

A Novel Comparative Linguistic Distance Measure Based on Hesitant Fuzzy Linguistic Term Sets and Its Application in Group Decision-Making

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ABSTRACT

The linguistic approaches are required in order to assess qualitative aspects of many real problems. In most of these problems, decision makers only adopt single and very simple terms which would not reflect exactly what the experts mean for many intricate applications. Frequently, the assessments of decision making problems involve comparative linguistic expressions. Accordingly, we propose a novel distance measure between hesitant fuzzy linguistic term sets (HFLTSs) to solve fuzzy group decision making (FGDM) problems. Firstly we define the characteristic functions to describe the HFLTSs transformed from comparative linguistic expressions. Then we construct a weighted HFLTSs graph containing all notes in the HFLTSs. Distances in the graph of individual assessments are defined by measures considering diversity and specificity of HFLTSs's granularity. We put forward a new approach to achieve aggregation results for group decision making to realize the minimal distances with individual assessments. Finally, numerical examples are illustrated.

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1. INTRODUCTION

Decision making is a common process to human beings. Decision making problems are usually defined in uncertain and imprecise situations. Different tools have been provided to solve such problems, such as fuzzy sets theory and the fuzzy linguistic approaches [1]. It seems natural to apply the computing with words (CWW) methodology in order to create and enrich decision models in which the information provided and manipulated has a qualitative nature [2]. CWW processes can be carried out by different linguistic representation models and computational models. Since Zadeh [1] proposed the fuzzy linguistic approach, many extensions such as the 2-tuple linguistic computational model [3], the proportional 2-tuple model [4], the continuous linguistic term [5] and numerical scales of 2-Tuple linguistic model [6, 7] have been introduced. However, these models have some limitations, mainly because they assess a linguistic variable by using single and very simple terms which may not reflect exactly what the experts mean [8, 9]. Considering the diversity in which different sources of knowledge exhibit, decision makers show their personal preferences when giving assessments. In order to rich linguistic expressions in different decision making situations, decision makers are permitted to use

context-free grammars to generate comparative linguistic expressions. For example, “lower than medium”, “greater than high”. But comparative linguistic expressions are hard to be directly participated in computing. Rordiguez *et al.* [9, 10] elicited computable information of comparative linguistic expressions and transformed to hesitant fuzzy linguistic term set (HFLTS). HFLTSs provide experts with greater flexibility. Many applications [11–20] were developed based on HFLTSs computational models.

In order to apply HFLTSs to solve decision making problems, studies of computational models about HFLTSs were developed. These are outlined below.

1. Envelope-based approaches.

Rodríguez, Martínez [9] defined the envelope of HFLTS. Some envelope-based approaches were proposed. Different kinds of aggregation operators are calculated by means of aggregating the envelopes for HFLTSs which are presented as linguistic intervals. Rodríguez, Martínez [10] aggregated the lower value of the linguistic intervals as the pessimistic perception and aggregated the greater value as the optimistic perception. Chen and Hong [21] performed the minimum operations and the maximum operations among the linguistic intervals and used the likelihood method for ranking the priority. Aggregating HFLTSs is transformed into aggregating two sets of different simple linguistic terms.

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2. All-elements-included approaches

The second kind approaches use the initial fuzzy representation of HFLTS in computing processes, which we call all-elements-included approach, while envelope-based operators only use the upper bound and the lower bound of an HFLTS. Wei, Zhao [22] defined two aggregation operators which use all elements in HFLTSs to obtain a new HFLTS. Liao, Xu [8] introduced the Hamming distance and the Euclidean distance for HFLTSs based on all-elements-included approaches. Liu and Rodríguez [23] defined a fuzzy envelope for HFLTS which is a trapezoidal fuzzy membership function obtained by aggregating the fuzzy membership functions of the linguistic terms of the HFLTS. Chen and Hong [21] proposed a method to aggregate the fuzzy sets in each HFLTS into a fuzzy set and performed the α -cut operations to these aggregated fuzzy sets to get intervals. The difficulty in this method is that the cardinalities of two HFLTSs are different. Xia and Xu [24] introduced the axioms of distance and similarity measures for hesitant fuzzy sets (HFSs) with different cardinalities. Rodríguez, Martínez [25] also discussed the comparison problem of two HFEs with different cardinalities. Liao, Xu [8] extended the shorter HFLTS by adding the average value in it until both HFLTSs have the same length.

Each of these two approaches has advantages and disadvantages. Envelope-based approach makes computation easier. But envelope-based approaches simply rely on the upper bound and the lower bound of HFLTS, which may lead to information distortion and/or loss [14]. The most important merit of HFLTSs is that they can reflect decision makers' uncertainty and hesitancy. Envelope-based approaches cannot reflect this merit. All-elements-included approaches make full use of information contained in HFLTSs, even use fuzzy membership function or possibility degree formulas of HFLTSs. But the process of decision making is complex. How to combine the advantages of these two approaches and provide a novel approach which can both simplify the computational complexity and identify difference in HFLTSs is our paper's main motivation.

Distance and similarity measures are fundamentally important in decision making and pattern recognition [8]. The individual assessments presented by HFLTS compose a graph and the individual assessments are the vertices in the graph. Liao and Xu [26] introduced a family of novel distance and similarity measures for HFLTSs from the geometric point of view. Falcó, García-Lapresta [27] applied the distance measure between two HFLTSs to solve decision problems in new majority judgment voting system. Wei, Zhao [15] introduced distance measures for extended HFLTS (EHFLTS) and developed a novel multi-criteria group decision making model. Wang and Xu [28] proposed two distinct consistency measures of extended hesitant fuzzy linguistic preference relations for group decision making. But the assumptions such as the equality among the distances between consecutive linguistic terms for all the agents [27], and the assumption that distance measures only be used to solve the MCDM problem with single expert/decision maker [26], will limit their scope of application.

