

Fractional Order Generalization of a New Three-scroll Chaotic Chua System and Simulation

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Abstract—Fractional-order differential systems are the generalization of integral-order differential systems that can be regarded as particular cases of the former. A three-scroll integral order chaotic Chua system and its generalization to fractional order are studied so as to construct a new fractional order Chua system. The theoretical analysis and numerical simulation of the new fractional Chua system dynamics are carried out. Fractional order range for system chaos generation is investigated and provided when integral order chaotic system being generalized to the fractional order systems, while the numeric simulation results are stated at the same time. The simulation results show that, with the decrease of fractional order, three-scroll attractor of integral-order system degenerates into two-scroll attractor of fractional-order system, then evolves into single-scroll periodic motion of two-scroll chaotic state, and finally trends towards the equilibrium point, i.e., the system features reverse Hopf bifurcation with which the equilibrium point area of chaotic attractor generation s will transfer.

Keywords—chaos; integral-order calculus, fractional-order calculus; chaotic Chua system; three-scroll attractor

I. INTRODUCTION

As a natural phenomenon widely exists, chaos has been studied by people in many researches. It has also been widely applied in non-linear science, computer science, and confidential communications and a wide range of engineering fields [1-3]. At the same time, recently after studying the application of fractional order calculus theory, people found that the introduction of fractional order calculus operator to chaotic system conduces to more accurate and objective revealing and description of chaotic system [2]. The relationship between fractional order system and integral order system can be regarded as the generalization of the former to the latter, while the later can be taken as the exception of the former. For example, literature [3] conducted a wide research on integral order chaotic system, and literature [2] also performed an analysis and research on many fractional order chaotic systems. However, there are few researches unifying the integral and fractional order chaotic systems together or generalizing integral order chaotic system to fractional ones, not to mention the dynamics characteristics and

dynamics evolution principles thereof. Based on the latter, this paper generalizes the integral order three-scroll chua system comprises of polynomial $ax + bx|x| + cx^3$ to the fractional order to form a new fractional order Chua system. At the same time theoretical analysis and numerical simulation of its dynamics characteristics are performed. When the given integral order chaotic system is generalized to the fractional one, the chaotic fractional order value range will be generated by the system as well as the experimental result of the chaotic systems to the numeric simulation. Results of the experiment showed that with the declination of order, its dynamics mainly evolves from three-scroll attractor of integral order system to two-scroll attractor, and then inclines to the equilibrium point after short periodic state following chaotic state.

II. THEOCRATIC BASIS OF FRACTIONAL ORDER CALCULUS

Different definitions can be gained if fractional order calculus is examined from different perspectives. Literature [2] and [4] provides Grunwald-Letnikov definition, Riemann- Letnikov definition and Caputo definition, wherein Riemann- Letnikov definition is based on the improvement of Grunwald-Letnikov definition and simplifies the counting of fractional order calculus, thus is widely used in engineering. Its expression is given as follows:

Definition of Riemann- Letnikov fractional order calculus is

$$D_{t_0}^q f(t) = D_{t_0}^m J_{t_0}^{m-q} f(t) = \begin{cases} \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-q)} \int_{t_0}^t (t-\tau)^{m-q-1} f(\tau) d\tau \right], & m-1 < q < m \\ \frac{d^m}{dt^m} f(t), & q = m \end{cases} \quad (1)$$

in which, $q \in R^+$, D^q is the q order differential operator, and $\Gamma(\bullet)$ is Gamma function.

In accordance with the definition of equation (1)

$$D_{t_0}^q J_{t_0}^q f(t) = f(t), \text{ Laplace transform of the both sides of the equation is performed, and the fractional order calculus complex frequency domain within in } S$$

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}} \right]_{t=0}$$

transform domain is defined as:

(for all q) (2)

III. BRIEF INTRODUCTION TO INTEGRAL ORDER CHUA SYSTEM COMPRISING POLYNOMIAL $ax + bx|x| + cx^3$

For the Chua system[3] generated from $ax + bx|x| + cx$, its equation is:

