x) \rightarrow (((A(x) \rightarrow \\
&\quad B(y)) \rightarrow (A^*(x) \rightarrow \alpha')) \rightarrow \alpha')) = \\
&(A(x) \rightarrow B(y)) \rightarrow \bigvee_{x \in X} (((A(x) \rightarrow B(y)) \rightarrow \\
&\quad (A^*(x) \rightarrow \alpha')) \rightarrow (A^*(x) \rightarrow \alpha')) = \\
&(A(x) \rightarrow B(y)) \rightarrow \bigvee_{x \in X} ((A(x) \rightarrow B(y)) \vee \\
&\quad (A^*(x) \rightarrow \alpha')) = I,
\end{aligned}$$

hence,

$$((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \bigvee_{x \in X} ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))')) \geqslant \alpha.$$

On the other hand, if $D(y) \in \mathscr{F}(Y)$ satisfies the condition (4), then

$$\alpha \rightarrow ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow D(y))) = I.$$

Since

$$\begin{aligned}
&((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))' \rightarrow D(y) = \\
&D(y)' \rightarrow ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha')) = \\
&(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow (D(y)' \rightarrow \alpha')) = \\
&(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow (\alpha \rightarrow D(y))) = \\
&(A(x) \rightarrow B(y)) \rightarrow (\alpha \rightarrow (A^*(x) \rightarrow D(y))) = \\
&\alpha \rightarrow ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow D(y))) = I,
\end{aligned}$$

it follows that

$$\bigvee_{x \in X} ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))' \leqslant D(y).$$

As a result,

$$B^* = \bigvee_{x \in X} ((A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow \alpha'))'.$$

$\square$

Corollary 3.1. *(algorithm formula of L_{18}-type triple I rule FMP) Let X, Y be two nonempty set, $A, A^* \in \mathscr{F}(X)$, $B, B^* \in \mathscr{F}(Y)$, then (1) can be computed as follows, for $y \in Y$,*

$$B^*(y) = \bigvee_{x \in X} ((A(x) \to B(y)) \to A^*(x)')'. \quad (7)$$

In the following, we shall discuss the recovery property of algorithm formula of L_{18}-type $\alpha-$triple I rule FMP. In (5), let $A^*(x) = A(x), x \in X$, we have that for any $y \in Y$,

$$\begin{aligned} B^*(y) &= \bigvee_{x\in X} ((A(x) \to B(y)) \to (A(x) \to \alpha'))' \\ &= \bigvee_{x\in X} ((A(x) \wedge B(y)) \to \alpha')' \\ &= \bigvee_{x\in X} (\alpha \to (A'(x) \vee B'(y)))' \\ &= \bigvee_{x\in X} ((\alpha \to B'(y)) \vee (\alpha \to A'(x)))' \\ &= \bigvee_{x\in X} ((\alpha \to B'(y))' \wedge (\alpha \to A'(x))') \\ &= (\alpha \to B'(y))' \wedge \bigvee_{x\in X} (\alpha \to A'(x))' \\ &= (\alpha \to B'(y))' \wedge (\bigwedge_{x\in X} (\alpha \to A'(x)))' \\ &= (\alpha \to B'(y))' \wedge (\alpha \to \bigwedge_{x\in X} A'(x))' \\ &= (\alpha \to B'(y))' \wedge (\alpha \to (\bigvee_{x\in X} A(x))')' \\ &= ((\alpha \to B'(y)) \vee (\alpha \to (\bigvee_{x\in X} A(x))'))' \\ &= (\alpha \to (B'(y) \vee (\bigvee_{x\in X} A(x))'))' \\ &= (\alpha \to (B(y) \wedge (\bigvee_{x\in X} A(x)))')'. \end{aligned}$$

According to the properties of lattice implication algebras, we can obtain the following conclusion about the recovery property of the algorithm.

Theorem 3.2. *For the algorithm formula of L_{18}-type α-triple I rule FMP, if for any $y \in Y$, $\alpha \to ((\bigvee_{x\in X} A(x)) \to B(y)) = I$ and $\alpha \vee B'(y) = I$, then when $A^*(x) = A(x), x \in X$, we have $B^*(y) = B(y), y \in Y$.*

Proof. According to the above discussion, the proof is easy. □

Corollary 3.2. *For the algorithm formula of L_{18}-type α-triple I rule FMP, if there exists $a \in X$ such that $A(a) = I$ and $\alpha = (a_9, b_1)$, and for any $y \in Y$, $B(y) \in \{(a_i, b_1); i = 1, 2, \cdots, 9\}$, then when $A^*(x) = A(x), x \in X$, we have $B^*(y) = B(y), y \in Y$.*

