Optimal Rational Function Approximation for Fractional Integral and Differential Operators

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Abstract—Bode diagram based rational function approximation method for fractional integral and differential operators are analyzed in detail. For the approximation rational function orders is the lowest under satisfying approximation accuracy in the approximation frequency interval, two steps are proposed: 1) Choose reasonable initial and terminal frequency of rational function logarithmic amplitude-frequency characteristic. 2) Set approximation error of logarithmic amplitude-frequency characteristic by taking the error between asymptote and exact value into account. Computation examples demonstrate the validity of this method.

Key words-Fractional integral and differential operators; Rational function approximation; Bode diagram; Optimal rational function approximation

I. INTRODUCTION

Fractional Order Calculus (FOC) was first proposed in 1695[1]. However, due to a lack of knowledge of its physical significance and geometric meaning, FOC has laid dormant for a long time. It's not until recent decades, as with the development of computer science and its application, FOC has gained much attention in scientific and engineering community[2]. FOC has broad application in complex systems modeling, analysis and identification, signal processing and automatic control[2]and so on. FOC can resolve some problems that traditional integer order calculus could not do. Some previously undiscovered or unexplained phenomenon can be discovered or explained by FOC. For example, the conventional automatic control system, its performance would be further improved by FOC modeling or FOC controller[4].

Fractional order integral or differential operator $1/s^\gamma$ ($0 < \gamma < 1$ and $-1 < \gamma < 0$ are integral and differential respectively) is the most common fractional filter in automatic control system. It should be discretized by an approximate model in digital control systems. Rational function of Laplace operator s is used to approximate $1/s^\gamma$ in continuous control systems. This paper is aimed at the rational function approximation for $1/s^\gamma$ .Existing methods[5-9] include continuous fractional expansion (CFE),

Carlson, Matsuda,Oustloup(CRONE),etc. A common feature of these methods is that they do not take how to obtain the rational function of lowest order under specified conditions (approximation frequency range and approximation error) into account, namely how to achieve the best rational function approximation. This paper proposes an optimal rational function approximation method for $1/s^\gamma$.

This paper is organized as follows: Section 2 deals with the proposal of rational function approximation based on Bode diagram for fractional integral operator. Section 3 describes the method of setting approximation error bandwidth. Section 4 gives out the solving issue and improving it. And section 5 is the conclusion part.

II. BODE DIAGRAM BASED RATIONAL FUNCTION APPROXIMATION FOR FRACTIONAL INTEGRAL OPERATOR

A. Mathematical model

Fractional integral operator is $1/s^\gamma$, where $0 < \gamma < 1$. Assume it is in series with proportional coefficient k_0 .The transfer function is $G_0(s) = k_0 / s^r$. It can be approximated by $R(s)$, where $R(s)$ is a rational function of s . And

$$R(s) = K \frac{(s+b_1)(s+b_2)...}{(s+a_1)(s+a_2)(s+a_3)...} \quad (1)$$

Where K is an undetermined proportion coefficient. a_i and b_i are undetermined constants. The frequency characteristic is $G_0(j\omega) = k_0 /(j\omega)^r$.

$$R(j\omega) = K \frac{(s+b_1)(s+b_2)...}{(s+a_1)(s+a_2)(s+a_3)...}\bigg|_{s=j\omega} \quad (2)$$

Thus the problem of $R(s)$ approximating $G_0(s)$ in s domain is transferred to $R(j\omega)$ approximating $G(j\omega)$ in the frequency domain. $R(s)$ approximating to $G_0(s)$ by Bode diagram of logarithmic amplitude-frequency characteristics (LAFC) in the paper.