$$\frac{dx}{dt} = \alpha[y - (ax + bx|x| + cx^3)] \quad (3a)$$

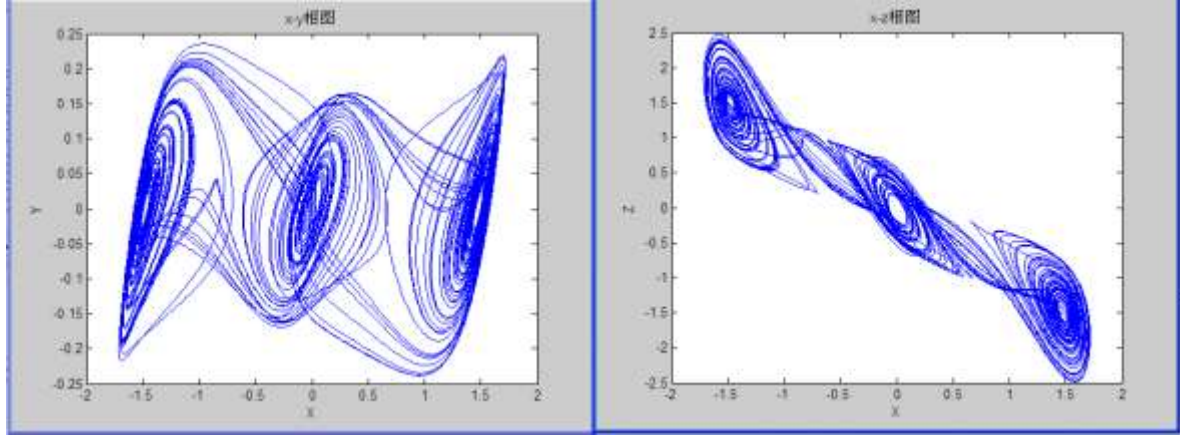


Figure 1. Simulation results of three-scroll Chua System Left: X-Y plane phase, Right: x-y-z space phase

It can be deduced from Fig. 1 that the system will generate three-scroll chaotic attractors which form scrolls at the domains corresponding to the three equilibrium points E0, E3 and E4, while the domains corresponding to E1 and E2 are key bands.

A. Researches on the dynamics of new fractional order calculus Chuan system comprising polynomial $ax + bx|x| + cx^3$

The three-scroll Chua system generated from $ax + bx|x| + cx$ will be generalized to the fractional order chaotic system whose dynamic equation can be described as:

$$\frac{d^q x}{dt^q} = \alpha[y - h(x)] \quad (4a)$$

$$\frac{dy}{dt} = x - y + z \quad (3b)$$

$$\frac{dz}{dt} = -\beta y \quad (3c)$$

where α and β are the control differences. when $\alpha = 12.8$, $\beta = 19.1$, $a=0.45$, $b=-1$, $c=0.47$, the equilibrium point of system(3) on x is $x_0 = 0$, $x_{1,2} = \pm 0.6463$, $x_{3,4} = \pm 1.4813$. Assuming the equilibrium point of the corresponding system (3)

as j ($j=0,1,2,3,4$). Computer numeric simulation results are shown in Fig. 1.

$$\frac{d^q y}{dt^q} = x - y + z \quad (4b)$$

$$\frac{d^q z}{dt^q} = -\beta y \quad (4c)$$

In system (4), q is the order number of fractional order system; α and β are the control parameter.

Apparently, if $h(x) = \frac{x - 2x^3}{7}$, $\beta = 100/7$, then it is the

traditional fractional order Chuan system studied in [5]. In this research, there is a fractional order chaotic system different from those in [5], that is, letting $h(x) = ax + bx|x| + cx^3$, other parameters of the system in consistence with equation (3), the dynamics features of system (3) generalized to system (4) are studied.

Firstly the stability of the fractional order non-linear system is studied. Most definitions of the non-linear system stability is based on the fundamental views of A.M.Lyapunov[6]. However, as mentioned in (7), exponential stability can not be used for studying the asymptotic stability of the fractional order system. For fractional order non-linear system, Matignon's fractional order system stability and the necessary and sufficient conditions of fractional order system stability, which are given as follows, are introduced:

Lemma1[7][8], for fractional order autonomous system,

$$D^q X = AX$$

$$X(0)=X^0(7)$$

in which, $0 < q < 1$, $X \in \mathbb{R}^n$ ($n \in \mathbb{N}$), $A \in \mathbb{R}^{n \times n}$,

then:

If and only if for any eigenvalue λ , $|\arg(\lambda)| > \frac{q\pi}{2}$, then system (7) is asymptotically stable.

If and only if for any eigenvalue λ , $|\arg(\lambda)| \geq \frac{q\pi}{2}$, then system (7) is stable.

Apparently got fractional order system in a chaotic state, there should be at least one eigenvalue in unstable domain, that is, the system is inclined to be in a chaotic state only if the system has equilibrium points in unstable

TABLE 1. The value range of q in different equilibrium point

equilibrium point	eigenvalue	$\text{Arg}(\lambda_i)$	The value range of q in stable conditon	The value range of q in instable conditon
E_0	$\lambda_0 = -7.2053$	$\text{Arg}(\lambda_0) = \pi$	$q < 0.9637$	$0.9637 < q < 1$
$E_{1,2}$	$\lambda_{1,2} = 0.2227 \pm j3.9012$ $\lambda_0 = 4.5596$	$\arg(\lambda_{1,2}) = \pm 1.5138$ $\text{Arg}(\lambda_0) = 0$	None	$0 < q < 1$
$E_{3,4}$	$\lambda_{1,2} = -1.1565 \pm j3.5018$ $\lambda_0 = -8.7310$ $\lambda_{1,2} = 0.1453 \pm j4.0318$	$\text{Arg}(\lambda_0) = \pi$ $\arg(\lambda_{1,2}) = \pm 1.5348$	$0.9771 < q < 1$	$0 < q < 0.9771$

