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Weighted Entropy Measure: A New Measure of Information with its Properties in Reliability Theory and Stochastic Orders

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ABSTRACT

The weighted entropy measure is a germane dynamic measure of uncertainty in reliability and survival studies. In this paper, the new results of weighted entropies with some characterizations are provided. Furthermore, **we** have presented some results for weighted entropy residual and weighted past residual of order statistics with some application of some reliability systems such as a series structure and a parallel structure. In addition, we introduced the lower bound for the weighted residual (past) entropy. Moreover, the stochastic orders based on weighted entropy are presented. Finally, we illustrate the usefulness of the proposed non-parametric estimators of weighted entropy by application to real data.

1. INTRODUCTION

The weighted distributions have been utilized in many applications such as distributions theory, reliability, probability, ecology, bio-statistics and applied.

Consider the distribution function $G(.)$ for a random variable $Y \geq 0$ with density function $g(.)$. Suppose $l_Y = \inf\left\{y \in \mathbb{R}^1 : G\left(y\right) > 0\right\}$, $u_Y = \sup\left\{y \in \mathbb{R}^1 : G\left(y\right) < 1\right\}$, $S_Y = (l_Y, u_Y)$ and $w(.) \in R^+$ be a weighted function. The weighted random variable Y^W, having probability density function as:

$$g^w\left(y\right) = w\left(y\right) g\left(y\right) \big/ E\left[w\left(y\right)\right], -\infty \leq y \leq \infty, \tag{1}$$

where $E\left[w\left(y\right)\right] \in \mathbb{R}^+$. Let Y represents the life length of a "unit" in reliability studies, and life distribution, with survival function $\overline{G}_Y$, hazard rate function $\overline{\varphi}_G(.) = g_Y(.)/\overline{G}_Y(.)$, reversed hazard rate $\varphi_G(.) = g_Y(.)/G_Y(.)$, the geometric vitality function $\vartheta(Y) = E(\ln\ Y | Y > 0)$ and mean revered residual lifetime as

$$\theta(t) = \mathbb{E}\left[\kappa - Y | Y \leq \kappa\right] = \int_0^\kappa \frac{G_Y(u)\, du}{G_Y(\kappa)}, \ \kappa \in \mathbb{R}^+. \tag{2}$$

As reported by Ebrahimi and Pellery [1] and Asha and Rejeesh [2], the differential entropy (H_Y) demonstrate the expected uncertainty of $g\left(y\right)$. In addition, it measures how the distribution spreads over its domain, where there is an inverse relationship between the value of H_Y and concentration of the probability mass of Y. H_Y sometimes called a dynamic measure of uncertainty or Shannon information measure.

The differential entropy of random variable Y can be defined in the continuous case as follows:

$$H_Y = E\left[-\ln g_Y(Y)\right] = -\int_0^\infty g_Y(u) \ln g_Y(u)\, du. \tag{3}$$

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Khinchin (1957) generalized Eq. (3) as

$$H_Y^{\phi} = E\left[\phi\left(g_Y\left(y\right)\right)\right] = \int_0^{\infty} g_Y\left(u\right)\phi\left(g_Y\left(u\right)\right)du.$$

Di Crescenzo and Longobardi [4] developed the following convex entropy measure:

$$H^{w}\left(Y\right) = -\int_0^{\infty} v g_Y\left(v\right)\ln\, g_Y\left(v\right)dv,$$

or equivalently:

$$H^{w}\left(Y\right) = -\int_0^{\infty} dy\int_y^{\infty} g_Y\left(v\right)\ln g_Y\left(v\right)dv. \tag{4}$$

The uncertainty of the residual lifetime is discussed in Di Crescenzo and Longobardi [5], with the following measure:

$$H\left(Y,\kappa\right) = 1 - E\left[\ln\,\overline{\varphi}_G\left(Y\right)\vert Y > \kappa\right] = -\int_{\kappa}^{\infty}\frac{g_Y\left(v\right)}{\overline{G}\left(v\right)}\ln\,\frac{g_Y\left(v\right)}{\overline{G}\left(v\right)}dv,\ \kappa\in\mathbb{R}^{+},$$

where $\kappa\in A = \left\{y\in\mathbb{R}^{+}\left\vert\overline{G}_Y\left(y\right) > 0\right.\right\}$. In addition, the past entropy has been widely researched. We can measure it as follows:

$$\overline{H}_G\left(Y,\kappa\right) = 1 - E\left[\ln\,\varphi_G\left(Y\right)\vert Y < \kappa\right] = -\int_0^{\kappa}\frac{g_Y\left(v\right)}{G\left(v\right)}\ln\,\frac{g_Y\left(v\right)}{G\left(v\right)}dv,\ \kappa\in\mathbb{R}^{+}.$$

Di Crescenzo and Longobardi [4] defined the convex residual entropy as

$$H^{w}\left(Y,\kappa\right) = -\int_{\kappa}^{\infty} y\frac{g_Y\left(y\right)}{\overline{G}\left(\kappa\right)}\ln\,\frac{y g_Y\left(y\right)}{\overline{G}\left(\kappa\right)}dy,\ \kappa\in\mathbb{R}^{+}.$$

Furthermore, let Y_1 and Y_2 be two random variables with distribution functions $G_1\left(.\right)$ and $G_2\left(.\right)$, densities functions $g_{Y_1}\left(.\right)$ and $g_{Y_2}\left(.\right)$ and survival functions $\overline{G}_1\left(.\right)$ and $\overline{G}_2\left(.\right)$ respectively. Kullback and Leibler [6] introduced an information distance between two distributions G_1 and G_2 as follows:

$$I_{Y_1,Y_2} = \int_0^{\infty} g_{Y_1}\left(u\right)\ln\,\frac{g_{Y_1}\left(u\right)}{g_{Y_2}\left(u\right)}du.$$

In addition, Ebrahimi and Kirmani [7] have demonstrated that the Kullback-Leibler discrimination information of Y_1 and Y_2 at time κ can be presented as

$$IR_{Y_1,Y_2}\left(\kappa\right) = \int_{\kappa}^{\infty}\frac{g_{Y_1}\left(u\right)}{\overline{G}_1\left(\kappa\right)}\ln\,\frac{g_{Y_1}\left(u\right)/\overline{G}_1\left(\kappa\right)}{g_Y\left(u\right)/\overline{G}_2\left(\kappa\right)}du. \tag{5}$$

We can use Eq. (5) to distinguish between two residual lifetimes those have both survived up to time κ, where $IR_{Y_1,Y_2}\left(\kappa\right)$ identifies with the relative entropy of $\left[Y_1 - \kappa\,\vert\,Y_1\rangle\kappa\right]$ and $\left[Y_2 - \kappa\,\vert\,Y_2\rangle\kappa\right]$.

The purpose of this study is to develop and add more properties, characterizations, order statistics, some inequalities and stochastic orders of weighted differential entropies measures. In Section 2, definitions, notation, basic properties and characterizations are illustrated. The weighted entropy (residual and past residual) of order statistics with some application of reliability systems such as a series structure and a parallel structure are given in Section 3. In addition, we provided the lower bound for the weighted residual (past) entropy. The stochastic orders based on weighted entropy are developed in Section 4. Lastly, in Section 5, the suggested estimators of weighted entropy are presented. Furthermore, we illustrate the usefulness of the proposed non-parametric estimators of weighted entropy by application to real data.

Throughout this article, the term entropy is used instead of differential entropy and using abbreviation *PL* for past lifetime, *WPE* for weighted past residual entropy, *WRE* for weighted residual entropy, *SS* for the series system, *PS* for the parallel system, *SE* for small than or equal.

2. THE WEIGHTED DIFFERENTIAL ENTROPY

The weighted differential entropy (WDE) defined by Das [8] for random variable Y *with weighted function* $w(x) = x$ as:

$$\xi^{w}(Y) = -\frac{\vartheta^{w}(Y) + \int_{\mathcal{L}} x f_Y(x) \ln f_Y(x)\, dx - \mathbb{E}[Y] \ln \mathbb{E}[Y]}{\mathbb{E}[Y]}, \quad (6)$$

$$= \frac{H^{w}(Y)}{\mathbb{E}[Y]} + \ln \mathbb{E}[Y] - \frac{\vartheta^{w}(Y)}{\mathbb{E}[Y]},$$

where

$$\vartheta^{w}(Y) = \int_{\mathcal{L}} xfY(x) \ln x dx =: \mathbb{E}(Y \ln Y). \quad (7)$$

whenever the integral $\int_{\mathcal{L}} \left(u^{\theta} f_Y(u) / \mathbb{E}\left(Y^{\theta}\right)\right)\left(1 \vee |\ln\left(u^{\theta} f_Y(u) / \mathbb{E}\left(Y^{\theta}\right)\right)|\right) du < \infty.$

As a general case, we can be defined the generalized the WDE as the following definition:

Definition 2.1. *Given a function* $y \in \mathcal{L} \mapsto w(y) \geq 0$, *and an RV* $Y\colon \mathcal{F} \to \mathcal{L}$, *with a probability density function* $g_Y(.)$, survival function $\overline{G}(.)$ and mean $\mathbb{E}(Y)$. Therefore, *the weighted differential entropy with weighted function* $w(y) = y^{\theta}$ *is defined as*

$$\begin{aligned}
\xi^{w}(Y) = \mathbb{E}\left[-\ln g_Y^{w}(Y)\right] &= \int_{\mathcal{L}} g_Y^{w}(u) \ln \frac{1}{g_Y^{w}(u)} du, \\
&= -\int_{\mathcal{L}} \frac{u^{\theta} g_Y(u)}{\mathbb{E}\left(Y^{\theta}\right)} \ln \frac{u^{\theta} g_Y(u)}{\mathbb{E}\left(Y^{\theta}\right)} du, \\
&= -\frac{1}{\mathbb{E}\left(Y^{\theta}\right)}\left[\theta \int_{\mathcal{L}} u^{\theta} g_Y(u) \ln u du + \int_{\mathcal{L}} u^{\theta} g_Y(u) \ln g_Y(u)\, du - \mathbb{E}\left(Y^{\theta}\right) \ln \mathbb{E}\left(Y^{\theta}\right)\right].
\end{aligned}$$

Now, let $Y_1, Y_2, ..., Y_n$ be a sample from the distribution F and $n \geq 3$. By using Vasicek [9], express Eq. (2.1) can be rewritten as

$$\xi^{w}(Y) = \int_0^1 \ln\left\{\frac{\partial}{\partial u} F^{-1}(u)\right\} du. \quad (8)$$

In addition, Das [8] have defined the weighted residual entropy as

$$\begin{aligned}
\xi^{w}(Y, \kappa) &= -\int_{\kappa}^{\infty} \frac{g_Y^{w}(v)}{\overline{G}^{w}(\kappa)} \ln \frac{g_Y^{w}(v)}{\overline{G}^{w}(\kappa)} dv, \\
&= -\frac{1}{E[Y|Y > \kappa]} \int_{\kappa}^{\infty} v \frac{g_Y(v)}{\overline{G}(\kappa)} \ln \frac{v g_Y(v)}{E[Y \mid Y > \kappa]\, \overline{G}(\kappa)} dv, \ \kappa \in \mathbb{R}^{+}.
\end{aligned}$$

If $f_X(x)$ is the actual density function of random variable X and $g_Y(x)$ is the density function determined by the researcher. Therefore, the weighted inaccuracy measure can be defined as

$$\mathbb{R}^{w}(X, Y) = -\int_{\mathcal{L}} x f_X(x) \ln g_Y(x)\, dx. \quad (9)$$

Next, we define the relative WDE of two densities.

Definition 2.2. Let X *and* Y *be two random variables with density function,* $s \in \mathcal{L} \mapsto f_X^{w}(s) \geq 0$ *and* $s \in \mathcal{L} \mapsto g_Y(s) \geq 0$, *and mean values* $\mathbb{E}_X(.)$ and $\mathbb{E}_Y(.)$, respectively. Therefore, *the relative* weighted differential entropy *of* $g_Y(x)$ *relative to* $f^{w}(x)$ *can be defined as*

$$\begin{aligned}
\mathbb{R}\left(X^{w} \parallel Y\right) &= \mathbb{E}\left[\ln \frac{s f_X(s)}{\mathbb{E}_X(s)\, g_Y(s)}\right], \\
&= \int_{\mathcal{L}} \frac{v f_X(v)}{\mathbb{E}_X(X)} \ln \frac{v f_X(v)}{\mathbb{E}_X(v)\, g_Y(v)} dv.
\end{aligned}$$

By using Eq. (4), we can define an alternative formulas of $\mathbb{R}\left(X^{w} \parallel Y\right)$ as follows

$$\mathbb{R}\left(X^{w} \parallel Y\right) = \ln \frac{1}{\mathbb{E}_X(X)} - \frac{H^{w}(X)}{\mathbb{E}_X(X)} - \int_{\mathcal{L}} \frac{x f_X(x)}{\mathbb{E}_X(x)} \ln x g_Y(x)\, dx.$$

Note that when $g_Y(x) \equiv f_X(x)$, then we have

$$\mathbb{R}\left(X^w \parallel X\right) = \ln \frac{1}{\mathbb{E}_X(X)} - \frac{H^w(X)}{\mathbb{E}_X(X)} - \int_{\mathcal{L}} \frac{x f_X(x)}{\mathbb{E}_X(x)} \ln x f_X(x)\, dx,$$
$$= \ln\left(\exp\left(-\frac{\vartheta^w(X) + 2H^w(X)}{\mathbb{E}_X(X)}\right)/\mathbb{E}_X(X)\right).$$

Remark 2.1. *By using Eqs. (8) and (9) we get the following relation*

$$\mathbb{R}\left(X^w \parallel Y^w\right) = \ln \frac{\mathbb{E}_Y(X)}{\mathbb{E}_X(X)} - \frac{H^w(X)}{\mathbb{E}_X(X)} + \frac{\mathbb{R}_X^w(X, Y)}{\mathbb{E}_X(X)}.$$

Furthermore, we can define the divergence between $f^w(x)$ and $g_Y(x)$ as follows

$$\mathbb{K}\left(X^w, Y\right) = \mathbb{R}\left(X^w \parallel Y\right) + \mathbb{R}^w(Y \parallel X), = \int_{\mathcal{L}} \left(f_X^w(x) - g_Y(x)\right) \ln \frac{f_X^w(x)}{g_Y(x)} dx,$$

it is a measure of the difficulty of discrimination between them.

Now, let X be RV with beta distribution as follows:

$$f_X(s) = s^{\alpha-1}(1-s)^{\beta-1} / B(\alpha, \beta),\ s \in \mathcal{L} \in (0, 1).$$

By using Eq. (3), we get that H_X satisfy the following equation:

$$H_X = \ln\left(\frac{B(\alpha, b)\, exp\left((\alpha+\beta-2)\Psi(\alpha+\beta)\right)}{exp\left((\alpha-1)\Psi(\alpha) + (\beta-1)\Psi(\beta)\right)}\right), \tag{10}$$

where $B(.,.)$ is beta function and $\Psi(.)$ is psi function.

The following theorem states that this relationship actually characterizes the beta distribution.

Characterization Theorem 2.1: *Any random variable Y with distribution function $\mathbb{K}$, density function $f_Y(x)$, mean $\mathbb{E}[Y]$, mode $\Gamma_{\mathbb{K}}(Y)$, geometric mean $\mathbb{G}[Y]$, entropy function H_Y and weighted differential entropy $\xi^w(Y)$ satisfying the following relationship:*

$$\xi^w(Y) = H_Y - (\alpha-1)\left[\frac{\Gamma_{\mathbb{K}}(Y) + \mathbb{E}[Y]}{\Gamma_{\mathbb{K}}(Y)}\right] + \ln \mathbb{E}[Y] - \ln\left(\mathbb{G}[Y]\right) + \left(\left(\mathbb{E}[Y] - 1\right)/\alpha\right),$$

is either degenerate or Y has a beta distribution. Indeed, the degenerate case should be subsumed in the beta distribution with $(\alpha, \beta) \in R^+$.

Proof. By using the following integral formula which is taken from Gradshteyn and Ryzhik ([10], formula 4.253(1), pp. 538):

$$\int_{\mathcal{L}} y^{\theta-1}\left(1-y^c\right)^{\lambda-1} \ln y dy = \frac{1}{c^2} B\left(\frac{\theta}{c}, \lambda\right)\left(\Psi\left(\frac{\theta}{c}\right) - \Psi\left(\frac{\theta}{c} + \lambda\right)\right),$$

provided that $Re(\theta) > 0$, $Re(\lambda) > 0$, $c > 0$. We denote the beta function with the symbol $B(\cdot,\cdot)$ and the digamma function with $\Psi(\cdot)$. Therefore, we have

$$\vartheta^w(Y) = \int_{\mathcal{L}} x f_X(x) \ln x dx,$$
$$= \frac{1}{B(\alpha, \beta)}\left[B(\alpha+1, \beta)\left(\Psi(\alpha+1) - \Psi(\alpha+\beta+1)\right)\right],$$
$$= \mathbb{E}[Y]\left(\Psi(\alpha+1) - \Psi(\alpha+\beta+1)\right).$$

By using Example 2.3 in Di Crescenzo and Longobard ([4], pp.682), Eq. (4) and the recurrence relation of the digamma function, we can rewrite $H^w(Y)$ as follows:

$$H^w(Y) = \mathbb{E}[Y]\left[\begin{array}{c} \ln B(\alpha, \beta) + (1-\alpha)\left(\Psi(\alpha) + \frac{1}{\alpha}\right) \\ +(\alpha+\beta-2)\left(\Psi(\alpha+\beta) + \frac{1}{\alpha+\beta}\right) + \Psi(\beta)(1-\beta) \end{array}\right]. \tag{11}$$

We can reduce Eq. (11) as,

$$H^w(Y) = \mathbb{E}[Y]\left[H_Y + (1-\alpha)\frac{1}{\alpha} + \frac{(\alpha+\beta-2)}{(\alpha+\beta)}\right], = \mathbb{E}[Y]\left[H_Y - (\alpha-1)\left[\frac{\Gamma_{\mathbb{K}}(Y) + \mathbb{E}[Y]}{\Gamma_{\mathbb{K}}(Y)}\right]\right].$$

This is true for $f\left(\Gamma_{\mathbb{K}}(.)\right)=\max_{-\infty<x<\infty} f_X(.)$. Furthermore by Eqs. (6) and (10) we have

$$\begin{aligned}\xi^w(Y) &= H_Y-(\alpha-1)\left[\frac{\Gamma_{\mathbb{K}}(Y)+\mathbb{E}[Y]}{\Gamma_{\mathbb{K}}(Y)}\right]+\ln \mathbb{E}[Y]-(\Psi(\alpha+1)-\Psi(\alpha+\beta+1)),\\ &= H_Y-(\alpha-1)\left[\frac{\Gamma_{\mathbb{K}}(Y)+\mathbb{E}[Y]}{\Gamma_{\mathbb{K}}(Y)}\right]+\ln \mathbb{E}[Y]-\ln(\mathbb{G}[Y])+\frac{1}{\alpha}(\mathbb{E}[Y]-1).\end{aligned}$$

In next results we study the closure transformation property of the weighted entropy. We can now proceed analogously to Di Crescenzo and Longobardi [4] and introduce the following theorem.

