

# Study on The Symmetry and Conserved Quantities for Hamilton Systems

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**Abstract**—The form invariance and the Lie symmetry are defined for Hamilton systems. A relation between the form invariance and the Lie symmetry is derived. The Hojman conserved quantity is constructed by using the generators of Lie symmetry. An approach to find Hojman conserved quantities in terms of the form invariance is presented. An example is given to illustrate the application of the results.

**Keywords**- mechanical systems; form invariance; Lie symmetry; Hojman conserved quantity; Hamilton system

## I. INTRODUCTION

The symmetry of mechanical systems is one of the most important subjects in physics, which has been investigated for a long time. The symmetry of a mechanical system will be useful for integrating the equations of motion, since it is closed to invariants of the system. The modern approaches of finding invariants are mainly in terms of the Noether symmetry, the Lie symmetry and the Mei form invariance<sup>[1-3]</sup>. In 1992, Hojman presented a new conservation law without using either Lagrangian or Hamilton solely based on the existence of symmetries<sup>[4]</sup>. This direct method has attracted much attention<sup>[5-10]</sup>. In this paper, we study Hojman conserved quantities by using the Mei form invariance for the Hamilton systems.

## II. DEFINITION AND CRITERION OF FORM INVARIANCE FOR HAMILTON SYSTEMS

If the differential equations of motion of a mechanical system can be written in the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad (1)$$

Then such system is called a lagrangian system. Introducing the generalized momentums and the Hamilton

$$P_s = \frac{\partial L}{\partial \dot{q}_s} \quad (2a)$$

$$H = \sum p_s \dot{q}_s - L \quad (2b)$$

Then (1) may be written in the canonical form

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \quad (3a)$$

$$\dot{p}_s = -\frac{\partial H}{\partial q_s} \quad (3b)$$

Where is the Hamilton. introduce the infinitesimal transformations with respect to time, generalized coordinates and generalized momentums

$$t^* = t + \Delta t \quad (4a)$$

$$q_s^*(t^*) = q_s(t) + \Delta q_s \quad (4b)$$

$$p_s^*(t^*) = p_s(t) + \Delta p_s \quad (4c)$$

Or their expansion formula

$$t^* = t + \varepsilon \tau(t, \mathbf{q}, \mathbf{p}) \quad (5a)$$

$$q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \mathbf{q}, \mathbf{p}) \quad (5b)$$

$$p_s^*(t^*) = p_s(t) + \varepsilon \eta_s(t, \mathbf{q}, \mathbf{p}) \quad (5c)$$

Where  $\varepsilon$  is an infinitesimal parameter,  $\tau$ ,  $\xi_s$  and  $\eta_s$  are called infinitesimal generators. Under the infinitesimal transformation (5), Hamilton  $H(t, \mathbf{q}, \mathbf{p})$  becomes  $H(t^*, \mathbf{q}^*, \mathbf{p}^*)$ .

Definition 1: Under the infinitesimal transformation (5), if the canonical (3) keep their form invariant,

$$\dot{q}_s = \frac{\partial H^*}{\partial p_s} \quad (6a)$$

$$\dot{p}_s = -\frac{\partial H^*}{\partial q_s} \quad (6b)$$

Where

$$H^* = H(t^*, \mathbf{q}^*, \mathbf{p}^*) \quad (7)$$

Then such invariance is called a form invariance of the Hamilton systems. Introduce the differential operator of the infinitesimal generators

$$X^{(0)} = \tau \frac{\partial}{\partial t} + \xi_k \frac{\partial}{\partial q_k} + \eta_k \frac{\partial}{\partial p_k} \quad (8)$$

$$X^{(1)} = X^{(0)} + (\dot{\xi}_k - \dot{q}_k \dot{\tau}) \frac{\partial}{\partial \dot{q}_k} + (\dot{\eta}_k + \dot{p}_k \dot{\tau}) \frac{\partial}{\partial \dot{p}_k} \quad (9)$$

Expanding  $H^*$ , one has

$$H^* = H(t, \mathbf{q}, \mathbf{p}) + \varepsilon [X^{(0)}(H)] + O(\varepsilon^2) \quad (10)$$

From (6)–(10), the following criterion can be obtained.