To consider the above value range of q comprehensively, when $0 < q < 0.9637$, $E_{0,3,4}$ is the asymptotically stable point, $E_{1,2}$ instable equilibrium points, in which the system may reveal chaotic state; when $0.9637 < q < 0.9771$, all the equilibrium points are instable except $E_{3,4}$, and the system may be in a chaotic state. If $0.9771 < q < 1$, all the equilibrium points of the system are instable, then the system may maintain a chaotic state, but this still depends on the system

domain. As for the dynamical system shown in equation (4), its equilibrium point should be studied first so as to determine the system stability or the conditions for chaos generation. Plugging $h(x) = ax + bx|x| + cx^3$ and control parameter into equation (4), the equilibrium points x_i ($i=0,1,2,3,4$) on x of the system respectively are:

$$x_0 = 0 \quad (5a)$$

$$x_{1,2} = \pm(-b - \sqrt{b^2 - 4ac})/(2c) = \pm 0.6463 \quad (5b)$$

$$x_{3,4} = \pm(-b - \sqrt{b^2 - 4ac})/(2c) = \pm 1.4813 \quad (5c)$$

Furthermore, equation (4) is linearized at the equilibrium point, acquiring the Jacobi matrix of the system, that is,

$$J(O) = \begin{bmatrix} -\frac{\alpha \partial h(x_i)}{\partial x_i} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \quad (i=0,1,2,3,4) \quad (6)$$

From (6), it can be inferred that

divergence operator which if is less than zero, the system is most likely to be in chaotic state, or otherwise will be diverged at disequilibrium points.

From the analysis above, when $0.9771 < q < 1$, all the equilibrium points of system (4) are inclined to diverging. The existence of chaotic attractor in this domain depends on the dissipative property of the system. After examining

system (4), we get

$$\nabla \bullet F = \frac{\partial}{\partial x} \left(\frac{d^q x}{dt^q} \right) + \frac{\partial}{\partial y} \left(\frac{d^q y}{dt^q} \right) + \frac{\partial}{\partial z} \left(\frac{d^q z}{dt^q} \right) \Big|_{E_j (j=0,1,2,3,4)} < 0 \quad (7)$$

The system is dissipative. When $t \rightarrow \infty$, each small volume element containing system trajectories converges to 0 exponentially, eventually resulting in all the trajectories being restricted in a collection of 0 volume and asymptotically fixed in one attractor, indicating the existence of the system chaotic attractor.

After analyzing the existence of chaos in system (4), is it possible to forecast the changes of the dynamics of the system generalized to fractional order chaotic system (4) giving the constant control difference and system topology. A numeric simulation is performed in the following content.

B. Numeric simulation of fractional order Chua system comprising polynomial $ax + bx|x| + cx$

1) Analysis of simulation environment

Presently there are mainly two methods for simulation calculation of fractional order system: predictor-corrector method and time domain- complex frequency domain conversion method. However, some literature [9] proved that simulation based on time domain- complex frequency domain conversion method may sometimes result in wrong conclusions and pointed out that some conclusions drawn by other researchers who have applied this method are incorrect[10]. In this research, predictor-corrector method[11] is applied for numeric simulation of system (5). In accordance with predictor-corrector algorithm[11], simulation step size $h=0.01$, simulation numeric point $N=4000$, simulation system initial value $(x_0, y_0, z_0)=(0.0021, 0.00321, 0.00123)$.

2) Simulation results

When $q = 0.98$, the simulation results are shown in Fig. 2.

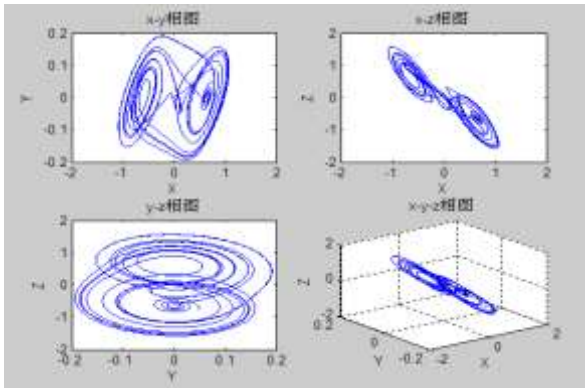


Figure 2. Attractor phase diagram generated by system (5) when $q=0.98$

When $q = 0.97$, the simulation results are shown in Fig. 3.