Theorem 2.2. *Suppose U is RV with density function $f_U(.)$ and $\psi(U)$ is strongly convex, strictly increasing, continuous and differentiable function with derivative $\frac{d}{du}\psi(u)$. Then*

$$\xi^w(\psi(U))=\xi_1^w\left(U|\psi^{-1}(0)\le U\le\psi^{-1}(\infty)\right)+\mathbb{E}^w\left[\ln\left|\frac{d}{dx}\psi(x)|\psi^{-1}(0)\le U\le\psi^{-1}(\infty)\right|\right]$$

where $E^w(U)=\int_0^\infty v f_U^w(v)\,dv$.

Proof. From Eq. (6) we have

$$\xi^w(\psi(U))=-\int_0^\infty f_U^w\left(\psi^{-1}(u)\right)|\frac{d}{du}\psi^{-1}(u)|\ln f_U^w\left(\psi^{-1}(u)\right)|\frac{d}{du}\psi^{-1}(u)|du.$$

We will make the following assumptions:

1. $\psi(u)$ is monotonically increasing in u.
2. $v=\psi(u)$,

Therefore, it clear that

$$\begin{aligned}\xi^w(\psi(U)) &= -\int_{\psi^{-1}(0)}^{\psi^{-1}(\infty)} f_U^w(v)\ln\left(f_U^w(v)|\frac{d}{dv}\psi(v)|^{-1}\right)dv.\\ &= \int_{\psi^{-1}(0)}^{\psi^{-1}(\infty)} f_U^w(v)\ln\left|\frac{d}{dv}\psi(v)\right|dv+\xi^w\left(U|\psi^{-1}(0)\le U\le\psi^{-1}(\infty)\right).\end{aligned}$$

Hence the proof is completed.

Proposition 2.3. *Let $\phi(U)=\alpha U^\beta$ whereas $\alpha,\beta>0$. From this we deduce that*

$$\xi^w(\phi(U))=\ln\alpha\beta+\frac{(\beta-1)\vartheta^w(U)}{\mathbb{E}(U)}+\xi^w(U).$$

Proof. From Eq. (6) we have

$$\begin{aligned}\xi^w(\phi(U)) &= \int_0^\infty f^w(v)\ln\left|\alpha\beta v^{\beta-1}\right|dv-\int_0^\infty f^w(v)\ln f^w(v)\,dv,\\ &= \ln\alpha\beta+(\beta-1)\int_0^\infty f^w(v)\ln v\,dv+\xi^w(U).\end{aligned}$$

By using Eq. (7), we get the required results.

From Definitions (2.1), it is easy to obtain the following characterizations:

Example 2.1: Suppose U be a random variable having Log-Normal with the following density function

$$f_U(u)=\frac{1}{\sqrt{2\pi}\sigma u}\exp\left(-(\ln u-\ln\mu)^2/2\sigma^2\right),\ \mu,\sigma,u>0,$$

with parameter $\mu,\sigma>0$. From Eq. (7) we get,

$$\vartheta^w(U)=\int_0^\infty v\frac{1}{\sqrt{2\pi}\sigma v}\exp\left(-(\ln v-\ln\mu)^2/2\sigma^2\right)\ln v\,dv.$$

Set $u = \ln v - \ln \mu$, then we have

$$\vartheta^{w}(U) = \frac{2}{\sqrt{2\pi}\sigma} \exp\,(2\ln\,\mu) \int_0^\infty \exp\left(2u - \frac{u^2}{2\sigma^2}\right) du.$$

It is follows from Gradshteyn and Ryzhik ([10], formula 3.322(1)) that

$$\int_a^\infty \exp\left(-\frac{x^2}{4\mu} - bx\right) dx = \sqrt{\pi\mu}\exp\,\left(\mu b^2\right)\left(1 - \Phi\left(b\sqrt{\mu} + \frac{a}{2\sqrt{\mu}}\right)\right), \ [Re\mu > 0, a \geq 0].$$

Therefore,

$$\vartheta^{w}(U) = \exp\,\left(2\left(\ln\,\mu + \sigma^2\right)\right)\left(1 - \Phi\left(-2\sqrt{1/2\sigma}\right)\right).$$

In addition,

$$\begin{aligned} H^{w}(U) &= -\int_0^\infty v\frac{1}{\sqrt{2\pi\sigma\,v}} \exp\left(-(\ln\,v - \ln\,\mu)^2/2\sigma^2\right)\left[-\ln\sqrt{2\pi\sigma} - \ln\,v - \frac{(\ln\,v - \ln\,\mu)^2}{2\sigma^2}\right] dv, \\ &= \ln\,\sqrt{2\pi\sigma}\mathbb{E}\,[U] + \vartheta^{w}(U) + \int_0^\infty v\frac{1}{\sqrt{2\pi\sigma v}} \exp\,\left(-(\ln\,v - \ln\,\mu)^2/2\sigma^2\right)\frac{(\ln\,v - \ln\,\mu)^2}{2\sigma^2} dv, \end{aligned}$$

with the same way, set $u = \ln\,v - \ln\,\mu$ and by using formula 3.462(1) in Gradshteyn and Ryzhik [10] we have

$$\begin{aligned} &\int_0^\infty v\frac{1}{\sqrt{2\pi\sigma v}} \exp\,\left(-(\ln\,v - \ln\,\mu)^2/2\sigma^2\right)\frac{(\ln\,v - \ln\,\mu)^2}{2\sigma^2} dv, \\ &= \left(\frac{1}{\sigma^2}\right)^{(-3/2)+3} \frac{2\mu}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{4}\right) D_{-3}(-\sigma). \end{aligned}$$

Therefore,

$$\begin{aligned} H^{w}(X) = \ln\left(\sqrt{2\pi\sigma}\right)\mu\exp\,\left(\sigma^2/2\right) &+ \exp\,\left(2\left(\ln\,\mu + \sigma^2\right)\right)\left(1 - \Phi\left(-2\sqrt{1/2\sigma}\right)\right) \\ &+ \left(\frac{1}{\sigma^2}\right)^{(-3/2)+3} \frac{2\mu}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{4}\right) D_{-3}(-\sigma), \\ &= \ln\left(\sqrt{2\pi\sigma}\right)\mathbb{E}\,[X] + \vartheta^{w}(X) + \sigma^{-3}\frac{2\mu}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{4}\right) D_{-3}(-\sigma), \end{aligned}$$

where $\Phi\,(.)$ is Error function, $\varpi\,(\mu,\sigma) = \left(\mu\exp\,\left(\sigma^2/2\right)\right)^{-1}$ and $D_x\left(y\right) = 2^{x/2}\exp\,\left(-y^2/4\right) F\left(\frac{-x}{2}, \frac{1}{2}, \frac{y^2}{2}\right)$ is Parabolic cylinder function. We denote a confluent hypergeometric function of the first kind with the symbol $F(.,.,.)$. Further,

$$\xi^{w}(U) = \varpi\,(\mu,\sigma)\,\alpha\,(\mu,\sigma) - \ln\,\varpi\,(\mu,\sigma) - \varpi\,(\mu,\sigma)\exp\,\left(2\left(\ln\,\mu + \sigma^2\right)\right)\left(1 - \Phi\left(-2\sqrt{1/2\sigma}\right)\right),$$

where $\alpha\,(\mu,\sigma) = \ln\,\sqrt{2\pi\sigma\mu}\exp\,\left(\sigma^2/2\right) + \exp\,\left(2\left(\ln\,\mu + \sigma^2\right)\right)\left(1 - \Phi\left(-2\sqrt{1/2\sigma}\right)\right) + \left(\frac{1}{\sigma^2}\right)^{(-3/2)+3} \frac{2\mu}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{4}\right) D_{-3}(-\sigma)$.