B. Amplitude frequency characteristics

Fig.1 shows LAFCs of $G_0(s)$ and its approximation rational function $R(s)$, where LAFC is a line L_0. Lines L_1 and L_2 are parallel with L_0. The LAFC of $R(s)$ is composed of polyline inside L_1 and L_2. The polyline is connected by horizontal lines and oblique lines with slope $-20dB/dec$ alternately. Maximum longitudinal distance between polyline and L_0 is maximum error (absolute value) δ. The area inside L_1 and L_2 is error band. The frequency corresponding to the intersection of the polyline with L_1 or L_2 is called handover frequency. From left to right, handover frequencies are divided into two groups: one group corresponding to the initial of horizontal segment, called as b_i, another is to the initial of oblique segment, denoted by a_i. Two special handover frequencies corresponding to initial and terminal of polyline, are denoted by Ω_1 and Ω_2 respectively. $[\omega_a, \omega_b]$ is approximation frequency range. $[\Omega_1, \Omega_2]$ is called rational function build frequency range. $[\omega_a, \omega_b] \in [\Omega_1, \Omega_2]$. At two small ranges by the right of Ω_1 and left of Ω_2, LAFC of $R(s)$ is within error band, its phase-frequency characteristic approximation error is larger than normal. In order to exclude these two small ranges from the approximate frequency range, the approximation frequency range should be less than the build frequency range. As shown in Fig.1.
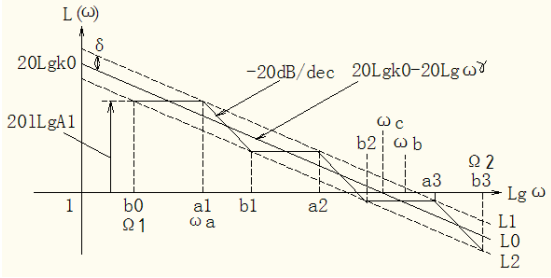

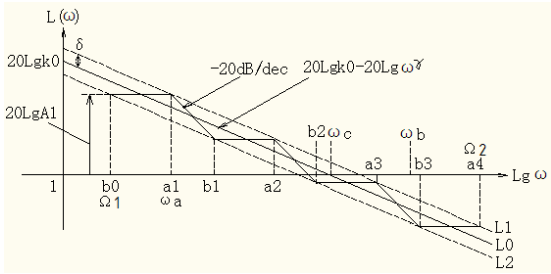Figure1. LAFCs of $G_0(s) = k_0 / s^\gamma$, $R(s)$, $\Omega_2 = b_j$, $j = 3$

Figure2. LAFCs of $G_0(s) = k_0 / s^\gamma$, $R(s)$, $\Omega_2 = a_j$, $j = 4$

C. Calculation of the handover frequency

After ω_a and ω_b have been set, handover frequencies is $a_1 = \omega_a$. In Fig.1, we have

$$20\lg k_0 - 20\lg a_1^\gamma + \delta = 20\lg k_0 - 20\lg a_1$$
$$20\lg k_0 - 20\lg b_1^\gamma - \delta = 20\lg k_0 - 20\lg b_1$$

Subtract two equations, we get the solution

$$b_1 = a_1 10^{\frac{\delta}{10(1-\gamma)}}.$$

Again, $20\lg k_0 - 20\lg b_1^\gamma - \delta = 20\lg k_0 - 20\lg a_2^\gamma + \delta$.

Predigest it as $20\lg(a_2/b_1)^\gamma = 2\delta$, Solution is

$$a_2 = b_1 10^{\frac{\delta}{10\gamma}}.$$ Similarly, we yield the recurrence formulas

$$b_i = a_i 10^{\frac{\delta}{10(1-\gamma)}}, \quad i = 1,2,3,... \tag{3}$$

$$a_{i+1} = b_i 10^{\frac{\delta}{10\gamma}}, \quad i = 1,2,3,... \tag{4}$$

Specifically, the adjacent handover frequency ratios are $b_i / a_i = 10$, when $\gamma = 5$, $\delta = 5$, respectively.

D. Calculation of Polyline initial and terminal frequencies

Polyline initial frequency is obtained by (4) when $i = 0$, $a_1 = b_0 10^{\frac{\delta}{10\gamma}}$,

$$\Omega_1 = b_0 = a_1 10^{\frac{-\delta}{10\gamma}} \tag{5}$$

The polyline terminal frequency is based on comparing values of a_{i+1} (or b_i) with ω_b. The steps are as follows:

1) In the case of $i + 1 = j$, obtained a_j. If $a_j \geq \omega_b$, then

$$\Omega_2 = b_j \tag{6}$$

2) In the case of $i = j$, obtained b_j. If $b_j \geq \omega_b$, then

$$\Omega_2 = a_{j+1} \tag{7}$$

E. Generation of approximation rational function

Approximation rational function can be obtained after calculating handover frequencies based on above algorithm. Each a_i is corresponding to an inertial element, which is a factor in denominator of $R(s)$. Similarly, every b_i is to a first-order differential element in numerator of $R(s)$ except polyline initial and terminal frequency.