Criterion 1: the infinitesimal transformation (5) is a Mei symmetric transformation of the system(3), if and only if the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy the following conditions

$$\frac{\partial X^{(0)}(H)}{\partial p_s} = 0 \quad (11a)$$

$$\frac{\partial X^{(0)}(H)}{\partial q_s} = 0 \quad (11b)$$

Proof: substituting (10) into (6), using (3), and neglecting the higher infinitesimal terms, (11) will be obtained.

### III. THE HOJMAN CONSERVED QUANTITY FOR HAMILTON SYSTEMS

The basic idea of the Lie symmetry is to keep the equations of motion(3) invariant under the infinitesimal transformations (5). For convenience, equation(3) are rewritten in following form

$$\dot{q}_s = g_s(t, \mathbf{q}, \mathbf{p}) \quad (12a)$$

$$\dot{p}_s = h_s(t, \mathbf{q}, \mathbf{p}) \quad (12b)$$

Where  $g_s = \partial H / \partial p_s$ , and  $h_s = -\partial H / \partial q_s$ .

Definition 2: the infinitesimal transformations (5) is a Lie,s symmetric transformation of the system(3), if and only if there exist function  $\tau$ ,  $\xi_s$  and  $\eta_s$  that satisfy the following determining equations

$$\frac{\bar{d}\xi_s}{dt} - g_s \frac{\bar{d}\tau}{dt} = \tau \frac{\partial g_s}{\partial t} + \xi_k \frac{\partial g_s}{\partial q_k} + \eta_k \frac{\partial g_s}{\partial p_k} \quad (s, k = 1, 2, \dots, n) \quad (13a)$$

$$\frac{\bar{d}\eta_s}{dt} - h_s \frac{\bar{d}\tau}{dt} = \tau \frac{\partial h_s}{\partial t} + \xi_k \frac{\partial h_s}{\partial q_k} + \eta_k \frac{\partial h_s}{\partial p_k} \quad (s, k = 1, 2, \dots, n) \quad (13b)$$

Where

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + g_s \frac{\partial}{\partial q_s} + h_s \frac{\partial}{\partial p_s} \quad (s = 1, 2, \dots, n) \quad (14)$$

In terms of the generators of the Lie symmetry for e(3), the following theorem concerning conserved quantities can be proved.

Theorem 1: the system(3) possesses the following conserved quantity

$$I = \frac{1}{\mu} \frac{\partial(\mu\tau)}{\partial t} + \frac{1}{\mu} \frac{\partial(\mu\xi_s)}{\partial q_s} + \frac{1}{\mu} \frac{\partial(\mu\eta_s)}{\partial p_s} - \frac{\bar{d}}{dt} \tau \quad (s = 1, 2, \dots, n) \quad (15)$$

If the infinitesimal the generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy the following determining (13), and the  $\mu(t, \mathbf{q}, \mathbf{p})$  admits the following equation

$$\frac{\bar{d}}{dt} \ln \mu = -\tau \frac{\partial g_s}{\partial q_s} - \frac{\partial h_s}{\partial p_s} \quad (16)$$

Proof: from (15), one has

$$\begin{aligned} \frac{\bar{d}I}{dt} &= \frac{\bar{d}}{dt} \left( \frac{1}{\mu} \frac{\partial \mu}{\partial t} \right) + \frac{\bar{d}}{dt} \frac{\partial \tau}{\partial t} + \frac{\bar{d}}{dt} \left( \frac{1}{\mu} \frac{\partial \mu}{\partial q_s} \xi_s \right) \\ &+ \frac{\bar{d}}{dt} \frac{\partial \xi_s}{\partial q_s} + \frac{\bar{d}}{dt} \left( \frac{1}{\mu} \frac{\partial \mu}{\partial \eta_s} \right) + \frac{\bar{d}}{dt} \frac{\partial \eta_s}{\partial p_s} - \frac{\bar{d}}{dt} \frac{\partial \tau}{\partial t} \quad (s = 1, 2, \dots, n) \end{aligned} \quad (17)$$

