



A Comprehensive Study on Power of Tests for Normality

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ARTICLE INFO

Article History

Received 14 Oct 2017

Accepted 17 May 2018

Keywords

Test of normality

Monte Carlo simulation

Power of test

The generalized lambda distribution

2000 Mathematics Subject Classification

62G10, 62P20, 62P30

ABSTRACT

Many statistical procedures assume that the underlying distribution is normal. In this paper, we consider the popular and powerful tests for normality and investigate the power values of these tests to detect deviations from normality. The family of four-parameter generalized lambda distributions (FMKL) for its high flexibility is considered as alternative distributions. We then compare the power values of normality tests against these alternatives and for different sample sizes. The considered tests are Kolmogorov-Smirnov, Anderson-Darling, Kuiper, Jarque-Bera, Cramer von Mises, Shapiro-Wilk and Vasicek. These tests are popular tests which are commonly used in practice and statistical software. The tests are described and then power values of the tests are compared against FMKL family by Monte Carlo simulation. The results are discussed and interpreted. Finally, we apply some real data examples to show the behavior of the tests in practice.

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1. INTRODUCTION

Ramberg and Schmeiser [1] introduced the four-parameter generalized lambda distribution (GLD) as

$$Q(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2},$$

where $Q(u)$ is the quantile function, $0 \leq u \leq 1$, λ_1, λ_2 are the location and scale parameters, and λ_3, λ_4 are shape parameters jointly related to the strengths of the lower and upper tails, respectively. For its high flexibility it is used in many fields such as modeling financial data.

Because of the limitations of the Ramberg and Schmeiser (RS) parameterizations, Freimer *et al.* [2] proposed a new parameterization called FMKL as

$$Q(u) = \lambda_1 + \frac{1}{\lambda_2} \left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4} \right),$$

where $0 \leq u \leq 1$, λ_1, λ_2 are the location and scale parameters, respectively. Also λ_3 and λ_4 determine the shape characteristics and for a symmetric distribution $\lambda_3 = \lambda_4$.

The five different shapes of the FMKL are: unimodal, U-shaped, J-shaped, S-shaped, and monotone, which may be symmetric and asymmetric with smooth, abrupt, truncated, long, medium or short tails.

In many situations, a goodness of fit test about the distribution of the population using observations is necessary. Since the normal distribution is widely used in many statistical procedures and also is the most fundamental distribution, test for the normal hypothesis is indispensable. Moreover, testing normality is one of the most areas of statistical research. For example in statistical modeling the normal assumption of the underlying error distribution must be checked. Therefore, many tests for normality are proposed by authors. A fair of normality tests can be found in the statistical literature. In many situations, a goodness of fit test about the distribution of the population is necessary. Since the normal distribution is the most basic distribution and use widely in statistics, test for the normal hypothesis has been studied by many authors. See for example, D'Agostino and Stephens [3], Huber-Carol *et al.* [4], Thode [5].

Recently, testing normality has been considered by Alizadeh Noughabi and Arghami [6], Harri and Coble [7], Sanqui *et al.* [8], Zamanzade and Arghami [9], Marmolejo-Ramos and González-Burgos [10], Joenssen and Vogel [11] and Wang [12].

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In this article, we consider seven popular (like Kolmogorov-Smirnov, Anderson-Darling) and powerfulness (like Shapiro-Wilk, Vasicek) normality tests and compare power values of these tests against the GLD (FMKL) with different parameters. We show that no single test procedure is uniformly more powerful than others. However, the powerful tests can be determined based on type of alternatives. Thus, tests for normality based on type of alternatives are classified.

The methodologies of the considered tests are presented in Section 2. Power values of the tests are compared with each other against the FMKL family by Monte Carlo simulation in Section 3. In Section 4, the applicability of the tests in practice is shown by real data. Finally, some conclusions are given in Section 5.

2. TESTS FOR NORMALITY

Given a random sample X_1, \dots, X_n from a continuous probability distribution F with a density $f(x)$, over the real line and with mean μ and variance $\sigma^2 < \infty$, the hypothesis of interest is

$$H_0 : f(x) = f_0(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad \text{for some } (\mu, \sigma) \in \Theta, x \in \mathbb{R}$$

where μ and σ are unspecified and $\Theta = \mathbb{R} \times \mathbb{R}^+$. The alternative to H_0 is

$$H_1 : f(x) \neq f_0(x; \mu, \sigma) \quad \text{for any } (\mu, \sigma) \in \Theta.$$

In this section, we consider seven popular tests for the above hypothesis. The considered tests are Cramer von Mises [13], Kolmogorov-Smirnov [14], Anderson-Darling [15], Kuiper [16], Shapiro-Wilk [17], Vasicek [18] and Jarque-Bera [19]. These tests are commonly used in practice and software. For example, the Shapiro-Wilks test is used in SAS software for testing normality. The description of each normality tests is presented in Table 1.

From the aforementioned tests, Vasicek's test, Shapiro-Wilk and Jarque-Bera test are specific in the sense that the null hypothesis is normal, while the rest are suitable for any null family of distributions. For further study about this tests, see D'Agostino and Stephens [3] and references there in.

3. SIMULATION STUDY

In this section, Type I error of the tests are obtained and then power values of the tests against flexible FMKL family are computed through a Monte Carlo simulation.

Table 1 | Tests of normality.

Test of normality	Test statistic	Notations
1- Cramer von Mises	$CH = \frac{1}{12n} + \sum_{i=1}^n \left(\frac{2i-1}{2n} - Z_i \right)^2 ; i = 1, \dots, n.$	$Z_i = \Phi\left(\frac{X_{(i)} - \bar{X}}{S_X}\right)$: where Φ is the cdf of standard normal distribution.
2- Kolmogorov-Smirnov	$D^+ = \max\left\{\frac{i}{n} - Z_i\right\}, D^- = \max\left\{Z_i - \frac{i-1}{n}\right\}, i = 1, \dots, n$ $D = \max(D^+, D^-)$	$Z_i = \Phi\left(\frac{X_{(i)} - \bar{X}}{S_X}\right)$: where Φ is the cdf of standard normal distribution.
3- Kuiper	$V = D^+ + D^-$	D^+ and D^- are as above.
4- Shapiro-Wilk	$W = \frac{\left(\sum_{i=1}^{[n/2]} a_{(n-i+1)} (X_{(n-i+1)} - X_{(i)})^2\right)}{\sum_{i=1}^n (X_{(i)} - \bar{X})^2}$	The coefficients a_i are tabulated in Pearson and Hartley [20]
5- Anderson-Darling	$A^2 = -n - \frac{\sum_{i=1}^n (2i-1) \{ \ln(Z_i) + \ln(1-Z_{n-i+1}) \}}{n}$	$Z_i = \Phi\left(\frac{X_{(i)} - \bar{X}}{S_X}\right)$: where Φ is the cdf of standard normal distribution.
6- Vasicek	$KL_{mn} = \frac{\exp(H(n, m))}{S_X}$ $H(n, m) = \frac{1}{n} \sum_{n=1}^n \log\left\{\frac{n}{2m} (X_{(i+m)} - X_{(i-m)})\right\}$	S_X^2 : sample variance m : positive integer $(m, \leq, \frac{n}{2})$
7- Jarque-Bera	$JB = n \left\{ \frac{c^2}{6} + \frac{(k-3)^2}{24} \right\}$	$c = \text{skewness}$ $k = \text{kurtosis}$

3.1. Type I Error of the Tests

Through a Monte Carlo simulation, we compute Type I error of the tests. Table 2 presents Type I error probabilities (the actual size of the tests), which have been obtained by 20 000 simulations. As is evident from Table 2 the actual size of the tests are quite acceptable.

3.2. Shapes of FMKL Family

The shapes returned by FMKL family are classified by Freimer *et al.* [2] as follows.

Class I ($\lambda_3 < 1, \lambda_4 < 1$): Unimodal densities with continuous tails. This class can be subdivided with respect to the finite or infinite slopes of the densities at the end points.

Class Ia ($\lambda_3, \lambda_4 \leq 0.5$),

Class Ib ($0.5 < \lambda_3 < 1, \lambda_4 \leq 0.5$),

Class Ic ($0.5 < \lambda_3 < 1, 0.5 < \lambda_4 < 1$).

Class II ($\lambda_3 > 1, \lambda_4 < 1$): Monotone pdfs similar to those of the exponential or Chi-square distributions. The left tail is truncated.

Class III ($1 < \lambda_3 < 2, 1 < \lambda_4 < 2$): U-shaped densities with both tails truncated.

Class IV ($\lambda_3 > 2, 1 < \lambda_4 < 2$): Rarely occurring S-shaped pdfs with one mode and one antimode. Both tails are truncated.