Transfer function of first inertial element is denoted as

$$G_1(s) = \frac{A_1}{1 + s/a_1} \tag{8}$$

As $20\lg A_1 = 20\lg k_0 - 20\lg a_1^\gamma + \delta$, $k_0 = \omega_c^\gamma$, then

$$A_1 = \left(\frac{\omega_c}{a_1}\right)^{\gamma} 10^{\frac{\delta}{20}} \qquad (9)$$

Similarly, b_i and a_i are differential and inertial element alternately, their coefficients are reciprocal and can be reduced in series product. Finally, for Fig.1 we have

$$R(s) = 10^{\frac{\delta}{20}} \left(\frac{\omega_c}{a_1}\right)^{\gamma} \frac{(1+s/b_1)(1+s/b_2)}{(1+s/a_1)(1+s/a_2)(1+s/a_3)} .$$

Fig. 2 shows

$$R(s) = 10^{\frac{\delta}{20}} \left(\frac{\omega_c}{a_1}\right)^{\gamma} \frac{(1+s/b_1)(1+s/b_2)(1+s/b_3)}{(1+s/a_1)(1+s/a_2)(1+s/a_3)} .$$

The general forms of above two expression are as follows:

$$R(s) = 10^{\frac{\delta}{20}} \left(\frac{\omega_c}{a_1}\right)^{\gamma} \frac{(1+s/b_1)(1+s/b_2)...(1+s/b_{j-1})}{(1+s/a_1)(1+s/a_2)...(1+s/a_j)} \quad (10)$$

$$R(s) = 10^{\frac{\delta}{20}} \left(\frac{\omega_c}{a_1}\right)^{\gamma} \frac{(1+s/b_1)(1+s/b_2)...(1+s/b_j)}{(1+s/a_1)(1+s/a_2)...(1+s/a_j)} \quad (11)$$

III. SETTING APPROXIMATION ERROR BANDWIDTH

For setting approximation frequency range $[\omega_a, \omega_b]$, we note that the wider approximation error bandwidth is, the less the number of polyline inflection point and handover frequency are, and the lower order of approximation function is. In order to reduce the orders and optimize rational approximation function, we should increase the approximation error bandwidth to the best of one's abilities within acceptable approximation error. Therefore, two problems should be considered:1)The error between the asymptote and exact value of LAFC.2) The relationship between pole frequency of LAFC approximation error curve and adjacent handover frequency.

A. The error between the asymptote and exact value of LAFC

LAFC of approximation rational function are represented by asymptote in Bode diagram. There is error between asymptote and accurate characteristics. The closer to the handover frequency, the greater the error is. Maximum error is on the handover frequency. For inertia element, the maximum error is $3dB$. For first-order differential element, the maximum error is $-3dB$. The asymptote is always outside of the accurate characteristics. If error bandwidth in Bode diagram is set directly according to the LAFC maximum acceptable approximation error, then it will result in higher order of $R(s)$. And the actual approximation error is less than the setting value, which means the approximation precision has margin. Eliminating this margin can reduce the orders of $R(s)$. Therefore, error band should be widened appropriately. Related problems are discussed as follows.

B. The relationship between approximation error curve pole frequency and the LAFC handover frequency

The LAFC expression of fractional integral operator k_0 / s^{γ} is $20\lg k_0 + 20\lg \omega^{-\gamma}$.

Assume $A_j / (1 + sT_j)$ is a transfer function of certain inertial element in the approximation rational function, its LAFC expression is $20\lg A_j - 20\lg \sqrt{1+(\omega T_j)^2}$. The first expression subtracts second one. Let derivative of the difference for ω is equal to zero, we obtain $\frac{d}{d\omega}[\lg \omega^{-\gamma} + \lg \sqrt{1+(\omega T_j)^2}] = 0$.

Solution of this equation is namely a pole frequency of approximation error curve, denote by ω^*. Then

$$\omega^* = \frac{1}{T_j} \sqrt{\frac{\gamma}{1-\gamma}} = \omega_j \sqrt{\frac{\gamma}{1-\gamma}} \qquad (12)$$

where ω_j is a handover frequency.

C. Setting approximation error bandwidth

Assume the maximum acceptable error of amplitude is μ. Convert μ to the error of LAFC in Bode diagram $\delta_0 = 20\lg \mu$. The error bandwidth (single side) should be taken $\delta = \delta_0 + \Delta\delta$. where δ is the difference between approximation function LAFC asymptote and LAFC of fractional integral operator at ω_j, δ_0 is the difference of approximation function LAFC exact value and fractional integral operator LAFC at ω^*. $\Delta\delta$ is a correction value related to ω^* and ω_j. There are three cases as follows.

(1) $\gamma = 0.5$, as shown in (12) $\omega^* = \omega_j$, the error between the asymptote and the exact value of the approximation function LAFC at ω_j equals to $3dB$.That is $\Delta\delta = \delta - \delta_0 = 3dB$.