Furthermore,

$$\mathbb{R}\left(X^{w} \parallel X\right) = \ln\,\left(\exp\,\left(-\beta_1\,(\mu,\sigma) + 2\beta_2\,(\mu,\sigma)\,\varpi\,(\mu,\sigma)\right)/\varpi\,(\mu,\sigma)\right),$$

where:

1. $\beta_1\,(\mu,\sigma) = \exp\,\left(2\left(\ln\,\mu + \sigma^2\right)\right)\left(1 - \Phi\left(-2\sqrt{1/2\sigma}\right)\right)$;
2. $\beta_2\,(\mu,\sigma) = \ln\left(\sqrt{2\pi\sigma}\right)\mathbb{E}\,[X] + \vartheta^{w}(X) + \left(\frac{1}{\sigma^2}\right)^{(-3/2)+3} \frac{2\mu}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{4}\right) D_{-3}(-\sigma)$

Example 2.2: Let X be random variable having Chi distribution with density function

$f_X(x) = \frac{2(\pi/2)^{\pi/2}}{\gamma^{\pi}\Gamma(\pi/2)} x^{\pi-1} \exp\left(-x^2\left(\pi/\left(2\gamma^2\right)\right)\right)$, $\gamma, x > 0$, π is a positive integer,with parameter $\mu, \gamma > 0$. From Eq. (7) we have

$$\vartheta^w(X) = \frac{2(\pi/2)^{\pi/2}}{\gamma^{\pi}\Gamma(\pi/2)} \int_0^{\infty} x^{\pi} \exp\left(-x^2\left(\pi/\left(2\gamma^2\right)\right)\right) \ln x dx,$$

take $u = x^2 \pi/\left(2\gamma^2\right)$, then we get

$$\vartheta^w(X) = \frac{\gamma(1/2\pi)^{1/2}}{\Gamma(\pi/2)} \int_0^{\infty} u^{\frac{\pi}{2}-0.5} \exp(-u) \ln\left(\left(u/\left(\pi/\left(2\gamma^2\right)\right)\right)^{1/2}\right) du.$$

By using Gradshteyn and Ryzhik ([10], 4.352(1)) for evaluate the below formula, we have the following result:

$$\vartheta^w(X) = \frac{\gamma(1/2\pi)^{1/2}}{\Gamma(\pi/2)} \Gamma\left(\frac{\pi+1}{2}\right)\left[\left(\psi\left(\frac{\pi+1}{2}\right)\right) - \ln\left(\pi/\left(2\gamma^2\right)\right)\right].$$

In addition, by Eq. (4) we have

$$H^w(X) = -\int_0^{\infty} x \frac{2(\pi/2)^{\pi/2}}{\gamma^{\pi}\Gamma(\pi/2)} x^{\pi-1} \exp\left(-x^2\left(\pi/\left(2\gamma^2\right)\right)\right)\left[\ln\frac{2(\pi/2)^{\pi/2}}{\gamma^{\pi}\Gamma(\pi/2)} + (\pi-1)\ln x - \frac{x^2\pi}{2\gamma^2}\right] dx,$$

since $\mathbb{E}(X) = \gamma\sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{\pi+1}{2}\right)}{\Gamma\left(\frac{\pi}{2}\right)}$.With this substitution we obtain

$$H^w(X) = \begin{matrix} \pi_2 - \pi_1(\pi-1)\int_0^{\infty} v^{\pi} \exp\left(-v^2\left(\pi/\left(2\gamma^2\right)\right)\right) \ln v dv \\ +\pi_3 \int_0^{\infty} v^{\pi+2} \exp\left(-v^2\left(\pi/\left(2\gamma^2\right)\right)\right) dv, \end{matrix}$$

where $\pi_1 = 2(\pi/2)^{\pi/2}/(\gamma^{\pi}\Gamma(\pi/2))$, $\pi_2 = -\ln\left(\pi_1\right)\gamma\left(\Gamma\left(\frac{\pi+1}{2}\right)/\Gamma\left(\frac{\pi}{2}\right)\right)\sqrt{2/\pi}$, $\pi_3 = \pi/\left(2\gamma^2\right)\pi_1$. Direct calculations give

$$H^w(X) = \frac{\gamma(\pi+1)\Gamma\left(\frac{\pi+1}{2}\right)}{2(\pi/2)^{1/2}\Gamma(\pi/2)} - \ln\left(\frac{2(\pi/2)^{\pi/2}}{\gamma^{\pi}\Gamma(\pi/2)}\right)\frac{\gamma\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{\pi+1}{2}\right)}{\Gamma\left(\frac{\pi}{2}\right)} - \frac{2(\pi/2)^{\pi/2}(\pi-1)}{\gamma^{\pi}\Gamma(\pi/2)}\int_0^{\infty} x^{\pi} \exp\left(-x^2\left(\pi/\left(2\gamma^2\right)\right)\right) \ln x dx.$$

Again, we the same way and continuing the simplification, we can conclude that

$$H^w(X) = -\left(\gamma\Gamma\left(\frac{\pi+1}{2}\right)/\Gamma(\pi/2)\right)\left(\begin{matrix} \ln\left(\frac{2(\pi/2)^{\pi/2}}{\gamma^{\pi}\Gamma(\pi/2)}\right)\sqrt{\frac{2}{\pi}} \\ +\frac{2(\pi-1)}{4\sqrt{(\pi/2)}}\left[\psi\left(\frac{\pi+1}{2}\right) - \ln\left(\pi/\left(2\gamma^2\right)\right)\right] - \frac{(\pi+1)}{2(\pi/2)^{1/2}} \end{matrix}\right).$$

Therefore,

$$\xi^w(X) = -\ln\left(2/\gamma^{\pi}\right) - \frac{2(\pi-1)}{4}\left[\psi\left(\frac{\pi+1}{2}\right) - \ln\left(\pi/\left(2\gamma^2\right)\right)\right] + \frac{(\pi+1)}{2}(1 - \ln(\pi/2))$$

$$+\ln\left(\gamma\Gamma\left(\frac{\pi+1}{2}\right)\right) - \psi\left(\frac{\pi+1}{2}\right) + \ln \pi/\left(2\gamma^2\right).$$

Continuing the simplification

$$\xi^w(X) = \ln\left(\frac{\pi\gamma^{\pi-1}}{4}\right) - \frac{2(\pi-1)}{4}\left[\psi\left(\frac{\pi+1}{2}\right) - \ln\left(\pi/\left(2\gamma^2\right)\right)\right]$$

$$+\frac{(\pi+1)}{2}(1 - \ln(\pi/2)) + \ln\Gamma\left(\frac{\pi+1}{2}\right) - \psi\left(\frac{\pi+1}{2}\right).$$

Moreover,

$$\mathbb{R}\left(X^{w} \parallel X\right) = -\frac{\pi\alpha_1\left[\left(\psi\left(\frac{\pi+1}{2}\right)\right) - \ln\ \left(\pi/\left(2\gamma^2\right)\right)\right] + 2H^{w}(X)}{\alpha_1} - \ln\ \left(\alpha_1\right),$$

where $\alpha_1 = \gamma\Gamma\left(\frac{\pi+1}{2}\right)\Big/\Gamma\left(\frac{\pi}{2}\right)\sqrt{2/\pi}$. Continuing the simplification

$$\mathbb{R}^{w}\left(X^{w} \parallel X\right) = \left(\frac{4\pi+1}{2}\right)\ln\ \left(\pi/\left(2\gamma^2\right)\right) + 2\ln\ 2 + \psi\left(\frac{\pi+1}{2}\right)(\pi-2) - (\pi+1) - \ln\ \left(\Gamma\left(\frac{\pi+1}{2}\right)\Gamma\left(\frac{\pi}{2}\right)\right).$$

Example 2.3: A random variable U has a Laplace (α), if it has density function as follows

$$f_U(u) = \tfrac{1}{2}\alpha\exp\ (-\alpha|u|)\,,\ \alpha, \infty > u > -\infty,$$

with parameter $\alpha > 0$. From Eq. (7) we obtain,

$$\vartheta^{w}(U) = \int_{-\infty}^{\infty} v\tfrac{1}{2}\alpha\exp\ (-\alpha|v|)\ln\ v dv = (\psi(2) - \ln\ \alpha)/\alpha,$$

Furthermore, direct calculations give

$$\begin{aligned} H^{w}(U) &= -\int_0^{\infty} x\alpha\exp\ (-\alpha x)\left(\ln\ \frac{\alpha}{2} - \alpha x\right)dx, \\ &= \frac{2 - \ln\ \alpha + \ln\ 2}{\alpha}\,nats = \frac{1 + H_U}{\alpha}\,nats. \end{aligned}$$

Since $\mathbb{E}[U] = 0$, we get that $\xi_1^{w}(U)$ and $\mathbb{R}_1^{w}\left(U^{w} \parallel U\right)$ can not be found.

3. CONNECTION TO RELIABILITY THEORY

Suppose $U_1, U_2, \ldots, U_n$ be i.i.d. lifetimes with probability density function $g(.)$, distribution $\mathbb{K}(.)$, survival function $\overline{\mathbb{K}}(.)$ and reversed hazard rate $\varphi_{\mathbb{K}}(.)$. Therefore, the probability that any two (or more) observation in random sample take the same magnitude (the same value is equal to zero). Therefore, there exists a unique ordered arrangement of the sample observation according to magnitude. Let $0 \le U_{(1)} \le U_{(2)} \le \ldots \le U_{(n)} < \infty$ be the corresponding order statistics. Therefore, $U_{(r)}$ defines the lifetime of an $(n-r+1)$out of n system. Write $g_{(r)}(.)$, $\mathbb{K}_{(r)}(.)$, $\varphi_{(r)}(.)$ and $\overline{\varphi}_{(r)}(.)$ as the distribution function, the probability density function, the RHR function of $U_{(r)}$ and the hazard rate of $U_{(r)}$ respectively. Then we have

$$\begin{aligned} g_{(r)}(\kappa) &= C_r[\mathbb{K}(\kappa)]^{r-1}[\overline{\mathbb{K}}(\kappa)]^{n-r}g(\kappa)\,,\ \kappa \in \mathbb{R}^{+}, \\ \mathbb{K}_{(r)}(\kappa) &= \textstyle\sum_{i=r}^{n}\binom{n}{i}[\mathbb{K}(\kappa)]^{i}[\overline{\mathbb{K}}(\kappa)]^{n-i}, \\ \varphi_{(r)}(\kappa) &= C_r\varphi_{\mathbb{K}}(\kappa)\,\beta^{r}\Big/\left(\textstyle\sum_{i=r}^{n}\binom{n}{i}\beta^{i}\right), \end{aligned} \tag{12}$$