Class V ($\lambda_3 > 2, \lambda_4 > 2$): Unimodal pdfs with both tails truncated.

Examples of each class of shapes are displayed in Figs. 1–7.

3.3. Power Comparison

To comparison of the power values of the tests, we select alternatives from FMKL family with different parameters. We compute the power values of the tests based on $CH, D, V, W, A^2, KL_{mn}$ and JB statistics against FMKL family by means of Monte Carlo simulations. As mentioned above, the alternatives can divide into five groups (Class I to V).

Power values of the tests were obtained by simulation in the following manner.

Under each alternative we generated 20 000 samples of size 10, 20, 30, 50. We calculated for each sample the statistics ($CH, D, V, W, A^2, KL_{mn}, JB$) and the power value of the corresponding test was computed by the frequency of the event “the value of the test statistic is in the critical region”. The required critical regions are given in the corresponding articles, but we obtained them by simulation, before power

Table 2 | Actual sizes of the tests for nominal significance level $\alpha = 0.05$ ($n = 20, \sigma = \text{standard deviation}$).

Test of normality	$\sigma = .01$	$\sigma = .25$	$\sigma = 1$	$\sigma = 5$	$\sigma = 10$	$\sigma = 100$
Cramer von Mises	0.0510	0.0506	0.0498	0.0492	0.0485	0.0509
Kolmogorov-Smirnov	0.0507	0.0492	0.0503	0.0505	0.0491	0.0511
Kuiper	0.0517	0.0509	0.0501	0.0493	0.0508	0.0487
Shapiro-Wilk	0.0503	0.0497	0.0499	0.0504	0.0497	0.0501
Anderson-Darling	0.0491	0.0505	0.0497	0.0506	0.0490	0.0508
Vasicek	0.0498	0.0501	0.0503	0.0499	0.0502	0.0496
Jarque-Bera	0.0503	0.0495	0.0501	0.0499	0.0502	0.0501

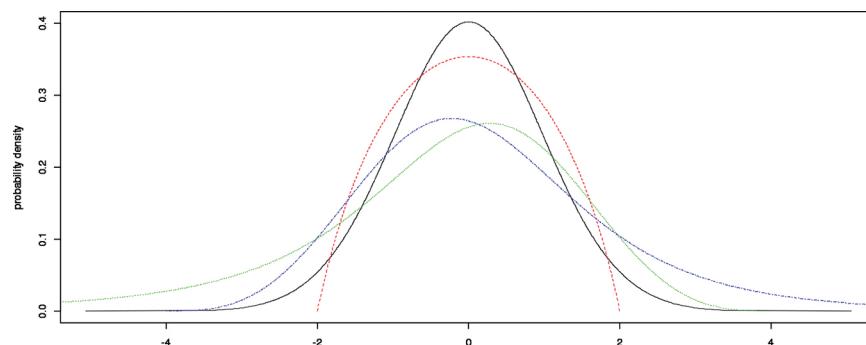


Figure 1 | Class Ia pdfs including the normal distribution.

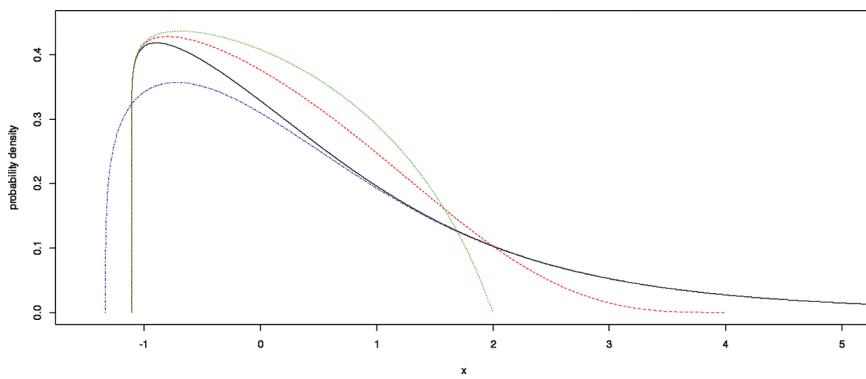


Figure 2 | Class Ib pdfs.

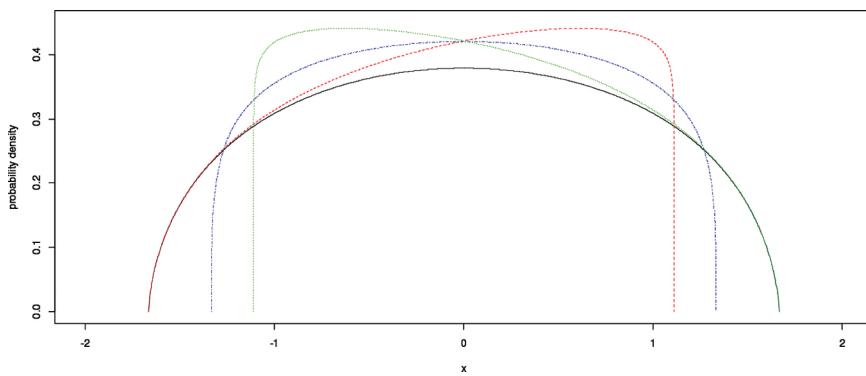


Figure 3 | Class Ic pdfs.

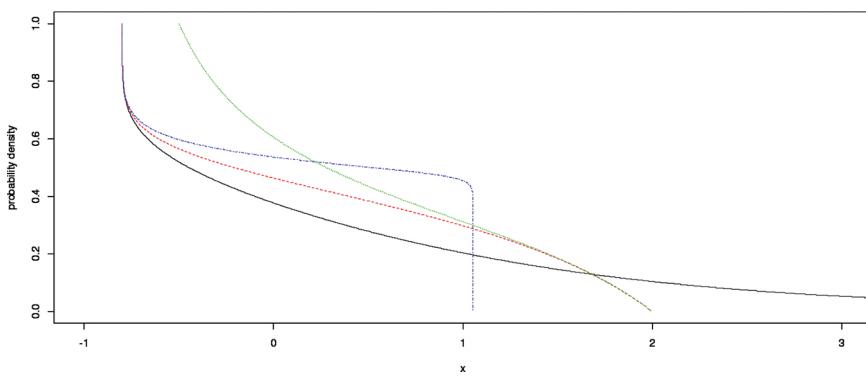


Figure 4 | Class II pdfs includes the exponential distribution.

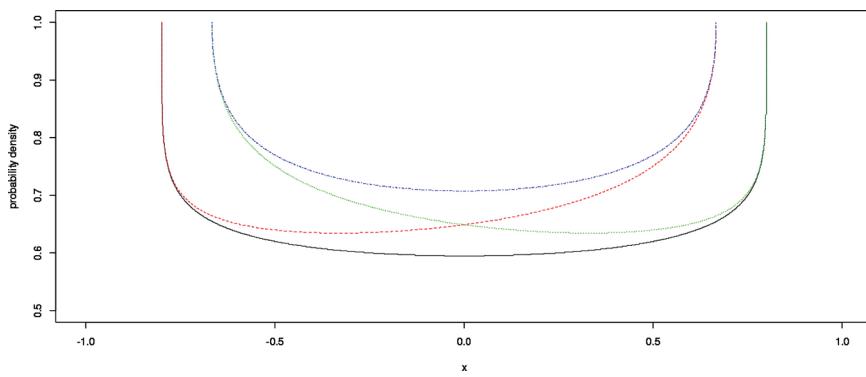


Figure 5 | Class III U-shaped pdfs.

simulations. The power values of the tests are presented in Tables 3–9. For each sample size and alternative, the bold type in these Tables indicates the test achieving the maximal power. For Vasicek's test, we used the recommended window sizes proposed by Vasicek [18], i.e., $m = 2, (3), 4$ for sample sizes $n = 10, (20, 30), 50$, respectively.

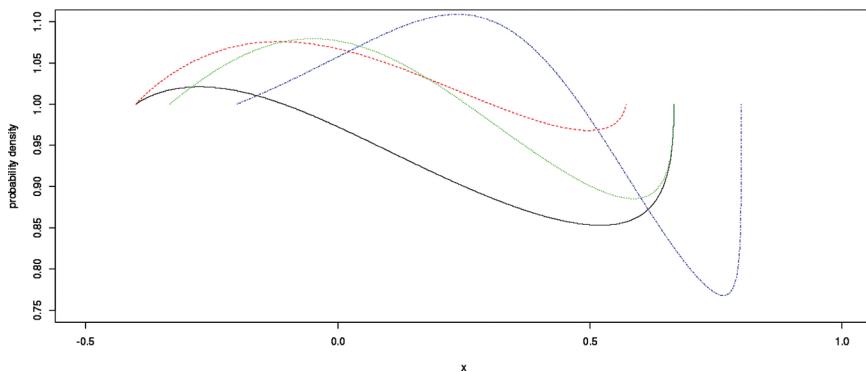


Figure 6 | Class IV S-shaped pdfs.