(2) $\gamma > 0.5$, as (12), $\omega^* > \omega_j$, the error between of the exact value of the approximation function LAFC and fractional integral operator LAFC is δ_0 at ω^* and is δ_1 at ω_j. Moreover, $\delta - \delta_1 = 3dB$ and $\delta_0 > \delta_1$. So we obtain $\Delta\delta = \delta - \delta_0 < 3dB$.

(3) $\gamma < 0.5$, as (12), $\omega^* < \omega_j$. Similarly get $\Delta\delta = \delta - \delta_0 < 3dB$.

The farther γ value deviates from 0.5, the smaller $\Delta\delta$ is. There is no analytical methods to determine $\Delta\delta$ value, but it can be selected by the experience or testing .For instance, when $\gamma = 0.4$ or $\gamma = 0.6$,

$\Delta\delta \approx 2.8dB$ .When $\gamma = 0.2$ or $\gamma = 0.8$, take $\Delta\delta \approx 2dB$, and so on.

IV. IMPROVEMENT

Let $G_0(s) = 1/s^{0.5}$. Find $R(s)$ to get the maximum approximation error of the LAFC (exact value) $\delta_0 = 2dB$ in the frequency range $[\omega_a, \omega_b] = [1,10^4]$ (rad/s) .we have $R(s) = 10^{\frac{\delta}{20}}\left(\frac{\omega_c}{a_1}\right)^{\gamma}\frac{(1+s/b_1)(1+s/b_2)}{(1+s/a_1)(1+s/a_2)}$ .LAFC and phase-frequency characteristics curves are as follows.

Here since before decreasing a_1 selecting $\Omega_2 = a_3$ is based on $b_2 > \omega_b$ but not $b_2 = \omega_b$, so that the decrease of a_1 cannot change Ω_2 and orders of $R(s)$. Contrarily if it is based on $b_2 = \omega_b$ (is in critical case) then decrease of a_1 will bring change of Ω_2 and increase of orders of $R(s)$. Commonly if selecting Ω_2 is based on equation of formula $a_j \geq \omega_b$ (or $b_j \geq \omega_b$) then decrease of a_1 will bring increase of $R(s)$ orders . Otherwise it is based on inequation then it is possible that the orders of $R(s)$ do not increase.
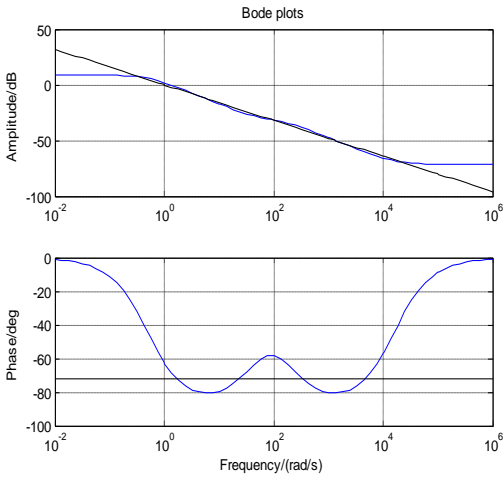

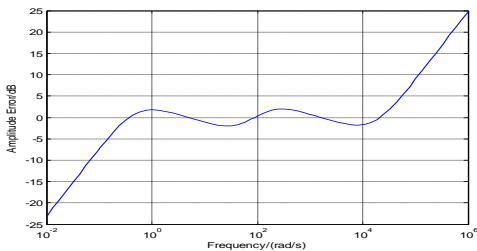Figure3. LAFC and phase-frequency characteristics curves

Figure4. Approximation error curve of LAFC

The approximation of fractional differential operator, $(1/k_0)s^{\gamma}$,can utilize above optimal rational function approximation method of fractional integral operator. Here reverse numerator and denominator of $(1/k_0)s^{\gamma}$,we get k_0/s^{γ} . Find the approximation rational function $R(s)$ of this k_0/s^{γ} .Furthermore, we reverse numerator and denominator of this $R(s)$ and change sign of order in MATLAB for drawing Bode diagram.

V. CONCLUSIONS

We have addressed fractional integral operator optimal rational function approximation method based on Bode diagram. The so-called optimal means: Under the condition of the approximate frequency range and the maximum acceptable amplitude approximation error (and corresponding to maximum phase error), the orders of the rational function is the lowest. The previous method can also be applied to fractional differential operator approximation. Calculation examples show availability of above works.

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