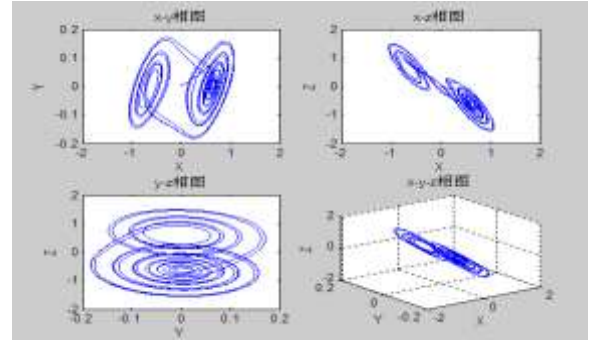


Figure 3. Attractor phase diagram generated by system (5) when $q=0.966$

When $q = 0.94$, the simulation results are shown in Fig. 4.

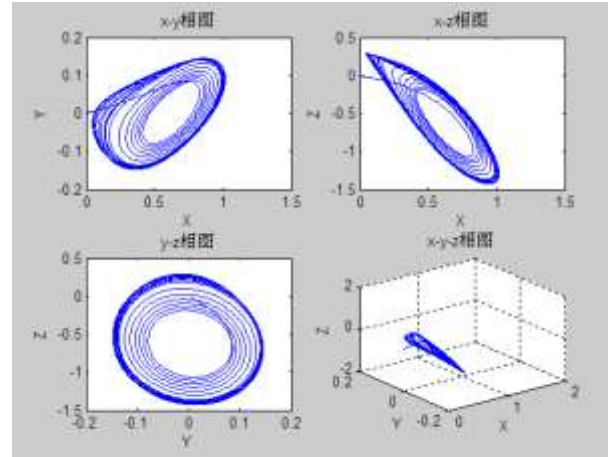


Figure 4. Attractor phase diagram generated by system (5) when $q=0.9637$

When $q = 0.9$, the simulation results are shown in Fig. 5.

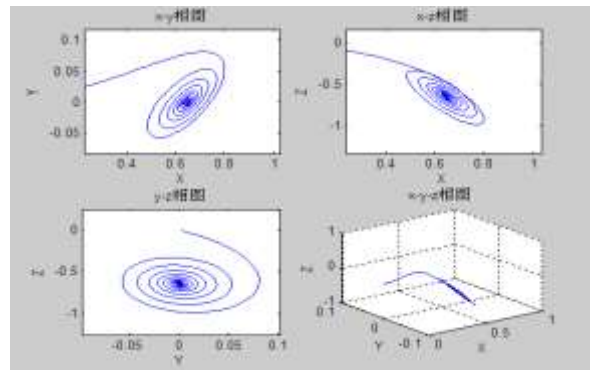


Figure 5. Attractor phase diagram generated by system (5) when $q=0.9$

After repeated simulation experiments, the following conclusions can be drawn:

It can be discovered by the the numerical simulation that for the fractional order system (4), any value of q within the domain $(0.9637-1)$ will result in chaos of system. Moreover, with the reduction of order, the system will firstly degrade from three-scroll chaotic state to two-scroll chaotic state, then evolves to single-scroll cyclical movement and eventually converges to equilibrium point

$E_{1,2}$, indicating the reverse Hopf bifurcation process of the system. At the same time, q , as the important control parameters of the system, can either control the system for generation of chaos or inhibit the system chaos, thus is of great theoretical and practical values. For example, NandKumaran and Kuruvilla once successfully inhibited the chaos in semiconductor lasers by reverse bifurcation [12,13].

IV. CONCLUSION

A new fractional order Chua system is constructed in this paper through polynomial $ax + bx|x| + cx$. After linear treatment of the system, conditions for system chaos are deducted by stability theorem of fractional nonlinear system. Simulation of the new fractional order Chua systems are performed by numeric simulation, and a comparison is made between the simulation results and the integral order chaotic system comprising polynomial $ax + bx|x| + cx$. When the integral order chaotic system is generalized to fractional order system, the two reveals the same dynamics features, yet there are also some distinct differences. In this research it is discovered that when the integral order three-scroll attractor is generalized to fractional order, the system shows revers Hopf bifurcation process and complicated dynamics as the order decreases[14-15].

Furthermore, When we studied the chaotic system, the fractional order system and can be regarded as the generalization of the integral order system. As the important control parameter, the fractional parameter q can not only control generation and suppression of chaotic behavior ,but also control complexity of the change of the chaotic system. At the same time, q can change the system topology and other control parameters at the least. Our analysis results provide a convenient method of design and implementation of the chaotic system . In addition, this results can applies to the field of communications, control engineering.

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