It is straightforward to show that for any  $A(t, \mathbf{q}, \mathbf{p})$

$$\frac{\bar{d}}{dt} \frac{\partial A}{\partial t} = \frac{\partial}{\partial t} \frac{\bar{d}A}{dt} - \frac{\partial g_s}{\partial t} \frac{\partial A}{\partial q_s} - \frac{\partial h_s}{\partial t} \frac{\partial A}{\partial p_s} \quad (s = 1, 2, \dots, n) \quad (18a)$$

$$\frac{\bar{d}}{dt} \frac{\partial \xi_s}{\partial q_s} = \frac{\partial}{\partial q_s} \frac{\bar{d}\xi_s}{dt} - \frac{\partial g_s}{\partial q_k} \frac{\partial \xi_k}{\partial q_s} - \frac{\partial h_s}{\partial q_k} \frac{\partial \xi_k}{\partial p_s} \quad (s = 1, 2, \dots, n) \quad (18b)$$

$$\frac{\bar{d}}{dt} \frac{\partial \eta_s}{\partial q_s} = \frac{\partial}{\partial p_s} \frac{\bar{d}\eta_s}{dt} - \frac{\partial g_s}{\partial p_k} \frac{\partial \eta_k}{\partial q_s} - \frac{\partial h_s}{\partial p_k} \frac{\partial \eta_k}{\partial p_s} \quad (s = 1, 2, \dots, n) \quad (18c)$$

And substituting (18) into (17) and using (13), we obtain

$$\begin{aligned} \frac{\bar{d}I}{dt} &= \frac{\bar{d}}{dt} \left( \frac{1}{\mu} \frac{\partial \mu}{\partial t} \tau \right) + \frac{\bar{d}}{dt} \left( \frac{1}{\mu} \frac{\partial \mu}{\partial q_s} \xi_s \right) + \frac{\bar{d}}{dt} \left( \frac{1}{\mu} \frac{\partial \mu}{\partial p_s} \eta_s \right) + \left( \frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} \right) \frac{\bar{d}\tau}{dt} \\ &+ \frac{\partial}{\partial t} \left( \frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} \right) \tau + \frac{\partial}{\partial q_k} \left( \frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} \right) \xi_k + \frac{\partial}{\partial q_k} \left( \frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} \right) \eta_k \quad (s = 1, 2, \dots, n) \end{aligned} \quad (19)$$

Finding the partial differential of (16) with respect to  $t$ ,  $q_k$  and  $p_k$  respectively, and substituting the results into (19), and using (13), one can get

$$\frac{\bar{d}I}{dt} = 0 \quad (20)$$

By virtue of above theorem, one can easily deduce following corollaries:

Corollary 1: the system (3) possesses the following conserved quantity

$$I = \frac{1}{\mu} \frac{\partial(\mu\xi_s)}{\partial q_s} + \frac{1}{\mu} \frac{\partial(\mu\eta_s)}{\partial p_s} \quad (s = 1, 2, \dots, n) \quad (21)$$

If the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy

$$\frac{\bar{d}\xi_s}{dt} = \frac{\partial g_s}{\partial q_k} \xi_k + \frac{\partial g_s}{\partial p_k} \eta_k \quad (s, k = 1, 2, \dots, n) \quad (22a)$$

$$\frac{\bar{d}\eta_s}{dt} = \frac{\partial h_s}{\partial q_k} \xi_k + \frac{\partial h_s}{\partial p_k} \eta_k \quad (s, k = 1, 2, \dots, n) \quad (22b)$$

And  $\mu(t, \mathbf{q}, \mathbf{p})$  admits the (16).

Corollary 2: the system (4) possesses the following conserved quantity

$$I = \frac{1}{\mu} \frac{\partial(\mu\tau)}{\partial t} + \frac{1}{\mu} \frac{\partial(\mu\eta_s)}{\partial p_s} - \frac{\bar{d}\tau}{dt} \quad (s = 1, 2, \dots, n) \quad (23)$$

If the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy

$$g_s \frac{\bar{d}\tau}{dt} + \tau \frac{\partial g_s}{\partial t} + \eta_k \frac{\partial g_s}{\partial p_k} = 0 \quad (s, k = 1, 2, \dots, n) \quad (24a)$$

$$\frac{\bar{d}\eta_s}{dt} - h_s \frac{\bar{d}\tau}{dt} - \tau \frac{\partial h_s}{\partial t} - \eta_k \frac{\partial h_s}{\partial p_k} = 0 \quad (24b)$$

And function  $\mu(t, \mathbf{q}, \mathbf{p})$  admits the (16).