and

$$\overline{\varphi}_{(r)}(\kappa) = g_{(r)}(\kappa)\big/\overline{\mathbb{K}}_{(r)}(\kappa)\,,$$

where $C_r = \dfrac{n!}{(r-1)!\,(n-r)!}$ and $\beta^{x} = \left(\mathbb{K}(\kappa)/\overline{\mathbb{K}}(\kappa)\right)^{x}$. The weighted residual entropy of order statistics U_r is given by

$$\xi_1^{w}\left(U_{(r)},\kappa\right) = -\int_{\kappa}^{\infty}\frac{g_{(r)}^{w}(u)}{\overline{\mathbb{K}}_{(r)}^{w}(\kappa)}\ln\ \frac{g_{(r)}^{w}(u)}{\overline{\mathbb{K}}_{(r)}^{w}(\kappa)}du, = \frac{-1}{\mathbb{E}\left[U_{(r)}|U_{(r)} > \kappa\right]}\int_{\kappa}^{\infty}\frac{yg_{(r)}\left(y\right)}{\overline{\mathbb{K}}_{(r)}(\kappa)}$$

$$\times\ln\ \frac{yg_{(r)}\left(y\right)}{\mathbb{E}\left[U_{(r)}|U_{(r)} > \kappa\right]\overline{\mathbb{K}}_{(r)}(\kappa)}dy.$$

Alternatively,

$$\xi_1^w\left(U_{(r)},\kappa\right) = \frac{-1}{\mathbb{E}\left[U_{(r)}|U_{(r)} > \kappa\right]} \tag{13}$$

$$\times\int_\kappa^\infty \frac{yg_{(r)}\left(y\right)}{\overline{\mathbb{K}}_{(r)}\left(\kappa\right)} \ln \frac{y\overline{\varphi}_{(r)}\left(y\right)\overline{\mathbb{K}}_{(r)}\left(y\right)}{\mathbb{E}\left[U_{(r)}|U_{(r)} > \kappa\right]\overline{\mathbb{K}}_{(r)}\left(\kappa\right)}dy,$$

$$= \ln\left[\mathbb{E}\left[U_{(r)}|U_{(r)} > \kappa\right]\overline{\mathbb{K}}_{(r)}\left(\kappa\right)\right]$$

$$-\frac{1}{\mathbb{E}\left[U_{(r)}|U_{(r)} > \kappa\right]}\int_\kappa^\infty \frac{yg_{(r)}\left(y\right)}{\overline{\mathbb{K}}_{(r)}\left(\kappa\right)} \ln\left(y\overline{\varphi}_{(r)}\left(y\right)\overline{\mathbb{K}}_{(r)}\left(y\right)\right)dy.$$

Now, we can proceed analogously to treatment of the weighted entropy of the order statistics of PL as follows

$$\begin{aligned}\xi_2^w\left(U_{(r)},\kappa\right) &= -\int_0^\kappa \frac{g_{(r)}^w\left(v\right)}{\mathbb{K}_{(r)}^w\left(\kappa\right)} \ln \frac{g_{(r)}^w\left(v\right)}{\mathbb{K}_{(r)}^w\left(\kappa\right)}dv,\\ &= \frac{-1}{\mathbb{E}\left[U_{(r)}|U_{(r)} < \kappa\right]}\int_0^\kappa \frac{vg_{(r)}\left(v\right)}{\mathbb{K}_{(r)}\left(\kappa\right)} \ln \frac{vg_{(r)}\left(v\right)}{\mathbb{E}\left[U_{(r)}|U_{(r)} < \kappa\right]\mathbb{K}_{(r)}^w\left(\kappa\right)}dv,\end{aligned}$$

which is equivalent to

$$\xi_2^w\left(U_{(r)},\kappa\right) = \frac{-1}{\mathbb{E}\left[U_{(r)}|U_{(r)} < \kappa\right]}\int_0^\kappa \frac{xg_{(r)}\left(x\right)}{\mathbb{K}_{(r)}\left(\kappa\right)} \ln \frac{x\varphi_{(r)}\left(x\right)\overline{\mathbb{K}}_{(r)}\left(x\right)}{\mathbb{E}\left[U_{(r)}|U_{(r)} < \kappa\right]\mathbb{K}_{(r)}^w\left(\kappa\right)}dx.$$

Direct calculations give

$$\xi_2^w\left(U_{(r)},\kappa\right) = \ln\left[\mathbb{E}\left[U_{(r)}|U_{(r)} < \kappa\right]\mathbb{K}_{(r)}\left(\kappa\right)\right] \tag{14}$$

$$-\frac{1}{\mathbb{E}\left[U_{(r)}|U_{(r)} < \kappa\right]}\int_0^\kappa \frac{vg_{(r)}\left(v\right)}{\mathbb{K}_{(r)}^w\left(\kappa\right)\left(\kappa\right)} \ln\left(v\varphi_{(r)}\left(v\right)\mathbb{K}_{(r)}\left(v\right)\right)dv,$$

for all $\kappa \geq 0$.

3.1. A Series Structure

It is to be noted that $U_{(1)}$ represents an age of the series system. By using Eq. (13), with simple calculation, we have the weighted the residual entropy of $U_{(1)}$ as

$$\xi_1^w\left(U_{(1)},\kappa\right) = \ln\left[\mathbb{E}\left[U_{(1)}|U_{(1)} > \kappa\right]\overline{\mathbb{K}}_{(1)}\left(\kappa\right)\right]$$

$$-\frac{1}{\mathbb{E}\left[U_{(1)}|U_{(1)} > \kappa\right]}\int_\kappa^\infty \frac{vg_{(1)}\left(v\right)}{\overline{\mathbb{K}}_{(1)}\left(\kappa\right)} \ln\left(v\overline{\varphi}_{(1)}\left(v\right)\overline{\mathbb{K}}_{(1)}\left(v\right)\right)dv, \tag{15}$$

$$= \ln\left[\mathbb{E}\left[U_{(1)}|U_{(1)} > \kappa\right]\overline{\mathbb{K}}_{(1)}\left(\kappa\right)\right]$$

$$-\frac{1}{\mathbb{E}\left[U_{(1)}|U_{(1)} < \kappa\right]}\int_0^\infty \frac{vn\left[\overline{\mathbb{K}}\left(v\right)\right]^{n-1}g\left(v\right)}{\left[\overline{\mathbb{K}}\left(\kappa\right)\right]^n} \ln\left(v\overline{\varphi}_{(1)}\left(v\right)\left[\overline{\mathbb{K}}\left(v\right)\right]^n\right)dv.$$

It follows from Proposition 1 in Bairamov et al. [11], and definition of mean residual lifetime of $(n-k+1)$-out-of-n system in Asadi and Bayramoglu [12] that

$$\mathbb{E}\left[U_{(1)}|U_{(1)} > \kappa\right] = \frac{\int_\kappa^\infty \overline{\mathbb{K}}_1\left(v\right)dv}{\overline{\mathbb{K}}_1\left(\kappa\right)} = \mathbb{M}_1\left(\kappa\right)\ say,$$

and

$$\overline{\mathbb{K}}(\kappa)=\left(\frac{\mathbb{M}_1(0)}{\mathbb{M}_1(\kappa)}\exp\left(-\int_0^\kappa \mathbb{M}_1^{-1}(\kappa)\right)\right)^{1/n},$$

Then, Eq. (15) can be written in the following form

$$\xi_1^w\left(U_{(1)},\kappa\right)=\ln\left(\mathbb{M}_1(0)\exp\left(-\int_0^\kappa \mathbb{M}_1^{-1}(\kappa)\right)\right)+\frac{\int_\kappa^\infty \frac{vn\left[\overline{\mathbb{K}}(v)\right]^{n-1}g(v)}{\left[\overline{\mathbb{K}}(\kappa)\right]^n}\ln\left(v\overline{\varphi}_{(1)}(v)\left[\overline{\mathbb{K}}(v)\right]^n\right)dv}{\mathbb{M}_1(0)\,exp\left(-\int_0^\kappa \mathbb{M}_1^{-1}(\kappa)\right)}.$$

Similarly, by Eq. (14), the WPE of $U_{(1)}$ follows

$$\begin{aligned}\xi_2^w\left(U_{(1)},\kappa\right)&=\ln\left[\mathbb{E}\left[U_{(1)}|U_{(1)}<\kappa\right]\mathbb{K}_{(1)}(\kappa)\right]\\&-\frac{1}{\mathbb{E}\left[U_{(1)}|U_{(1)}<\kappa\right]}\int_0^\kappa\frac{vg_{(1)}(v)}{\mathbb{K}_{(1)}(\kappa)}\ln\left(v\varphi_{(1)}(v)\mathbb{K}_{(1)}(v)\right)dv,\\&=\ln\left[\mathbb{E}\left[U_{(1)}|U_{(1)}<\kappa\right]\mathbb{K}_{(1)}(\kappa)\right]\\&-\frac{1}{\mathbb{E}\left[U_{(1)}|U_{(1)}<\kappa\right]}\int_0^\kappa\frac{vn\left[\overline{\mathbb{K}}(v)\right]^{n-1}g(v)}{1-\left[\overline{\mathbb{K}}(\kappa)\right]^n}\ln\left(v\varphi_{(1)}(v)\left[1-\overline{\mathbb{K}}^n(v)\right]\right)dv.\end{aligned}$$