Corollary 3.3. *For the algorithm formula of L_{18}-type triple I rule FMP, if there exists $a \in X$ such that $A(a) = I$, then when $A^*(x) = A(x), x \in X$, we have $B^*(y) = B(y), y \in Y$.*

Corollary 3.4. *For the algorithm formula of L_{18}-type triple I rule FMP, if for any $y \in Y$, $B(y) \leqslant \bigvee_{x\in X} A(x)$, then when $A^*(x) = A(x), x \in X$, we have $B^*(y) = B(y), y \in Y$.*

Definition 3.3. *(L_{18}-type α-triple I rule FMT)Let X, Y be two nonempty set, $A, A^* \in \mathscr{F}(X)$, $B, B^* \in \mathscr{F}(Y)$, then A^* in (2) is the largest L_{18}-fuzzy subset in $\mathscr{F}(X)$ such that for any $x \in X, y \in Y$*

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \geqslant \alpha. \quad (8)$$

Remark 3.2. *If $\alpha = I$, we call L_{18}-type α-triple I rule FMT as L_{18}-type triple I rule FMT.*

Theorem 3.3. *(algorithm formula of L_{18}-type α-triple I rule FMT)Let X, Y be two nonempty set, $A, A^* \in \mathscr{F}(X)$, $B, B^* \in \mathscr{F}(Y)$, then (2) can be computed as follows, for $x \in X$,*

$$A^*(x) = \bigwedge_{y\in Y} ((A(x) \to B(y)) \to (\alpha \to B^*(y))). \quad (9)$$

Proof. Similar to the proof of Theorem 3.2. □

Corollary 3.5. *(algorithm formula of L_{18}-type triple I rule FMT) Let X, Y be two nonempty set, $A, A^* \in \mathscr{F}(X)$, $B, B^* \in \mathscr{F}(Y)$, then (2) can be computed as follows, for $x \in X$,*

$$A^*(x) = \bigwedge_{y\in Y} ((A(x) \to B(y)) \to B^*(x)). \quad (10)$$

In the following, we shall discuss the recovery property of algorithm formula of L_{18}-type $\alpha-$triple I rule FMT. In (5), let $B^*(y) = B(y), y \in Y$, we have that for any $x \in X$,

$$\begin{aligned} A^*(x) &= \bigwedge_{y\in Y} ((A(x) \to B(y)) \to (\alpha \to B(y))) \\ &= \bigwedge_{y\in Y} (\alpha \to (A(x) \vee B(y))) \\ &= \alpha \to \bigwedge_{y\in Y} (A(x) \vee B(y)) \\ &= (\alpha \to A(x)) \vee (\alpha \to \bigwedge_{y\in Y} B(y)). \end{aligned}$$

Therefore, we can obtain the following conclusion about the recovery property of the algorithm formula of L_{18}-type α-triple I rule FMT.

Theorem 3.4. *For the algorithm formula of L_{18}-type α-triple I rule FMT, if for any $x \in X$, $\alpha \to (\bigwedge_{y\in Y} B(y)) \leqslant A(x)$ and $\alpha \vee A(x) = I$, then when $B^*(y) = B(y), y \in Y$, we have $A^*(x) = A(x), x \in X$.*

Corollary 3.6. *For the algorithm formula of L_{18}-type α-triple I rule FMT, if there exists $a \in Y$ such that $B(a) = O$ and $\alpha = (a_9, b_1)$, and for any $x \in X$, $A(x) \in \{(a_i, b_2); i = 1, 2, \cdots, 9\}$, then when $B^*(y) = B(y), y \in Y$, we have $A^*(x) = A(x), x \in X$.*

Corollary 3.7. *For the algorithm formula of L_{18}-type triple I rule FMT, if there exists $a \in Y$ such that $B(a) = O$, then when $B^*(y) = B(y), y \in Y$, we have $A^*(x) = A(x), x \in X$.*

Corollary 3.8. *For the algorithm formula of L_{18}-type triple I rule FMT, if for any $x \in X$, $A(x) \geqslant \bigwedge_{y\in Y} B(y)$, then when $B^*(y) = B(y), y \in Y$, we have $A^*(x) = A(x), x \in X$.*

4. Conclusion

In this present work, we investigate an triple I fuzzy reasoning method based on a linguistic truth-valued lattice implication algebra L_{18}. L_{18}-type α-triple I rule FMP and L_{18}-type α-triple I rule FMT are defined, and the corresponding algorithm formulas are given respectively. Some sufficient conditions which make the algorithm formulas satisfy recovery properties are given.

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