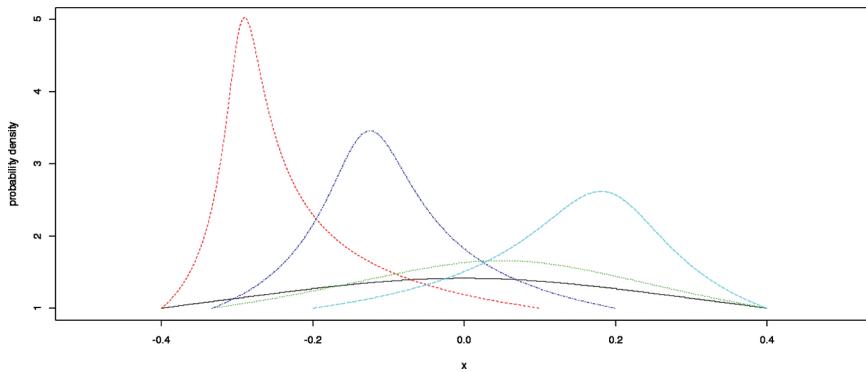


Figure 7 | Class V pdfs.

Table 3 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group Ia.

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
10	-0.2	-0.2	0	22.21	0.1528	0.1549	0.1735	0.1780	0.0866	0.1771	0.2036
	0.25	0.25	0	2.54	0.0457	0.0463	0.0461	0.0441	0.0558	0.0427	0.0331
	0.5	0.5	0	2.08	0.0464	0.0479	0.0452	0.0459	0.0876	0.0454	0.0227
	-0.2	-0.1	2.63	35.40	0.1313	0.1315	0.1467	0.1512	0.0787	0.1507	0.1736
	-0.2	0	4.65	73.8	0.1278	0.1280	0.1419	0.1474	0.0843	0.1519	0.1743
	-0.2	0.25	6.99	157.98	0.1546	0.1561	0.1792	0.1879	0.1309	0.1994	0.2097
	-0.2	0.5	7.25	189.98	0.2075	0.2184	0.2480	0.2616	0.2083	0.2811	0.2780
	0	-0.1	-2.81	17.83	0.0931	0.0938	0.1017	0.1057	0.0653	0.1097	0.1277
	0	0.25	1.05	3.70	0.0765	0.0741	0.0830	0.0872	0.664	0.0907	0.0944
	0	0.5	0.566	2.40	0.1063	0.1092	0.1263	0.1340	0.1116	0.1438	0.1425
	0.25	-0.1	-2.13	11.21	0.1071	0.1043	0.1226	0.1292	0.0888	0.1360	0.1478
	0.25	0.5	-0.041	2.25	0.0509	0.0538	0.0540	0.0530	0.0769	0.0533	0.0363
	0.5	-0.1	-1.15	4.09	0.1513	0.1562	0.1777	0.1908	0.1486	0.2052	0.2050
20	-0.2	-0.2	0	22.21	0.2405	0.2575	0.2906	0.3083	0.1376	0.3224	0.3672
	0.25	0.25	0	2.54	0.0464	0.0488	0.0447	0.0419	0.0668	0.0377	0.0186
	0.5	0.5	0	2.08	0.0501	0.0692	0.0612	0.0657	0.1628	0.0649	0.0064
	-0.2	-0.1	2.63	35.40	0.1955	0.2089	0.2391	0.2543	0.1131	0.2657	0.3110
	-0.2	0	4.65	73.8	0.1914	0.1970	0.2303	0.2485	0.1281	0.2684	0.3047
	-0.2	0.25	6.99	157.98	0.2755	0.2729	0.3330	0.3613	0.2637	0.4003	0.4123
	-0.2	0.5	7.25	189.98	0.3790	0.4069	0.4731	0.5137	0.4726	0.5747	0.5334
	0	-0.1	-2.81	17.83	0.1195	0.1257	0.1436	0.1566	0.0712	0.1747	0.2127
	0	0.25	1.05	3.70	0.0981	0.0925	0.1119	0.1231	0.0908	0.1481	0.1627
	0	0.5	0.566	2.40	0.1741	0.1744	0.2234	0.2496	0.2360	0.3039	0.2719

(continued)

Table 3 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group Ia. (Continued)

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
30	0.25	-0.1	-2.13	11.21	0.1738	0.1688	0.2091	0.2309	0.1564	0.2661	0.2822
	0.25	0.5	-0.041	2.25	0.0664	0.0686	0.0738	0.0773	0.1322	0.0807	0.0375
	0.5	-0.1	-1.15	4.09	0.2680	0.2819	0.3443	0.3824	0.3481	0.4431	0.4049
50	-0.2	-0.2	0	22.21	0.3199	0.3535	0.3903	0.4132	0.2093	0.4289	0.4805
	0.25	0.25	0	2.54	0.0449	0.0502	0.0480	0.0444	0.0741	0.0365	0.0105
	0.5	0.5	0	2.08	0.0653	0.0937	0.0849	0.0934	0.2309	0.0986	0.0028
	-0.2	-0.1	2.63	35.40	0.2695	0.2970	0.3300	0.3527	0.1789	0.3711	0.4263
	-0.2	0	4.65	73.8	0.2668	0.2774	0.3254	0.3470	0.1962	0.3763	0.4219
	-0.2	0.25	6.99	157.98	0.3847	0.3763	0.4599	0.4975	0.3857	0.5528	0.5561
	-0.2	0.5	7.25	189.98	0.5343	0.5667	0.6508	0.6999	0.6765	0.7739	0.7052
	0	-0.1	-2.81	17.83	0.1556	0.1656	0.1919	0.2087	0.1006	0.2288	0.2744
	0	0.25	1.05	3.70	0.1292	0.1153	0.1526	0.1687	0.1196	0.2096	0.2200
	0	0.5	0.566	2.40	0.2589	0.2591	0.3383	0.3878	0.3729	0.4775	0.3946
	0.25	-0.1	-2.13	11.21	0.2399	0.2235	0.2933	0.3236	0.2312	0.3816	0.3894
	0.25	0.5	-0.041	2.25	0.0867	0.0862	0.0996	0.1076	0.1776	0.1133	0.0354
	0.5	-0.1	-1.15	4.09	0.3914	0.4127	0.5011	0.5576	0.5337	0.6466	0.5699

Table 4 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group Ib.

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
10	0.6	-0.2	-7.26	198.82	0.2239	0.2423	0.2734	0.2894	0.2451	0.3098	0.2951
	0.6	-0.1	-0.94	3.29	0.1687	0.1799	0.2047	0.2190	0.1846	0.2398	0.2266
	0.6	0	-0.42	2.16	0.1222	0.1292	0.1457	0.1556	0.1400	0.1704	0.1624
	0.6	0.25	0.094	2.18	0.0595	0.0607	0.0639	0.0662	0.0942	0.0691	0.0492
	0.6	0.5	0.045	2.03	0.0446	0.0515	0.0469	0.0486	0.0941	0.0479	0.0221
	0.75	-0.2	-7.33	212.90	0.2540	0.2866	0.3140	0.3339	0.3070	0.3631	0.3360
	0.75	-0.1	-0.69	2.60	0.1926	0.2121	0.2406	0.2584	0.2373	0.2809	0.2614
	0.75	0	-0.24	1.95	0.1447	0.1587	0.1805	0.1925	0.1837	0.2120	0.1953
	0.75	0.25	0.18	2.12	0.0757	0.0801	0.0833	0.0871	0.1201	0.0929	0.0681
	0.75	0.5	0.12	1.99	0.0534	0.0605	0.0563	0.0583	0.1109	0.0579	0.0294
	0.9	-0.2	-7.49	229.48	0.2764	0.3180	0.3471	0.3689	0.3569	0.4001	0.3577
	0.9	-0.1	-0.49	2.20	0.2223	0.2554	0.2803	0.2993	0.2882	0.3257	0.2921
20	0.9	0	-0.09	1.84	0.1673	0.1906	0.2109	0.2278	0.2290	0.2482	0.2186
	0.9	0.25	0.26	2.11	0.0913	0.0995	0.1047	0.1121	0.1460	0.1220	0.0869
	0.9	0.5	0.19	1.97	0.0633	0.0747	0.0710	0.0739	0.1349	0.0780	0.0397
	0.6	-0.2	-7.26	198.82	0.4219	0.4648	0.5269	0.5732	0.5592	0.6399	0.5715
	0.6	-0.1	-0.94	3.29	0.3120	0.3412	0.4032	0.4495	0.4458	0.5179	0.4501
	0.6	0	-0.42	2.16	0.2105	0.2228	0.2727	0.3105	0.3202	0.3750	0.3107
	0.6	0.25	0.094	2.18	0.0843	0.0886	0.0988	0.1083	0.1745	0.1209	0.0550
	0.6	0.5	0.045	2.03	0.0582	0.0787	0.0710	0.0755	0.1849	0.0763	0.0061
	0.75	-0.2	-7.33	212.90	0.4789	0.5543	0.6034	0.6534	0.6786	0.7196	0.6282
	0.75	-0.1	-0.69	2.60	0.3719	0.4323	0.4801	0.5309	0.5751	0.6107	0.5143
	0.75	0	-0.24	1.95	0.2621	0.3037	0.3553	0.4024	0.4517	0.4827	0.3776
	0.75	0.25	0.18	2.12	0.1165	0.1257	0.1473	0.1640	0.2680	0.1927	0.0905
	0.75	0.5	0.12	1.99	0.0759	0.1001	0.0943	0.1053	0.2443	0.1145	0.0126
	0.9	-0.2	-7.49	229.48	0.5260	0.6127	0.6513	0.6996	0.7457	0.7658	0.6586
	0.9	-0.1	-0.49	2.20	0.4261	0.5124	0.5553	0.6099	0.6801	0.6868	0.5609
	0.9	0	-0.09	1.84	0.3172	0.3835	0.4262	0.4784	0.5613	0.5623	0.4276