Eq. (12) implies that

$$\xi_2^w\left(U_{(1)},\kappa\right)=\ln\left(\mathbb{E}\left[U_{(1)}|U_{(1)}<\kappa\right]\left(1-\overline{\mathbb{K}}^n(\kappa)\right)\right)\tag{16}$$

$$+\frac{\int_0^{\overline{\mathbb{K}}(\kappa)}\overline{\mathbb{K}}^{-1}(u)\,n\,[u]^{n-1})\ln\left(n\overline{\mathbb{K}}^{-1}(u)\frac{g\left(\overline{\mathbb{K}}^{-1}(u)\right)}{u((u)^{-n}-1)}\left(1-u^n\right)\right)du}{\mathbb{E}\left[U_{(1)}|U_{(1)}>\kappa\right]\left(1-\overline{\mathbb{K}}^n(\kappa)\right)}.$$

According to Eqs. (4–5) in Tavangar and Asadi [13], the mean *PL* of *series* system can be obtained as follows:

$$P_1(\kappa)=\mathbb{E}\left[\kappa-U_{(1)}|U_{(1)}<\kappa\right],\tag{17}$$

$$=\frac{\sum_{l=1}^{n}\binom{n}{l}\alpha_\kappa^l S_l(\kappa)}{\sum_{l=1}^{n}\binom{n}{l}\alpha_\kappa^l},$$

where $\alpha(.)=\mathbb{K}(.)/\overline{\mathbb{K}}(.)$ and

$$S_\pi(\kappa)=\int_0^\kappa\sum_{l=1}^{\pi}\binom{\pi}{l}\left(\frac{\mathbb{K}(\kappa-v)}{\mathbb{K}(\kappa)}\right)^l\left(1-\frac{\mathbb{K}(\kappa-v)}{\mathbb{K}(\kappa)}\right)^{\pi-l}dv,3.7\tag{18}$$

Equations (17) and (18) demonstrate that

$$\xi_2^w\left(U_{(1)},\kappa\right)=\ln\left((\kappa-P_1(\kappa))\left(1-\overline{\mathbb{K}}^n(\kappa)\right)\right)$$

$$+\frac{\int_0^{\overline{\mathbb{K}}(\kappa)}\overline{\mathbb{K}}^{-1}(u)\,n\,[u]^{n-1})\ln\left(n\overline{\mathbb{K}}^{-1}(u)\frac{g\left(\overline{\mathbb{K}}^{-1}(u)\right)}{u((u)^{-n}-1)}\left(1-u^n\right)\right)du}{(\kappa-P_1(\kappa))\left(1-\overline{\mathbb{K}}^n(\kappa)\right)}.$$

3.2. A Parallel Structure

It is to be noted that $U_{(n)}$ refers to the lifetime of *PS* with survival function $\overline{\mathbb{K}}_{(n)}(.) = 1-\mathbb{K}^n(.)$. Based on Eq. (13), we can define the weighted the residual entropy of $U_{(n)}$ as

$$\xi_1^w\left(U_{(n)},\kappa\right) = \ln\left[\mathbb{E}\left[U_{(n)}|U_{(n)}>\kappa\right]\left[1-\mathbb{K}^n(\kappa)\right]\right]$$

$$-\frac{1}{\mathbb{E}\left[U_{(n)}|U_{(n)}>\kappa\right]\left[1-\mathbb{K}^n(\kappa)\right]}\int_\kappa^\infty vn\mathbb{K}^{n-1}(v)g(v)\ln\left(vn\mathbb{K}^{n-1}(v)g(v)\right)dv.$$

Applying Theorem 2.1, pp. 477 in Asadi and Bayramoglu [14] and Eq. (18), it obtains that,

$$\mathbb{E}\left[U_{(n)}|U_{(n)}>\kappa\right] = \mathbb{B}_{(n)}(\kappa)+\kappa,$$

where $\mathbb{B}_{(n)}(\kappa)$ is mean residual lifetime of *PS*, it can be found as follows

$$\mathbb{B}_{(n)}(\kappa) = \frac{\sum_{l=1}^{n-1}\binom{n}{l}\alpha^l(\kappa)\sum_{s=1}^{n}(-1)^{s-1}\binom{n-l}{s}\beta_s(\kappa)}{\sum_{l=1}^{n-1}\binom{n}{l}\alpha^l(\kappa)},$$

and $\beta_j(\kappa) = \int_\kappa^\infty \frac{\overline{\mathbb{K}}^j(v)\,dv}{\overline{\mathbb{K}}^j(\kappa)}$. Now, it is evident that

$$\xi_1^w\left(U_{(n)},\kappa\right) = \ln\left[\left(\mathbb{B}_{(n)}(\kappa)+\kappa\right)\left[1-\mathbb{K}^n(\kappa)\right]\right]$$

$$-\frac{n}{\left(\mathbb{B}_{(n)}(\kappa)+\kappa\right)\left[1-\mathbb{K}^n(\kappa)\right]}\int_\kappa^\infty v\mathbb{K}^{n-1}(v)g(v)\ln\left(vn\mathbb{K}^{n-1}(v)g(v)\right)dv.$$

By using Eq. (4) in Asadi (2006, pp. 1200), we have the mean *PL* of the components of *PS* as follows

$$\theta_n(\kappa) = \mathbb{E}\left[\kappa-U_{(n)}|U_{(n)}\le\kappa\right] = \int_0^\kappa \frac{\mathbb{K}^n(v)\,dv}{\mathbb{K}^n(\kappa)}. \tag{19}$$

Putting $r=n$ in Eq. (14) and using Eqs. (12 and 14), we get the WPE of $U_{(n)}$ as follows

$$\begin{aligned}\xi_2^w\left(U_{(n)},\kappa\right) &= \ln\left[\left(\kappa-\theta_n(\kappa)\right)\mathbb{K}_{(n)}(\kappa)\right] - \frac{1}{\kappa-\theta_n(\kappa)}\int_0^\kappa \frac{vg_{(n)}(v)}{\mathbb{K}_{(n)}^w(\kappa)}\ln\left(v\varphi_{(n)}(v)\mathbb{K}_{(n)}(v)\right)dv,\\ &= \ln\left[\left(\kappa-\theta_n(\kappa)\right)\mathbb{K}_{(n)}(\kappa)\right] - \frac{1}{\kappa-\theta_n(\kappa)}\int_0^\kappa \frac{vg_{(n)}(v)}{\mathbb{K}_{(n)}(\kappa)\mathbb{E}\left[U_{(n)}|U_{(n)}<\kappa\right]}\ln\left(vn\varphi_{\mathbb{K}}(v)\mathbb{K}^n(v)\right)dv,\end{aligned}$$

which is equivalent to,

$$\xi_2^w\left(U_{(n)},\kappa\right) = \ln\left[\left(\kappa-\theta_n(\kappa)\right)\mathbb{K}_{(n)}(\kappa)\right]$$

$$-\frac{n}{\left(\kappa-\theta_n(\kappa)\right)\left(\mathbb{K}^n(\kappa)\left(\kappa-\theta_n(\kappa)\right)\right)}\int_0^\kappa v\mathbb{K}^{n-1}(v)g(v)\ln\left(vn\varphi_{\mathbb{K}}(v)\mathbb{K}^n(v)\right)dv,$$

for all $\kappa\ge 0$.

3.3. Some inequalities

Next, we derive the upper bound of *WRE* of $U_{(r)}$.It is obvious that

$$\xi_1^w\left(U_{(r)},\kappa\right)=\ln\mathbb{E}\left[U_{(r)}|U_{(r)}>\kappa\right]+\ln\overline{\mathbb{K}}_{(r)}(\kappa)$$
$$-\frac{1}{\mathbb{E}\left[U_{(r)}|U_{(r)}>\kappa\right]}\int_\kappa^\infty\frac{vg_{(r)}(v)}{\overline{\mathbb{K}}_{(r)}(\kappa)}\ln\left(v\overline{\varphi}_{(r)}(v)\overline{\mathbb{K}}_{(r)}(v)\right)dv,$$

since $\ln\mathbb{E}\left[U_{(r)}|U_{(r)}>\kappa\right]\geq 0$ and

$$\kappa\geq 0\Rightarrow\ln\left[\overline{\mathbb{K}}_{(r)}(\kappa)\right]\leq 0,$$

by using Gupta et al. [16], we can deduce that

$$\xi_1^w\left(U_{(r)},\kappa\right)\leq\ln\mathbb{E}\left[U_{(r)}|U_{(r)}>\kappa\right].$$

For $r=1$, we have

$$\xi_1^w\left(U_{(1)},\kappa\right)\leq\ln\mathbb{M}_1(\kappa).$$

In addition, we know that $\ln\left[\left(\mathbb{B}_{(n)}(\kappa)+\kappa\right)\left[1-\mathbb{K}^n(\kappa)\right]\right]\leq 0$. Hence, we have

$$\xi_1^w\left(U_{(n)},\kappa\right)\leq\ln\left[\left(\mathbb{B}_{(n)}(\kappa)+\kappa\right)\left[1-\mathbb{K}^n(\kappa)\right]\right].$$

In next result, we derive the lower bound for *WPE* of $U_{(n)}$.