Corollary 3: the system (4) possesses the following conserved quantity

$$I = \frac{1}{\mu} \frac{\partial(\mu\tau)}{\partial t} + \frac{1}{\mu} \frac{\partial(\mu\xi_s)}{\partial q_s} - \frac{\bar{d}\tau}{dt} \quad (s = 1, 2, \dots, n) \quad (25)$$

If the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy

$$h_s \frac{\bar{d}\tau}{dt} + \tau \frac{\partial h_s}{\partial t} + \eta_k \frac{\partial h_s}{\partial p_k} = 0 \quad (s, k = 1, 2, \dots, n) \quad (26a)$$

$$\frac{\bar{d}\xi_s}{dt} - g_s \frac{\bar{d}\tau}{dt} - \tau \frac{\partial g_s}{\partial t} - \xi_k \frac{\partial g_s}{\partial p_k} = 0 \quad (26b)$$

And function  $\mu(t, \mathbf{q}, \mathbf{p})$  admits (16).

#### IV. NECESSARY AND SUFFICIENT CONDITION UNDER WHICH THE FORM INVARIANCE IS A LIE SYMMETRY

From the deductions of (11) and (13), it can be seen that the form invariance is generally different from the Lie symmetry. For seeking their relations, the equation (3) may be rewritten as follows:

$$F(t, \mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}) = \dot{q}_s \frac{\partial H}{\partial p_s} = 0 \quad (27a)$$

$$G(t, \mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}) = \dot{p}_s \frac{\partial H}{\partial p_s} = 0 \quad (27b)$$

Then for the system (3), the determining equations of a Lie, s symmetry have new form

$$X^{(1)}(F) = 0 \quad (28a)$$

$$X^{(1)}(G) = 0 \quad (28b)$$

Some direction calculations yield

$$\begin{aligned} \frac{\partial X^{(0)}(H)}{\partial p_s} &= \dot{\xi}_s - \dot{q}_s \dot{\tau} - X^{(1)}(F) \\ &+ \frac{\partial \tau}{\partial p_s} \frac{\partial H}{\partial t} + \frac{\partial \xi_k}{\partial p_s} \frac{\partial H}{\partial q_k} + \frac{\partial \eta_k}{\partial p_s} \frac{\partial H}{\partial p_k} \end{aligned} \quad (29a)$$

$$\begin{aligned} \frac{\partial X^{(0)}(H)}{\partial q_s} &= -\dot{\eta}_s - \dot{p}_s \dot{\tau} + X^{(1)}(G) \\ &+ \frac{\partial \tau}{\partial q_s} \frac{\partial H}{\partial t} + \frac{\partial \xi_k}{\partial q_s} \frac{\partial H}{\partial q_k} + \frac{\partial \eta_k}{\partial q_s} \frac{\partial H}{\partial p_k} \end{aligned} \quad (29b)$$

Equation(29) demonstrates the relation between the form invariance and the Lie symmetry. From the relation, the following proposition can be derived.

Proposition 1: for the Hamilton system, the necessary and sufficient condition under which the form invariance is a Lie symmetry is that the following relations hold

$$\dot{\xi}_s - \dot{q}_s \dot{\tau} - \frac{\partial \tau}{\partial p_s} \frac{\partial H}{\partial t} + \frac{\partial \xi_k}{\partial p_s} \frac{\partial H}{\partial q_k} + \frac{\partial \eta_k}{\partial p_s} \frac{\partial H}{\partial p_k} = 0 \quad (30a)$$

$$\dot{\eta}_s - \dot{p}_s \dot{\tau} - \frac{\partial \tau}{\partial q_s} \frac{\partial H}{\partial t} - \frac{\partial \xi_k}{\partial q_s} \frac{\partial H}{\partial q_k} - \frac{\partial \eta_k}{\partial q_s} \frac{\partial H}{\partial p_k} = 0 \quad (30b)$$

Let the sum of the coefficient of the terms which dependent on or respectively equals zero, and the sum of the remainder of terms equals also zero, one has

$$\dot{\tau} = 0 \quad (31a)$$

$$\dot{\xi}_s + \frac{\partial \tau_0}{\partial p_s} \frac{\partial H}{\partial t} + \frac{\partial \xi_k}{\partial p_s} \frac{\partial H}{\partial q_k} + \frac{\partial \eta_k}{\partial p_s} \frac{\partial H}{\partial p_k} = 0 \quad (31b)$$

$$\dot{\eta}_s - \frac{\partial \tau_0}{\partial q_s} \frac{\partial H}{\partial t} - \frac{\partial \xi_k}{\partial q_s} \frac{\partial H}{\partial q_k} - \frac{\partial \eta_k}{\partial q_s} \frac{\partial H}{\partial p_k} = 0 \quad (31c)$$

Proof: substitution of (11) and (30) into (29) leads to  $X^{(1)}(F) = 0$  and  $X^{(1)}(G) = 0$ . According to the determining (28), we know that the form invariance is a Lie symmetry.