(continued)

Table 4 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group Ib. (Continued)

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
	0.9	0.25	0.26	2.11	0.1477	0.1674	0.1954	0.2236	0.3580	0.2680	0.1264
	0.9	0.5	0.19	1.97	0.0949	0.1212	0.1220	0.1406	0.3147	0.1610	0.0245
30	0.6	-0.2	-7.26	198.82	0.5934	0.6519	0.7169	0.7668	0.7718	0.8354	0.7499
	0.6	-0.1	-0.94	3.29	0.4511	0.4982	0.5795	0.6416	0.6520	0.7308	0.6246
	0.6	0	-0.42	2.16	0.3098	0.3333	0.4146	0.4781	0.5053	0.5834	0.4624
	0.6	0.25	0.094	2.18	0.1205	0.1212	0.1474	0.1652	0.2653	0.1912	0.0620
	0.6	0.5	0.045	2.03	0.0805	0.1122	0.1061	0.1189	0.2861	0.1281	0.0043
	0.75	-0.2	-7.33	212.90	0.6659	0.7511	0.7929	0.8412	0.8734	0.8981	0.8023
	0.75	-0.1	-0.69	2.60	0.5417	0.6214	0.6830	0.7461	0.7896	0.8241	0.6938
	0.75	0	-0.24	1.95	0.4024	0.4666	0.5349	0.6067	0.6774	0.7111	0.5350
	0.75	0.25	0.18	2.12	0.1742	0.1840	0.2267	0.2652	0.4118	0.3261	0.1180
	0.75	0.5	0.12	1.99	0.1069	0.1420	0.1491	0.1752	0.3886	0.2025	0.0116
	0.9	-0.2	-7.49	229.48	0.7231	0.8172	0.8443	0.8878	0.9296	0.9356	0.8355
	0.9	-0.1	-0.49	2.20	0.6087	0.7052	0.7479	0.8078	0.8695	0.8784	0.7382
	0.9	0	-0.09	1.84	0.4727	0.5774	0.6254	0.6976	0.7937	0.7981	0.6065
	0.9	0.25	0.26	2.11	0.2332	0.2681	0.3154	0.3724	0.5553	0.4579	0.1776
	0.9	0.5	0.19	1.97	0.1454	0.1909	0.2056	0.2475	0.4999	0.2963	0.0253
50	0.6	-0.2	-7.26	198.82	0.8089	0.8683	0.9064	0.9419	0.9561	0.9758	0.9312
	0.6	-0.1	-0.94	3.29	0.6671	0.7373	0.8026	0.8672	0.8979	0.9378	0.8517
	0.6	0	-0.42	2.16	0.4852	0.5344	0.6301	0.7168	0.7781	0.8425	0.6888
	0.6	0.25	0.094	2.18	0.1834	0.1859	0.2449	0.2908	0.4652	0.3788	0.1042
	0.6	0.5	0.045	2.03	0.1144	0.1770	0.1741	0.2184	0.5080	0.2845	0.0188
	0.75	-0.2	-7.33	212.90	0.8758	0.9351	0.9514	0.9734	0.9880	0.9916	0.9596
	0.75	-0.1	-0.69	2.60	0.7653	0.8572	0.8882	0.9345	0.9669	0.9772	0.9053
	0.75	0	-0.24	1.95	0.6089	0.7146	0.7725	0.8515	0.9194	0.9356	0.7887
	0.75	0.25	0.18	2.12	0.2825	0.3179	0.3852	0.4729	0.6954	0.6126	0.2144
	0.75	0.5	0.12	1.99	0.1714	0.2372	0.2559	0.3257	0.6598	0.4337	0.0422
	0.9	-0.2	-7.49	229.48	0.9219	0.9685	0.9734	0.9884	0.9964	0.9974	0.9756
	0.9	-0.1	-0.49	2.20	0.8404	0.9238	0.9354	0.9674	0.9895	0.9907	0.9390
	0.9	0	-0.09	1.84	0.7120	0.8372	0.8610	0.9229	0.9743	0.9767	0.8601
	0.9	0.25	0.26	2.11	0.3854	0.4586	0.5274	0.6340	0.8434	0.7846	0.3480
	0.9	0.5	0.19	1.97	0.2419	0.3130	0.3565	0.4512	0.7868	0.5925	0.0852

Table 5 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group Ic.

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
10	0.6	0.6	0	1.99	0.0494	0.0587	0.0519	0.0525	0.1019	0.0508	0.0230
	0.6	0.9	-0.14	1.92	0.0608	0.0715	0.0678	0.0704	0.1384	0.0744	0.0321
	0.75	0.6	0.07	1.94	0.0514	0.0612	0.0563	0.0583	0.1167	0.0590	0.0258
	0.75	0.75	0	1.89	0.0550	0.0690	0.0621	0.0638	0.1283	0.0640	0.0250
	0.75	0.9	-0.07	1.86	0.0586	0.0736	0.0661	0.0678	0.1414	0.0693	0.0261
	0.9	0.9	0	1.83	0.0594	0.0769	0.0683	0.0709	0.1518	0.0737	0.0269
20	0.6	0.6	0	1.99	0.0614	0.0866	0.0791	0.0871	0.2162	0.0866	0.0053
	0.6	0.9	-0.14	1.92	0.0908	0.1216	0.1187	0.1349	0.3255	0.1536	0.0123
	0.75	0.6	0.07	1.94	0.0707	0.0994	0.0931	0.1029	0.2637	0.1110	0.0070
	0.75	0.75	0	1.89	0.0739	0.1115	0.1018	0.1167	0.3050	0.1303	0.0066
	0.75	0.9	-0.07	1.86	0.0852	0.1247	0.1184	0.1387	0.3531	0.1583	0.0088
	0.9	0.9	0	1.83	0.0889	0.1339	0.1255	0.1495	0.3830	0.1743	0.0066
30	0.6	0.6	0	1.99	0.0819	0.1255	0.1143	0.1353	0.3322	0.1498	0.0031
	0.6	0.9	-0.14	1.92	0.1267	0.1803	0.1876	0.2284	0.5056	0.2760	0.0124
	0.75	0.6	0.07	1.94	0.1010	0.1483	0.1440	0.1755	0.4253	0.2066	0.0063
	0.75	0.75	0	1.89	0.1079	0.1763	0.1654	0.2035	0.4877	0.2465	0.0058
	0.75	0.9	-0.07	1.86	0.1266	0.1955	0.1931	0.2407	0.5529	0.2966	0.0094
	0.9	0.9	0	1.83	0.1304	0.2121	0.2101	0.2660	0.6005	0.3292	0.0105
50	0.6	0.6	0	1.99	0.1228	0.2037	0.1969	0.2546	0.5836	0.3422	0.0204
	0.6	0.9	-0.14	1.92	0.2169	0.3165	0.3411	0.4448	0.8110	0.5892	0.0777
	0.75	0.6	0.07	1.94	0.1653	0.2574	0.2627	0.3424	0.7086	0.4626	0.0455
	0.75	0.75	0	1.89	0.1770	0.2964	0.3009	0.3944	0.7868	0.5399	0.0577
	0.75	0.9	-0.07	1.86	0.2163	0.3457	0.3596	0.4695	0.8470	0.6249	0.0908
	0.9	0.9	0	1.83	0.2273	0.3696	0.3847	0.5076	0.8852	0.6821	0.1092

Table 6 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group II.