Proposition 3.1: *Suppose* $U\geq 0$ *be a random variable with distribution function* $\mathbb{K}(v)$. *Then*

$$\xi_2^w\left(U_{(n)},\kappa\right)\geq\frac{-n^2E\left[u^2\mathbb{K}^{2n-2}(u)\,g(u)\,|U\leq\kappa\right]}{\left(\kappa-\theta_n(\kappa)\right)^2\mathbb{K}^{n-1}(\kappa)}.$$

Proof. Using Eq.(2), inequality $-\ln y\geq 1-y$, for $y\geq 0$, and since

$$\int_0^\kappa v\mathbb{K}^{n-1}(v)\,g(v)\,dv\leq\int_0^\kappa nv^2\mathbb{K}^{2n-2}(v)\,g^2(v)\,dv.$$

The result follows.

4. STOCHASTIC ORDERS BASED ON WEIGHTED ENTROPY

In this section, we explore the possibility of **application** of stochastic orders.

Definition 4.1. *Assume* $U_1\geq 0$ *and* $U_2\geq 0$ *be two random variables with density functions* g_1 *and* g_2, *distribution functions* $\mathbb{G}_{U_1}$ *and* $\mathbb{G}_{U_2}$, *reliability functions* $\overline{\mathbb{G}}_{U_1}=1-\mathbb{G}_{U_1}$ *and* $\overline{\mathbb{G}}_{U_2}=1-\mathbb{G}_{U_2}$, *the weighted entropy functions* $\xi_{g_1}^w(.)$ *and* $\xi_{g_2}^w(.)$, the convex residual entropy functions $H_{g_1}^w(U_1,t)$ and $H_{g_2}^w(U_2,t)$ and the weighted inaccuracy measures $\mathbb{R}\left(U_1^w\parallel U_1\right)$ and $\mathbb{R}\left(U_2^w\parallel U_2\right)$, respectively. We say that U_1 is $\mathbb{SE}$ to U_2 in the:

- *weighted entropy ordering* $(U_1\leq_{\xi^w}U_2)$ if $\xi_{g_1}^w(x)\leq\xi_{g_2}^w(x)$, for all $x\geq 0$.
- *weighted inaccuracy ordering* $(U_1\leq_{\mathbb{R}^w}U_2)$ if $\mathbb{R}\left(U_1^w\parallel U_1\right)\leq\mathbb{R}\left(U_2^w\parallel U_2\right)$.
- convex residual entropy *ordering* $(U_1\leq_{c^w}U_2)$ if $H_{g_1}^w(U_1,t)\leq H_{g_2}^w(U_2,t)$.
- less uncertainty *ordering* $(U_1\leq_{\mathbb{U}}U_2)$ if $H(U_1)\leq H(U_2)$.

Definition 4.2. Let U_1 and U_2 be two random variables, then U_1 is said to be $\mathbb{SE}$ to U_2 in the convex order $(U_1\leq_{cx}U_2)$. If

$$E[\phi(U_1)]\leq E[\phi(U_2)],$$

This is true for any convex functions ϕ.

Definition 4.3. The random variable U_1 is said to be increasing hazard rate, *IHR*, if, and only if,

$$\overline{\mathbb{G}}_{U_1}(u+v)/\overline{\mathbb{G}}_{U_1}(u) \text{ is decreasing in } u \geq 0, \text{ for all } v \geq 0.$$

Next result discusses the closure under increasing linear transformation of $\leq_{\xi^w}$:

Theorem 4.1. Suppose U_1 and U_2 are to be two random variables, let we define new functions as

$$V_1 = \alpha_1 U_1^{\beta_1} \text{ and } V_2 = \alpha_2 U_2^{\beta_2}, \text{ for all } \alpha_1, \alpha_2 \in \mathbb{R}^+ \text{and } \beta_1, \beta_2 \in \mathbb{R}^+.$$

Let (i) $U_1 \leq_{\xi^w} U_2$, (ii) $\alpha_1 \leq \alpha_2$, (iii) $\beta_1 \leq \beta_2$. Then $V_1 \leq_{\xi^w} V_2$ if $U_1 \leq_{cx} U_2$.

Proof. Due to fact that $\varphi(x) = x \ln x$ is convex function, and when $U_1 \leq_{cx} U_2$ we get that $E[U_1] = E[U_2]$, if we suppose that $\xi^w_{U_1}(u)$ is decreasing in u, and let $U_1 \leq_{\xi^w} U_2$, $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. By apply Eq. (7) and Proposition 2.3, we have $V_1 \leq_{\xi^w} V_2$.

Corollary 4.1. Suppose the relationship between two random variables U_1 and U_2 as follows:

$$U_1 \leq_{\xi^w} U_2.$$

Define $V_1 = \alpha U_1^{\beta}$ and $V_2 = \alpha U_2^{\beta}$, $\alpha, \beta \in \mathbb{R}^+$. Then $U_1 \leq_{\xi^w} U_2$ if $U_1 \leq_{cx} U_2$.

In next theorem, we explain preservation properties and application of $\leq_{\xi^w}$, $\leq_{\mathbb{R}^w}$ and $\leq_{\mathbb{U}}$ between two exponential RV's if their scale parameters are ordered.

Theorem 4.2. Let two absolutely continuous random variables U_1 and U_2 with density function

$$f_i(x) = \alpha_i \exp\left(-\alpha_i x\right), \ \alpha_i, x > 0 \text{ and } i = 1, 2.$$

- If $\alpha_1 \geq \alpha_2$, then $U_1 \leq_{\xi^w} U_2$.
- If $\alpha_1 \leq \alpha_2$, then $U_1 \leq_{\mathbb{R}^w} U_2$.
- If $X \leq_{\mathbb{R}^w} Y$ then $H_{U_2} \leq_{\mathbb{U}} H_{U_1}$.

Proof. The result is obtained immediately from Remark 2.1.

Many studies explain the properties of repairable systems such as minimal repair. If the system has the virtual age $\mathbb{U}_{n-1}$ immediately after the $(n-1)^{th}$ repair, the functioning system obtained has the *nth* failure-time $\mathbb{Y}_n$ distributed as

$$\Pr\left[\mathbb{Y}_n \leq y | \mathbb{U}_{n-1} = u\right] = \frac{G\left(y+u\right) - G(u)}{\overline{G}(u)}, \tag{20}$$

where $G\left(y\right)$ is the failure time distribution of a new system $\left(\mathbb{U}_0 = 0\right)$. Let n^{th} repair cannot remove the damages incurred before the $(n-1)^{th}$ repair and α_n be the degree of the n^{th} repair, now the time between $(n-1)^{th}$ failure and nth failure reduce from $\mathbb{Y}_n$ to $\alpha_n \mathbb{Y}_n$. If $\alpha_n = 1$ for all $n \geq 1$ then it agrees with a minimal repair model.

Suppose $\mathbb{V}_n = \sum_{i=1}^{n} \mathbb{Y}_i\,(n \geqq 1)$ with $\mathbb{V}_0 = 0$ which represents the time elapsed since the system was put in operation, or the associated counting process $\mathbb{N}(t) = \sup\left\{n \geqq 1 : \mathbb{V}_{n-1} \leqq t\right\}$. Kijima [17] proved that $\mathbb{N}(t)$ (or $\left\{\mathbb{V}_n\right\}_0^\infty$) is a non-homogeneous Poisson process when $\alpha_n = 1$ for all $n \geq 1$. Ebrahimi and Pellerey [1] defined the following definition:

Definition 4.4. A point process $\{\mathbb{N}(t), t \geq 0\}$ consisting of interarrival times $\mathbb{Y}_1, \mathbb{Y}_2, \ldots$ is increasing $\left(\text{decreasing}\right)$ in the

1. convex residual entropy *order if*

$$H^w\left(\mathbb{B}_i, t\right) \leq (\geq) H^w\left(\mathbb{B}_j, t\right), \text{ for all } t \in R^+.$$

2. *weighted entropy order if*

$$\xi^w\left(\mathbb{B}_i, t\right) \leq (\geq) \xi^w\left(\mathbb{B}_j, t\right)$$

and $1 \leq i \leq j \leq n$, where f_k is the conditional probability density function of $\mathbb{Y}_k = \mathbb{V}_k - \mathbb{V}_{k-1}$ for all $k = 1, 2, \ldots$, given $\mathbb{V}_{k-1} = v_{k-1}, \ldots, \mathbb{V}_1 = v_1$, and

$$\mathbb{B}_k \stackrel{st}{=} \left[\mathbb{Y}_k | \mathbb{V}_{k-1} = v_{k-1}, \ldots, \mathbb{V}_1 = v_1\right].$$

From Definition 5.3, we can note that if a point process is increasing (decreasing) means the uncertainty of the distribution is increasing (decreasing), i.e., the process is deterioration (improving).

Lemma 4.1. Let $\mathbb{B}_k$ be as defined in Definition 5.3. Then, for $k = 1, 2, ...,$

1. $H^w\left(\mathbb{B}_k, t\right) = H^w\left(X, t + v_{k-1}\right);$
2. $\xi^w\left(\mathbb{B}_k, t\right) = \xi^w\left(X, t + v_{k-1}\right).$

Proof. Refer to Ebrahimi and Pellery [1], Theorem 2.5.