Particularly, if  $\tau = 0$ , then the conditions (30) become

$$\dot{\xi}_s + \frac{\partial \xi_k}{\partial p_s} \frac{\partial H}{\partial q_k} + \frac{\partial \eta_k}{\partial p_s} \frac{\partial H}{\partial p_k} = 0 \quad (32a)$$

$$\dot{\eta}_s - \frac{\partial \xi_k}{\partial q_s} \frac{\partial H}{\partial q_k} - \frac{\partial \eta_k}{\partial q_s} \frac{\partial H}{\partial p_k} = 0 \quad (32b)$$

#### V. HOJMAN CONSERVED QUANTITY DEDUCED FROM FORM INVARIANCE

The Hojman conserved quantity can be located by using the form invariance.

Proposition 2: For the Hamilton system, under the infinitesimal transformation (5), if the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy (11) and (30), and there exists a function  $\mu = \mu(t, \mathbf{q}, \mathbf{p})$  admits the equation (16), then form invariance leads to the Hojman conserved quantity(15).

Proof: if the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy (11) and (30), by using proposition 1, we now that the generators are also Lie symmetry. And we can subsequently obtain the conserved quantity (15) by using the theorem 1.

Proposition 3: For the Hamilton system, under the infinitesimal transformation(5), if the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy (11) and (32), and there exists a function  $\mu = \mu(t, \mathbf{q}, \mathbf{p})$  admits the (16), then form invariance leads to the Hojman conserved quantity (21).

Proof: if the infinitesimal generators  $\tau$ ,  $\xi_s$  and  $\eta_s$  satisfy (11) and (32), proposition 1 means that the generators are also Lie symmetrical. Corollary 1 yields subsequently the conserved quantity (21).

## VI. AN ILLUSTRATIVE EXAMPLE

As an illustration of the theory developed in the preceding sections, consider the case of a simple degree of freedom linear damped oscillator

$$\ddot{q} + \gamma \dot{q} = 0 \quad (33)$$

First, transform Eq.(33) into a Hamilton system, and its Lagrangian is

$$L = \frac{1}{2} e^{\lambda t} \dot{q}^2 \quad (34)$$

Therefore

$$p = \frac{\partial L}{\partial \dot{q}} = e^{\lambda t} \dot{q}^2 \quad (35a)$$

$$H = p\dot{q} - L = \frac{1}{2} e^{-\lambda t} p^2 \quad (35b)$$

Eq.(11) leads to

$$\frac{\partial \tau}{\partial p} \left( -\frac{1}{2} p^2 \right) - p\tau + \frac{\partial \eta}{\partial p} p + \eta = 0 \quad (36a)$$

$$\frac{\partial \tau}{\partial q} \left( -\frac{1}{2} p^2 \right) + \frac{\partial \eta}{\partial q} = 0 \quad (36b)$$

It can be easily verified that

$$\tau = 0, \quad \xi = 1, \quad \eta = 0; \quad (37)$$

$$\tau = 1, \quad \xi = q, \quad \eta = \frac{1}{2} p^2 \quad (38)$$

Are the solution sets of (36). Since the generator (37) satisfies the (13), so it is also the Lie symmetry of the system (34). However, the generators (38) do not satisfy the (13), so it is not the Lie symmetry of the system (34). From (16), one has

$$\frac{d\tau}{dt} \ln \mu = 0 \quad (39)$$

Equation (39) exists a solution

$$\mu = e^{-\lambda t} p + \gamma p \quad (40)$$

Inserting (37) and (40) into (25) leads to conserved quantity

$$I = \frac{\gamma}{e^{-\lambda t} p + \gamma p} \quad (41)$$

## VII. CONCLUSIONS

For Hamilton systems, we present an approach to find Hojman conserved quantities in terms of the form invariance.

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