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
10	1.25	-0.2	-8.16	280.37	0.3265	0.3885	0.4109	0.4369	0.4507	0.4708	0.4064
	1.25	-0.1	-0.14	1.81	0.2700	0.3236	0.3481	0.3722	0.3875	0.4037	0.3448
	1.25	0	0.20	1.80	0.2153	0.2567	0.2775	0.2998	0.3207	0.3293	0.2718
	1.25	0.25	0.44	2.17	0.1251	0.1417	0.1524	0.1630	0.2148	0.1792	0.1259
	1.25	0.5	0.35	2.00	0.0841	0.1006	0.1002	0.1080	0.1861	0.1159	0.0609
	1.25	0.75	0.218	1.86	0.0715	0.0907	0.0852	0.0892	0.1768	0.0944	0.0394
	1.25	0.95	0.12	1.80	0.0658	0.0849	0.0793	0.0846	0.1808	0.0934	0.0348
	2	-0.2	-10.64	443.42	0.3924	0.4727	0.4942	0.5218	0.5579	0.5563	0.4749
	2	-0.1	0.376	1.85	0.3283	0.3984	0.4228	0.4486	0.4854	0.4824	0.4028
	2	0	0.64	2.14	0.2761	0.3349	0.3562	0.3815	0.4211	0.4168	0.3395
	2	0.25	0.75	2.5	0.1712	0.2042	0.2191	0.2372	0.2952	0.2598	0.1867
	2	0.5	0.61	2.24	0.1190	0.1389	0.1423	0.1535	0.2292	0.1651	0.0948
	2	0.75	0.46	2.02	0.0879	0.1096	0.1082	0.1153	0.2067	0.1249	0.0559
	2	0.95	0.36	1.91	0.0773	0.0965	0.0923	0.1000	0.1968	0.1075	0.0445
20	1.25	-0.2	-8.16	280.37	0.6143	0.7226	0.7500	0.7978	0.8623	0.8521	0.7284
	1.25	-0.1	-0.14	1.81	0.5149	0.6280	0.6565	0.7136	0.8017	0.7829	0.6353
	1.25	0	0.20	1.80	0.4177	0.5265	0.5611	0.6216	0.7316	0.7043	0.5273
	1.25	0.25	0.44	2.17	0.2315	0.2795	0.3133	0.3560	0.5276	0.4262	0.2121
	1.25	0.5	0.35	2.00	0.1496	0.1876	0.1988	0.2314	0.4470	0.2694	0.0556
	1.25	0.75	0.218	1.86	0.1125	0.1614	0.1627	0.1934	0.4370	0.2281	0.188
	1.25	0.95	0.12	1.80	0.1033	0.1587	0.1505	0.1805	0.4494	0.2152	0.109
	2	-0.2	-10.64	443.42	0.7103	0.8211	0.8345	0.8724	0.9264	0.9137	0.7930
	2	-0.1	0.376	1.85	0.6275	0.7484	0.7664	0.8142	0.8888	0.8698	0.7169
	2	0	0.64	2.14	0.5395	0.6675	0.6915	0.7463	0.8423	0.8153	0.6311
	2	0.25	0.75	2.5	0.3402	0.4260	0.4578	0.5154	0.6812	0.6018	0.3391
	2	0.5	0.61	2.24	0.2205	0.2696	0.2957	0.3343	0.5493	0.3945	0.1197
	2	0.75	0.46	2.02	0.1650	0.2104	0.2240	0.2613	0.5116	0.3073	0.0461
	2	0.95	0.36	1.91	0.1330	0.1869	0.1913	0.2254	0.4948	0.2663	0.0251
30	1.25	-0.2	-8.16	280.37	0.8203	0.9067	0.9152	0.9446	0.9756	0.9716	0.8903
	1.25	-0.1	-0.14	1.81	0.7230	0.8401	0.8535	0.8989	0.9520	0.9467	0.8201
	1.25	0	0.20	1.80	0.6124	0.7423	0.7692	0.8332	0.9193	0.9054	0.7202
	1.25	0.25	0.44	2.17	0.3549	0.4479	0.4875	0.5646	0.7556	0.6759	0.3159
	1.25	0.5	0.35	2.00	0.2314	0.2950	0.3263	0.3921	0.6730	0.4717	0.0755
	1.25	0.75	0.218	1.86	0.1726	0.2546	0.2684	0.3310	0.6716	0.4151	0.0257
	1.25	0.95	0.12	1.80	0.1699	0.2634	0.2668	0.3406	0.6981	0.4303	0.0195
	2	-0.2	-10.64	443.42	0.8910	0.9546	0.9577	0.9751	0.9915	0.9886	0.9317
	2	-0.1	0.376	1.85	0.8347	0.9240	0.9288	0.9545	0.9852	0.9785	0.8916
	2	0	0.64	2.14	0.7498	0.8671	0.8747	0.9204	0.9680	0.9616	0.8240
	2	0.25	0.75	2.5	0.5037	0.6379	0.6692	0.7457	0.8844	0.8365	0.5026
	2	0.5	0.61	2.24	0.3409	0.4246	0.4733	0.5489	0.7872	0.6398	0.1726
	2	0.75	0.46	2.02	0.2594	0.3369	0.3739	0.4488	0.7409	0.5351	0.0647
	2	0.95	0.36	1.91	0.2114	0.3039	0.3276	0.4058	0.7343	0.4959	0.0402
50	1.25	-0.2	-8.16	280.37	0.9685	0.9912	0.9912	0.9963	0.9997	0.9995	0.9882
	1.25	-0.1	-0.14	1.81	0.9328	0.9800	0.9814	0.9922	0.9989	0.9985	0.9770
	1.25	0	0.20	1.80	0.8616	0.9483	0.9512	0.9787	0.9967	0.9958	0.9385
	1.25	0.25	0.44	2.17	0.5714	0.7194	0.7429	0.8415	0.9628	0.9390	0.5867
	1.25	0.5	0.35	2.00	0.3989	0.5047	0.5584	0.6732	0.9212	0.8031	0.2253
	1.25	0.75	0.218	1.86	0.3115	0.4453	0.4843	0.6161	0.9284	0.7756	0.1641
	1.25	0.95	0.12	1.80	0.2931	0.4568	0.4857	0.6223	0.9430	0.7948	0.1778
	2	-0.2	-10.64	443.42	0.9908	0.9984	0.9977	0.9996	0.9999	0.9999	0.9962
	2	-0.1	0.376	1.85	0.9783	0.9955	0.9959	0.9985	0.9999	0.9999	0.9924
	2	0	0.64	2.14	0.9497	0.9869	0.9864	0.9963	0.9996	0.9995	0.9775
	2	0.25	0.75	2.5	0.7702	0.8976	0.9009	0.9541	0.9924	0.9883	0.8129
	2	0.5	0.61	2.24	0.5746	0.7035	0.7426	0.8353	0.9720	0.9262	0.4443
	2	0.75	0.46	2.02	0.4377	0.5643	0.6216	0.7389	0.9589	0.8649	0.2695
	2	0.95	0.36	1.91	0.3741	0.5193	0.5706	0.7036	0.9585	0.8503	0.2320

Table 7 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group III.

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
10	1.25	1.25	0	1.76	0.0663	0.0877	0.0785	0.0857	0.1919	0.0911	0.0317
	1.25	1.5	-0.09	1.76	0.0713	0.0904	0.0828	0.0894	0.1918	0.0954	0.0341
	1.25	1.75	-0.17	1.79	0.0732	0.0948	0.0888	0.0939	0.1982	0.0999	0.0366
	1.5	1.5	0	1.75	0.0697	0.0933	0.0851	0.0911	0.2000	0.0966	0.0313
	1.75	1.5	0.08	1.76	0.0703	0.0906	0.0834	0.0914	0.1943	0.0971	0.0315
	1.75	1.75	0	1.77	0.0677	0.0888	0.0805	0.0870	0.1889	0.0923	0.0312
20	1.25	1.25	0	1.76	0.1127	0.1743	0.1656	0.2022	0.4926	0.244	0.0116
	1.25	1.5	-0.09	1.76	0.1104	0.1723	0.1665	0.2051	0.4929	0.2481	0.0107
	1.25	1.75	-0.17	1.79	0.1168	0.1755	0.1735	0.2127	0.4990	0.2559	0.0125
	1.5	1.5	0	1.75	0.1119	0.1811	0.1727	0.2101	0.5066	0.2536	0.0093
	1.75	1.5	0.08	1.76	0.1127	0.1736	0.1663	0.2027	0.4934	0.2470	0.0109
	1.75	1.75	0	1.77	0.1059	0.1660	0.1593	0.1960	0.4770	0.2345	0.0093
30	1.25	1.25	0	1.76	0.1737	0.2781	0.2825	0.3595	0.7270	0.4581	0.0220
	1.25	1.5	-0.09	1.76	0.1798	0.2898	0.2985	0.3775	0.7436	0.4783	0.0235
	1.25	1.75	-0.17	1.79	0.1812	0.2841	0.2954	0.3744	0.7354	0.4714	0.0246
	1.5	1.5	0	1.75	0.1773	0.2880	0.2964	0.3777	0.7462	0.4778	0.0238
	1.75	1.5	0.08	1.76	0.1705	0.2716	0.2763	0.3538	0.7280	0.4543	0.0200
	1.75	1.75	0	1.77	0.1705	0.2705	0.2752	0.3495	0.7169	0.4409	0.0186
50	1.25	1.25	0	1.76	0.3023	0.4837	0.5126	0.6588	0.9572	0.8309	0.2068
	1.25	1.5	-0.09	1.76	0.3199	0.4928	0.5231	0.6727	0.9641	0.8387	0.2224
	1.25	1.75	-0.17	1.79	0.3294	0.5013	0.5359	0.6825	0.9621	0.8443	0.2245
	1.5	1.5	0	1.75	0.3131	0.4922	0.5196	0.6689	0.9640	0.8392	0.2176
	1.75	1.5	0.08	1.76	0.3053	0.4854	0.5174	0.6570	0.9581	0.8294	0.2104
	1.75	1.75	0	1.77	0.2948	0.4714	0.5000	0.6467	0.9504	0.8143	0.1966