Theorem 4.3. The stochastic point process $\mathbb{N}(t) = \sup\left\{n \geqq 1 : \mathbb{V}_{n\ 1} \leqq t\right\}$ consisting of the time of *nth* failure $\mathbb{V}_{k-1}$, $k = 1, 2, ...$generated by a minimal repair policy is increasing (decreasing) in

1. convex residual entropy *order if* $G(.)$ is *IHR*,
2. *weighted entropy order if* $\xi^w(U)$ is increasing for all $u \geq 0$.

Proof. Similarly to lemma 5.1, we have $H^w\left(\mathbb{B}_{k+1}, t\right) = H^w\left(U, t + v_k\right)$. In addition, by Theorem 3.1 in Di Crescenzo and Longobardi [4] we conclude that

$$H^w\left(U, t + v_k\right) = \left(t + v_k\right)\left[1 - \ln\ \lambda\left(t + v_k\right)\right] + \frac{t + v_k}{\lambda\left(t + v_k\right)}\frac{d}{dt}H\left(U, t + v_k\right) + \frac{1}{\overline{G}\left(t + v_k\right)}I\left(t + v_k\right), 4.2 \tag{21}$$

where

$$I(t) = \int_t^\infty \overline{G}(u)\, H(U, u)\, du - \int_t^\infty \overline{G}(u) \ln \frac{\overline{G}(u)}{\overline{G}(t)} du,$$

and by Theorem 2.1 in Ebrahimi and Pellery [1] we have

$$H^w\left(U, t + v_k\right) = \left(t + v_k\right) H\left(U, t + v_k\right) + \frac{1}{\overline{G}\left(t + v_k\right)}I\left(t + v_k\right)\ .4.3 \tag{22}$$

By Theorem 2.5 in Ebrahimi and Pellery [1] we conclude that $H\left(U, t + v_{n-1}\right) \leq H\left(U, t + v_n\right)$. By using Eqs. (20–22), and when a continuous distribution $G(.)$ is IHR. It is obvious that

$$\overline{G}\left(t + v_k\right) \leq \overline{G}\left(t + v_{k-1}\right),$$

and

$$I\left(t + v_k\right) \geq I\left(t + v_{k-1}\right).$$

Then, we get $H^w\left(U, t + v_k\right) \geq H^w\left(U, t + v_{k-1}\right)$. This complete the proof.

5. ENTROPY ESTIMATION

In this section, we introduce four the non-parametric estimators of Eq. (6) by using the same idea in Vasicek [9], Van Es [18], Ebrahimi *et al.* [19] and Al-Omari [20].

Let $Z_1, Z_2, ..., Z_n$ is a sequence of the random sample with the distribution G and let $Z_{(1)}, Z_{(2)}, ..., Z_{(n)}$ be the corresponding order statistics. Besides the sample distribution function $G_n(z) = n^{-1}\sum_{i=1}^{n} \mathbf{1}_{Z_k \leq z, 1 \leq k \leq n}$. Then, the on-parametric estimators can be expressed as:

1. **Weighted Vasicek Entropy** ($V\xi^w_{(\delta,n)}(Z)$): We can estimate of Eq. (8) by replacing $G^w(t)$ by the empirical distribution $G^w_n(t)$, and using a difference operator in place of the differential operator. Thus, $V\xi^w_{(m,n)}(Z)$ estimator of Eq. (8) can be represented as follows

$$V\xi^w_{(\delta,n)}(Z) = n^{-1}\sum_{i=1}^{n} \ln\left\{\frac{n\sum_{k=1}^{n} w\left(Z_{(k)}\right)}{\sum_{j=1}^{i} w\left(Z_{(j)}\right) 2\delta}\left(Z_{(i+\delta)} - Z_{(i-\delta)}\right)\right\}, \tag{23}$$

where $\delta \in \mathbb{N}^{+}$ know as a window size, $\delta < n/2$, $Z_{(s)} = Z_{(1)}$ if $s < 1$ and $Z_{(s)} = Z_{(n)}$ if $s > n$.

2. **Weighted δ-spacings Entropy** ($SE\xi^{w}_{(\delta,n)}(Z)$): Estimates of weighted entropy based on sample δ-spacings which introduced by Van Es [18], we can provide $VE\xi^{w}_{(\delta,n)}(Z)$ estimator of Eq. (6) as

$$SE\xi^{w}_{(\delta,n)}(Z) = n^{-1}\sum_{i=1}^{n-\delta}\ln\left\{\frac{n\sum_{j=1}^{i}Z_{(j)}}{\sum_{k=1}^{n}Z_{(k)}\delta}\left(Z_{(i+\delta)} - Z_{(i)}\right)\right\} - \psi(\delta) + \ln\,\delta + E[Z], \tag{24}$$

where $\psi(.)$ is the digamma function and $\ln\,\delta - \psi(\delta)$ corrected bias entropy estimator.

3. **Weighted Small weights Entropy** ($WS\xi^{w}_{(\delta,n)}(Z)$): As assign smaller weights in Vasicek [9], we obtain

$$WS\xi^{w}_{(\delta,n)}(Z) = n^{-1}\sum_{i=1}^{n}\ln\left\{\frac{n\sum_{k=1}^{n}Z_{(k)}}{\sum_{j=1}^{i}Z_{(j)}\alpha_i\delta}\left(Z_{(i+\delta)} - Z_{(i-\delta)}\right)\right\}, \tag{25}$$

where

$$\alpha_k = \begin{cases} \dfrac{\delta + k - 1}{\delta}, & 1 \le k \le \delta \\ 2, & \delta + 1 \le k \le n - \delta\ . \\ \dfrac{\delta + n - k}{\delta}, & n - \delta + 1 \le k \le n \end{cases}$$

4. **Modified Small weights Entropy** ($MS\xi^{w}_{(\delta,n)}(Z)$): As assign smaller weights in Ibrahimi et al. [21] we get

$$MS\xi^{w}_{(\delta,n)}(Z) = n^{-1}\sum_{i=1}^{n}\ln\left\{\frac{n\sum_{k=1}^{n}w\left(Z_{(k)}\right)}{\sum_{j=1}^{i}w\left(Z_{(j)}\right)\beta_i\delta}\left(Z_{(i+\delta)} - Z_{(i-\delta)}\right)\right\}, \tag{26}$$

where

$$\beta_i = \begin{cases} \frac{3}{2}, & 1 \le i \le \delta, \\ 2, & \delta + 1 \le i \le n - \delta,\ . \\ \frac{1}{2}, & n - \delta + 1 \le i \le n. \end{cases}$$

Example 5.1: Let Z be random variable having exponential distribution with density function $f_Z(x) = \alpha\exp(-\alpha z)$, $\alpha, z > 0$ with parameter $\alpha > 0$. From Eq. (7) we obtain,

$$\vartheta^{w}(Z) = \frac{1}{\alpha}\left(\psi(2) - \ln\,\alpha\right),$$

where $\psi(.)$ is Euler's psi function. By Example (2.1, a) in Di Crescenzo and Longobardi [4] we get,

$$H^{w}(Z) = \frac{2 - \ln\,\alpha}{\alpha}\ \text{and}\ \xi^{w}_{1}(Z) = 2 - \ln\,\alpha - \psi(2).$$

Therefore,

$$\mathbb{R}\left(Z^{w} \parallel Z\right) = -4\left(1 - \ln \alpha\right) - \psi\left(2\right) = -4H_{Z} - \psi\left(2\right).$$

Suppose $\alpha \in [0, 100]$. $H^{w}(Z)$, $\xi^{w}(Z)$ and $\mathbb{R}\left(Z^{w} \parallel Z\right)$ with weighted function $w(z) = z$ is evaluated in Table 1.

Table 1 Measures of weighted entropies of exponential distribution.

α	$\hat{H}^{w}(Z)$	$\hat{\xi}^{w}(Z)$	$\hat{\mathbb{R}}^{w}(Z^{w} \parallel Z)$
0.25	13.5452	2.9635	−9.9680
0.5	5.3863	2.2704	−7.1954
1	2	1.5772	−4.4228
5	0.0781	−0.0322	2.0150
10	-0.0303	−0.7254	4.7876

Now, a real data is illustrated to investigate the performance of suggested estimators.

Data Set: The following data set which is taken from Smith and Naylor [22], it represents the strength of 1.5 cm glass fibers measured at the National Physical Laboratory, England.

Data Set: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89

Shanker et al. [23] show the exponential density function ($Exp\,(0.663647)$) provided a better fit for this data. We compute the exact value of the weight entropy measure by the real data and compare this measure with $V\xi^{w}_{(\delta,n)}(Z)$, $SE\xi^{w}_{(\delta,n)}(Z)$, $WS\xi^{w}_{(\delta,n)}(Z)$ and $MS\xi^{w}_{(\delta,n)}(Z)$ which shows in Table 2.

Table 2 Weighted entropy measure for exponential distribution (0.663647).

δ	$\xi^{w}_{(\delta,n)}(Z)$	$V\xi^{w}_{(\delta,n)}(Z)$	$SE\xi^{w}_{(\delta,n)}(Z)$	$WS\xi^{w}_{(\delta,n)}(Z)$	$MS\xi^{w}_{(\delta,n)}(Z)$
1	1.9872	1.210967749	1.244680937583	1.232972422	1.237538804
2	1.9872	1.200196229	1.231653171583	1.943017269	1.298445816
3	1.9872	1.245455104	1.462350850583	2.308545839	1.323498873
4	1.9872	1.279410563	1.650373401583	2.537544185	1.337842382
5	1.9872	1.282220574	1.820823663583	2.721711858	1.415075846

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