Table 8 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group IV.

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
10	2.5	1.25	0.35	1.92	0.0685	0.0869	0.0815	0.0864	0.1820	0.0932	0.0369
	2.5	1.5	0.25	1.87	0.0672	0.0857	0.0752	0.0795	0.1754	0.0853	0.0311
	2.5	1.75	0.18	1.86	0.0575	0.0747	0.0687	0.0714	0.1590	0.0736	0.0261
	3	1.25	0.44	2.03	0.0661	0.0825	0.0783	0.0809	0.1677	0.0854	0.0323
	3	1.25	0.34	1.97	0.0604	0.0741	0.0690	0.0727	0.1540	0.0774	0.0277
	3	1.75	0.26	1.95	0.0586	0.0724	0.0648	0.0678	0.1481	0.0694	0.0246
	5	1.25	0.67	2.50	0.0530	0.0661	0.0601	0.0621	0.1343	0.0635	0.0263
	5	1.5	0.56	2.39	0.0532	0.0645	0.0576	0.0604	0.1273	0.0615	0.0236
	5	1.75	0.46	2.34	0.0474	0.0565	0.0507	0.0515	0.1151	0.0532	0.0219
	20	1.25	0.35	1.92	0.1145	0.1654	0.1603	0.1939	0.4554	0.2313	0.0131
20	2.5	0.25	1.87	0.0952	0.1449	0.1416	0.1711	0.4300	0.2030	0.0099	
	2.5	1.75	0.18	1.86	0.0942	0.1404	0.1342	0.1583	0.3993	0.1871	0.0075
	3	1.25	0.44	2.03	0.1022	0.1504	0.1439	0.1733	0.4118	0.2042	0.0121
	3	1.25	0.34	1.97	0.0902	0.1341	0.1265	0.1489	0.3877	0.1742	0.0083
	3	1.75	0.26	1.95	0.0775	0.1186	0.1091	0.1303	0.3477	0.1504	0.0064
	5	1.25	0.67	2.50	0.0758	0.1131	0.1030	0.1213	0.3225	0.1388	0.0064
	5	1.5	0.56	2.39	0.0680	0.1015	0.0907	0.1052	0.2980	0.1203	0.0052
	5	1.75	0.46	2.34	0.0681	0.0938	0.0865	0.1010	0.2751	0.1143	0.0065
	30	1.25	0.35	1.92	0.1738	0.2648	0.2717	0.3435	0.6897	0.4327	0.0213
	2.5	0.25	1.87	0.1564	0.2444	0.2474	0.3072	0.6673	0.3904	0.0160	
30	2.5	1.75	0.18	1.86	0.1418	0.2238	0.2234	0.2811	0.6302	0.3560	0.0132
	3	1.25	0.44	2.03	0.1556	0.2317	0.2403	0.3040	0.6417	0.3813	0.0183
	3	1.25	0.34	1.97	0.1342	0.2117	0.2111	0.2672	0.6145	0.3388	0.0116
	3	1.75	0.26	1.95	0.1154	0.1876	0.1839	0.2333	0.5668	0.2941	0.0080
	5	1.25	0.67	2.50	0.1119	0.1726	0.1676	0.2126	0.5252	0.2703	0.0067
	5	1.5	0.56	2.39	0.0959	0.1552	0.1465	0.1855	0.4937	0.2362	0.0056
	5	1.75	0.46	2.34	0.0966	0.1460	0.1442	0.1758	0.4481	0.2154	0.0066

(continued)

Table 8 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group IV. (Continued)

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
50	2.5	1.25	0.35	1.92	0.3061	0.4544	0.4933	0.6279	0.9408	0.7960	0.1815
	2.5	1.5	0.25	1.87	0.2696	0.4270	0.4552	0.5921	0.9287	0.7646	0.1541
	2.5	1.75	0.18	1.86	0.2345	0.3853	0.4065	0.5364	0.9041	0.7130	0.1246
	3	1.25	0.44	2.03	0.2677	0.4100	0.4421	0.5778	0.9203	0.7560	0.1402
	3	1.25	0.34	1.97	0.2221	0.3669	0.3821	0.5091	0.8971	0.6924	0.1115
	3	1.75	0.26	1.95	0.1933	0.3213	0.3346	0.4537	0.8640	0.6332	0.0836
	5	1.25	0.67	2.50	0.1781	0.2920	0.3015	0.4102	0.8381	0.5889	0.0675
	5	1.5	0.56	2.39	0.1568	0.2665	0.2699	0.3702	0.8056	0.5374	0.0509
	5	1.75	0.46	2.34	0.1555	0.2471	0.2532	0.3415	0.7590	0.4906	0.0438

Table 9 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group V.

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
10	2.5	2.5	0	1.91	0.0508	0.0634	0.0574	0.0593	0.1235	0.0605	0.0247
	2.5	3	-0.08	1.98	0.0462	0.0547	0.0487	0.0497	0.1086	0.0494	0.0215
	2.5	4	-0.2	2.17	0.0462	0.0508	0.0469	0.0478	0.0874	0.0480	0.0240
	2.5	10	-0.33	3.02	0.1550	0.1582	0.1775	0.1808	0.1764	0.1827	0.1225
	3	2.5	0.08	1.98	0.0475	0.0529	0.0482	0.0495	0.1033	0.0506	0.0224
	3	3	0	2.06	0.0394	0.0459	0.0413	0.0414	0.0874	0.0419	0.0200
	3	4	-0.11	2.24	0.0437	0.0468	0.0434	0.0422	0.0673	0.0417	0.0262
	3	5	-0.18	2.44	0.0556	0.0543	0.0565	0.0548	0.0675	0.0534	0.0350
	3	10	-0.22	3.14	0.2118	0.2105	0.2384	0.2394	0.1953	0.2377	0.1732
	5	2.5	0.27	2.36	0.0591	0.0603	0.0588	0.0584	0.0859	0.0571	0.0331
	5	3	0.18	2.44	0.0592	0.0576	0.0580	0.0564	0.0699	0.0544	0.0361
	5	4	0.06	2.66	0.0650	0.0679	0.0661	0.0619	0.0501	0.0553	0.0477
	5	5	0	2.90	0.0837	0.0877	0.0889	0.0834	0.0459	0.0711	0.0657
	5	10	-0.007	3.83	0.3532	0.3628	0.3867	0.3711	0.2141	0.3341	0.2725
20	2.5	2.5	0	1.91	0.0671	0.1027	0.0921	0.1078	0.2980	0.1243	0.0054
	2.5	3	-0.08	1.98	0.0584	0.0814	0.0722	0.0825	0.2372	0.0915	0.0048
	2.5	4	-0.2	2.17	0.0537	0.0686	0.0602	0.0655	0.1755	0.0692	0.0076
	2.5	10	-0.33	3.02	0.3245	0.3243	0.4015	0.4188	0.4262	0.4170	0.1666
	3	2.5	0.08	1.98	0.0586	0.0811	0.0720	0.0818	0.2320	0.0898	0.0042
	3	3	0	2.06	0.0430	0.0595	0.0515	0.0563	0.1750	0.0619	0.0046
	3	4	-0.11	2.24	0.0421	0.0518	0.0440	0.0467	0.1169	0.0457	0.0068
	3	5	-0.18	2.44	0.0715	0.0766	0.0832	0.0825	0.1160	0.0737	0.0226
	3	10	-0.22	3.14	0.4228	0.4102	0.4996	0.5063	0.4310	0.4862	0.2540
	5	2.5	0.27	2.36	0.0713	0.0814	0.0884	0.0930	0.1732	0.0924	0.0167
	5	3	0.18	2.44	0.0723	0.0755	0.0800	0.0823	0.1122	0.0729	0.0200
	5	4	0.06	2.66	0.0802	0.0853	0.0851	0.0778	0.0613	0.0570	0.0263
	5	5	0	2.90	0.1200	0.1433	0.1333	0.1117	0.1350	0.0655	0.0370
	5	10	-0.007	3.83	0.6405	0.6593	0.7047	0.6772	0.4580	0.5857	0.4217
30	2.5	2.5	0	1.91	0.0963	0.1551	0.1461	0.1823	0.4836	0.2291	0.0060
	2.5	3	-0.08	1.98	0.0792	0.1202	0.1101	0.1334	0.3929	0.1659	0.0027
	2.5	4	-0.2	2.17	0.0672	0.0924	0.0868	0.1024	0.2960	0.1197	0.0033
	2.5	10	-0.33	3.02	0.4937	0.4925	0.6021	0.6278	0.6249	0.6310	0.2283
	3	2.5	0.08	1.98	0.0723	0.1140	0.1088	0.1361	0.3959	0.1684	0.0039
	3	3	0	2.06	0.0568	0.0857	0.0759	0.0924	0.3040	0.1114	0.0023
	3	4	-0.11	2.24	0.0513	0.0665	0.0623	0.0680	0.2101	0.0718	0.0044
	3	5	-0.18	2.44	0.1012	0.1044	0.1156	0.1205	0.2059	0.1094	0.0158
	3	10	-0.22	3.14	0.6192	0.5951	0.7129	0.7219	0.6376	0.7009	0.3593
	5	2.5	0.27	2.36	0.1054	0.1189	0.1293	0.1431	0.2887	0.1505	0.0123
	5	3	0.18	2.44	0.0979	0.1009	0.1140	0.1181	0.2093	0.1081	0.0159
	5	4	0.06	2.66	0.1013	0.1124	0.1111	0.1022	0.1351	0.0669	0.0154
	5	5	0	2.90	0.1508	0.1838	0.1678	0.1504	0.1413	0.0796	0.0224
	5	10	-0.007	3.83	0.8310	0.8439	0.8816	0.8601	0.7556	0.7767	0.5465
50	2.5	2.5	0	1.91	0.1472	0.2572	0.2582	0.3570	0.7916	0.5213	0.0492
	2.5	3	-0.08	1.98	0.1124	0.1926	0.1830	0.2580	0.7001	0.3961	0.0254
	2.5	4	-0.2	2.17	0.0952	0.1453	0.1387	0.1894	0.5684	0.2821	0.0128
	2.5	10	-0.33	3.02	0.7504	0.7489	0.8554	0.8832	0.8954	0.9051	0.5397
	3	2.5	0.08	1.98	0.1121	0.1932	0.1847	0.2604	0.6965	0.3970	0.0245

(continued)

Table 9 | Power comparisons of the tests at significance level 0.05 for sample sizes $n = 10, 20, 30$ and 50 under alternatives from group V . (Continued)

n	λ_3	λ_4	α_3	α_4	D	V	CH	A^2	KL_{mn}	W	JB
3	3	0	2.06	0.0723	0.1236	0.1121	0.1580	0.5702	0.2573	0.0090	
3	4	-0.11	2.24	0.0617	0.0873	0.0805	0.1074	0.4141	0.1532	0.0045	
3	5	-0.18	2.44	0.1514	0.1622	0.1831	0.2023	0.4169	0.2204	0.0179	
3	10	-0.22	3.14	0.8553	0.8323	0.9188	0.9295	0.9015	0.9295	0.6700	
5	2.5	0.27	2.36	0.1540	0.1875	0.2103	0.2590	0.5529	0.3328	0.0259	
5	3	0.18	2.44	0.1477	0.1598	0.1794	0.2019	0.4190	0.2214	0.0186	
5	4	0.06	2.66	0.1456	0.1603	0.1600	0.1536	0.2932	0.1068	0.0089	
5	5	0	2.90	0.2347	0.3042	0.2761	0.2515	0.3472	0.1249	0.0086	
5	10	-0.007	3.83	0.9684	0.9720	0.9845	0.9796	0.9629	0.9555	0.7671	

Table 10 | The best test in term of power performance in different groups

Groups (Alternatives)						
Ia JB, W	Ib W, KL_{mn}	Ic KL_{mn}	II KL_{mn}	III KL_{mn}	IV KL_{mn}	V KL_{mn}

From Tables 3-9, we can see that the power values of the tests are considerably different. We can select the tests which are, generally, the most powerful against the alternatives from the given groups.

In group Ia, it is seen that the tests JB and W generally have the most power and the test KL_{mn} has the most power against the symmetric alternatives with parameters $\lambda_3 = \lambda_4 = 0.25, 0.5$. The difference powers between JB (or W) with the other tests are considerable.

In group Ib, sometimes the test W gives a higher power and sometimes the test KL_{mn} does. When the sample size increases the difference power values become substantial. It is seen that, in groups Ia and Ib, when $\lambda_4 > 0$, the test KL_{mn} is powerful and otherwise the test W is powerful.

In group Ic, a uniform superiority of KL_{mn} test to other tests is obvious. The difference of power values between the KL_{mn} test and the other tests is substantial. We note that in this group(Ic) $0.5 < \lambda_4 < 1$.

In group II, the KL_{mn} test has the most power and the other tests have low powers. The difference power values between the KL_{mn} test and the other tests are substantial. When $\lambda_4 < 0$, we can see that the difference power values between KL_{mn} and W are small.

In groups III and IV, since $\lambda_4 > 0$ the KL_{mn} test again has the most power. The difference powers between the KL_{mn} test and the other tests are substantial. For these groups, the test KL_{mn} , based on the sample entropy, has very good power values but the other tests have low powers.

Finally, in group V, the test based on KL_{mn} statistic has generally the most power. We can see that when $\lambda_4 > 5$, the KL_{mn} test dose not achieve the most power, but when sample size increases the power of KL_{mn} test increases and the difference powers between KL_{mn} test and the other tests become small. Therefore, we can conclude that the KL_{mn} test is powerful in this group.

Briefly, the best test in term of power for different groups is presented in Table 10.

It should be noted that the KS and JB tests have the least power values in groups Ia, Ib, II and Ic, III, IV, V, respectively.

4. ILLUSTRATION WITH SOME REAL DATA

In this section, we apply some real data examples to show the behavior of the tests. The histograms of the considered data sets are displayed in Figs. 8 and 9.

Example 1. The following data are 100 breaking strengths of yarn presented by Duncan [21]:

66, 117, 132, 111, 107, 85, 89, 79, 91, 97, 138, 103, 111, 86, 78, 96, 93, 101, 102, 110, 95, 96, 88, 122, 115, 92, 137, 91, 84, 96, 97, 100, 105, 104, 137, 80, 104, 104, 106, 84, 92, 86, 104, 132, 94, 99, 102, 101, 104, 107, 99, 85, 95, 89, 102, 100, 98, 97, 104, 114, 111, 98, 99, 102, 91, 95, 111, 104, 97, 98, 102, 109, 88, 91, 103, 94, 105, 103, 96, 100, 101, 98, 97, 101, 102, 98, 94, 100, 98, 99, 92, 102, 87, 99, 62, 92, 100, 96, 98.

Puig and Stephens [22] used the Empirical distribution function (EDF) tests for fitting a normal distribution for these data. They concluded that all the test statistics have significance levels below 0.01 so that the normal assumption is rejected at this level of significance.

We apply the tests for testing the assumption that the data are from a normal distribution. We obtain the maximum likelihood estimator of FMKL family as

$$\hat{\lambda}_1 = 98.6370, \hat{\lambda}_2 = 0.2715, \hat{\lambda}_3 = -0.2678, \hat{\lambda}_4 = -0.3866.$$

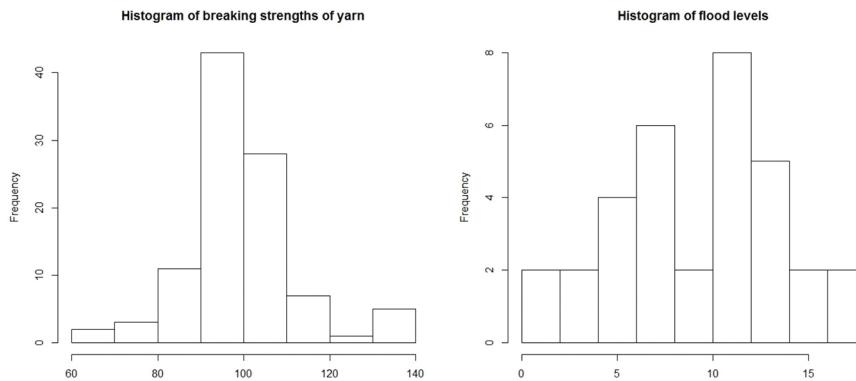


Figure 8 | Histograms for data in Examples 1 and 2.

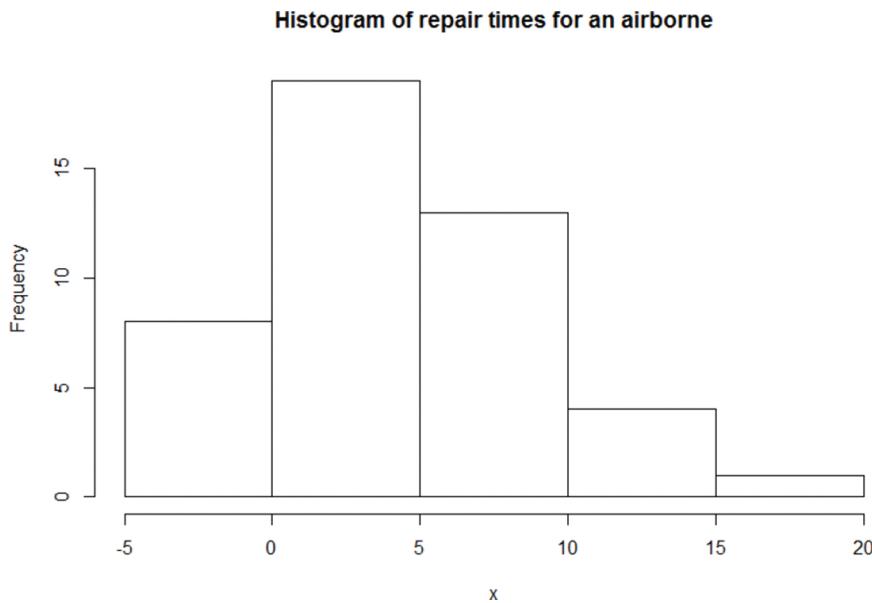


Figure 9 | Histograms for data set in Example 3.

Table 11 | Comparison of p -values of the tests in Example 1.

Test	Value of the test statistic	p -value
D	0.1569	0.0000
V	0.2462	0.0000
CH	0.4503	0.0000
A^2	2.7180	0.0000
KL_{mn}	0.1887	0.0006
W	0.9904	0.0000
JB	31.5526	0.0009

Since $\hat{\lambda}_3, \hat{\lambda}_4 \leq 0.5$, density of these data belong to class Ia. Based on our simulations, in this class generally the test JB has the most power and we should use this test. Moreover, if we assume approximately $\hat{\lambda}_3 \approx \hat{\lambda}_4$, the test based on KL_{mn} is appreciate.

The values of the test statistics are

$$D = 0.1569, V = 0.2462, CH = 0.4503, A^2 = 2.7180, KL_{mn} = 0.1887, W = 0.9904, JB = 31.5526,$$

and then the p -values obtained are tabulated in Table 11.

Therefore, the normal assumption is rejected by the EDF statistics. Also, the tests KL_{mn} , W and JB reject the normal assumption.

Example 2. Bain and Engelhardt [23] presented the following dataset, consisting of 33 difference in flood levels between stations on a river: 1.96, 1.97, 3.60, 3.80, 4.79, 5.66, 5.76, 5.78, 6.27, 6.30, 6.76, 7.65, 7.84, 7.99, 8.51, 9.18, 10.13, 10.24, 10.25, 10.43, 11.45, 11.48, 11.75, 11.81, 12.34, 12.78, 13.06, 13.29, 13.98, 14.18, 14.40, 16.22, 17.06.

They suggested that the Laplace distribution might provide a good fit. Puig and Stephens [22] concluded that D and W^2 reject the Laplace distribution for the data at 0.05 level and the other tests accept the Laplace assumption.

We apply the tests for testing the assumption that the data are from a normal distribution. We obtain the maximum likelihood estimator of FMKL family as

$$\hat{\lambda}_1 = 8.0967, \hat{\lambda}_2 = 0.1564, \hat{\lambda}_3 = 0.9997, \hat{\lambda}_4 = 0.6779.$$

Since $0.5 < \hat{\lambda}_3, \hat{\lambda}_4 < 1$, density of these data belong to class Ic. Based on our simulations, in this class the test KL_{mn} has the most power and we should use this test.

The values of the test statistics are

$$D = 0.0929, V = 0.1722, CH = 0.0416, A^2 = 0.2467, KL_{mn} = 0.2175, W = 0.9766, JB = 1.3645,$$

and then the p -values are obtained and presented in Table 12.

Therefore, the normal assumption is accepted by all tests at the significance level of 0.05. Based on our simulation results presented in Section 3, we can accept the result obtained by KL_{mn} test.

Example 3. The following data represent active repair times (in hours) for an airborne communication transceiver:

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

von Alven [24] fitted the lognormal distribution for these data. Chhikara and Folks [25] fitted the Inverse Gaussian (IG) distribution and justified it by the Kolmogorov–Smirnov statistic. Finally, Lee *et al.* [26] tested the lognormal and IG distributions which were both accepted.

We obtain the maximum likelihood estimator of FMKL family as

$$\hat{\lambda}_1 = 1.2184, \hat{\lambda}_2 = 0.6639, \hat{\lambda}_3 = 1.4792, \hat{\lambda}_4 = -0.6779.$$

Since $\hat{\lambda}_3 > 1$ and $\hat{\lambda}_4 < 1$, density of these data belong to class II. Based on our simulations, in this class the test based on KL_{mn} statistic has the most power and we should use this test.

In this case, we obtained the values of the test statistics for normal model as

$$D = 0.2465, V = 0.4614, CH = 0.8793, A^2 = 5.0138, KL_{mn} = 0.9404, W = 0.6317, JB = 177.19,$$

and then the p -values are obtained and presented in Table 13.

Therefore, the normal assumption is rejected by all tests at the significance level of 0.05. Based on our power study, we accept the result obtained by KL_{mn} test.

Table 12 | Comparison of p -values of the tests in Example 2.

Test	Value of the test statistic	p-value
D	0.0929	0.6599
V	0.1722	0.5707
CH	0.0416	0.6455
A^2	0.2467	0.7489
KL_{mn}	0.2175	0.3734
W	0.9766	0.6798
JB	1.3645	0.3857

Table 13 | Comparison of p -values of the tests in Example 2.

Test	Value of the test statistic	p-value
D	0.2465	0.0000
V	0.4614	0.0000
CH	0.8793	0.0000
A^2	5.0138	0.0000
KL_{mn}	0.9404	0.0000
W	0.6317	0.0000
JB	177.19	0.0000

5. CONCLUSIONS

In this paper, we considered seven popular tests for normality, namely, Kolmogorov-Smirnov, Anderson-Darling, Kuiper, Jarque-Bera, Cramer von Mises, Shapiro-Wilk and Vasicek. These tests are commonly used in practice and statistical software and therefore power values of these tests against various alternatives are important. Here, we considered the family of four-parameter GLDs which is called FMKL family, as alternatives for tests for normality.

The FMKL family is divided to five groups and therefore we computed the power values of the tests against these five groups using Monte Carlo computations for sample sizes $n = 10, 20, 30$ and $n = 50$. Differences in power of the seven tests are considerable and each of the tests JB , W and KL_{mn} can be most powerful depending on the type of alternatives. In group I, unimodal densities with continuous tails, the tests JB and W have the most power and when $\lambda_3 = \lambda_4$ (symmetric distribution) or $\lambda_4 > 0$ the test KL_{mn} has the most power. In groups II-V, monotone, U-shaped, S-shaped densities and unimodal pdfs with both tail truncated, the KL_{mn} test generally is most powerful.

Based on these observations, we can formulate the following recommendations for the application of the tests in practice.

- 1-Use the statistics JB or W , if the assumed alternatives are in groups Ia and Ib with the exception of the cases where $\lambda_3 = \lambda_4$ (symmetric distribution) and $\lambda_4 > 0$.
- 2-Use the statistic KL_{mn} , based on the sample entropy, if the assumed alternatives are not in groups Ia and Ib with the exception of the cases where $\lambda_3 = \lambda_4$ (symmetric distribution) and $\lambda_4 > 0$.

ACKNOWLEDGEMENT

The author is thankful to the referees and editor-in-chief of the journal for their valuable comments and suggestions, which improved the presentation of this paper greatly.

Declarations: The author has any competing interests in the manuscript.

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