Inspired by decision making models based on distance measures, we look for other measures, based on fuzzy sets and fuzzy logic,

to identify the differences of HFLTSs to avoid complex computation. We will propose some novel distance measures from the view of graphs. As distance and similar measures are the foundation of many decision making models, it would be interesting to integrate the geometric distance and envelope-based approaches into HFLTSs decision making approaches.

In this paper, we focus on investigating the distance measures for HFLTSs, and their properties. Then we define a set of aggregation operators especially for fuzzy group decision-making (FGDM).

The remainder of the paper is organized as follows. In Section 2, we present a brief review of HFLTSs. In Section 3, we give a novel distance measure between HFLTSs and distance measures. We present a weighted HFLTSs graph composed by linguistic expressions as its vertices to obtain a distance measure between any two vertices. In Section 4, an approach based on distance measures to solve FGDM problems is provided. Finally examples are illustrated in Section 5. The last section draws our conclusion.

2. PRELIMINARIES

2.1. Comparative Linguistic Expressions and HFLTSs

In the traditional CWW, assessment is selected from a predefined linguistic term set. This kind assessment is not powerful enough in reflecting a decision maker's hesitance. Hence, more attention is paid to another possibility for generating more elaborate linguistic expressions, which refers to a context-free grammar [29]. In this section we recall some concepts of HFLTS. HFLTS was proposed by Rodríguez, Martínez [10] to model comparative linguistic expression using a context-free grammar. A context-free grammar [9] G_H is a 4-tuple (V_N, V_T, I, P) , where V_N is the set of non-terminal symbols, V_T is the set of terminals' symbols, I is the starting symbol, and P is the production rules.

Comparative linguistic expressions cannot be directly used in decision making process. Elicitation is the necessary process to get the formal representation to suit the CWW [30]. Comparative linguistic expressions can be converted into HFLTSs by the transformation function E_{GH} [10]:

- i. $E_{GH}(s_i) = \{s_i | s_i \in S\}$,
- ii. $E_{GH}(\text{less than } s_i) = \{s_j | s_j \in S \text{ and } s_j \leq s_i\}$,
- iii. $E_{GH}(\text{greater than } s_i) = \{s_j | s_j \in S \text{ and } s_j \geq s_i\}$,
- iv. $E_{GH}(\text{between } s_i \text{ and } s_j) = \{s_k | s_k \in S \text{ and } s_i \leq s_k \leq s_j\}$.

The basic concepts and operations of HFLTSs are defined as follows.

Definition 1. [10] Let G_H be a context-free grammar and $S = \{s_0, \dots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$$V_N = \{ \langle \text{primary term} \rangle, \langle \text{composite term} \rangle, \\ \langle \text{unary relation} \rangle, \langle \text{binary relation} \rangle, \langle \text{conjunction} \rangle \}$$

$V_T = \{ \text{lower than, greater than, at least, at most, between, and, } s_0, \dots, s_g \}$

$$I \in V_N$$

- $P = \{ I : := \langle \text{primary term} \rangle \mid \langle \text{composite term} \rangle$
- $\langle \text{composite term} \rangle : := \langle \text{unary relation} \rangle,$
- $\langle \text{binary relation} \rangle \mid \langle \text{binary relation} \rangle$
- $\langle \text{primary term} \rangle \langle \text{conjunction} \rangle \langle \text{primary term} \rangle$
- $\langle \text{primary term} \rangle : := s_0 | s_1 \cdots | s_g$
- $\langle \text{unary relation} \rangle : := \text{lower than, greater than, at least, at most}$
- $\langle \text{binary relation} \rangle : := \text{between}$
- $\langle \text{conjunction} \rangle : := \text{and} \}$

For example, by using the previous grammar G_H and linguistic terms $s_i \in S$, the expert may express their preferences about an item by comparative linguistic expressions, such as:

$$P^1 = \begin{pmatrix} - & \text{between high and very high} & \text{very high} \\ \text{at most low} & - & \text{high} \\ \text{at most low} & \text{between very low and low} & - \end{pmatrix}$$

Definition 2. [9] let $S = \{s_0, s_1, \dots, s_g\}$ be a linguistic term set. A HFLTS H_s is an ordered finite subset of consecutive linguistic terms of the linguistic term set S .

Definition 3. [9] let $S = \{s_0, s_1, \dots, s_g\}$ be a linguistic term set. The upper bound H_{s^+} and the lower bound H_{s^-} of the HFLTS H_s , respectively, are defined as follows:

- i. $H_{s^+} = \max(s_i) = s_j, \forall i, s_i \leq s_j, s_i \in H_s \text{ and } s_j \in H_s,$
- ii. $H_{s^-} = \min(s_i) = s_j, \forall i, s_i \geq s_j, s_i \in H_s \text{ and } s_j \in H_s.$

Definition 4. [9] Let $S = \{s_0, s_1, \dots, s_g\}$ be a linguistic term set. The envelope $env(H_s)$ of an HFLTS H_s is a linguistic interval which is obtained by means of the upper bound H_{s^+} and the lower bound H_{s^-} of an HFLTS H_s , shown as follows:

$$env(H_s) = [H_{s^-}, H_{s^+}]$$

Many computational models of HFLTSs are based on the envelope $env(H_s)$ of an HFLTS H_s . They treat HFLTSs as linguistic intervals.

2.2. The Distance Measure Between HFLTSs

Traditional work on distance measures was based on Hamming distance and Euclidean distance. Liao, Xu [8] adopted the Hamming distance and Euclidean distance.

Definition 5. [8] Let $S = \{s_\alpha \mid \alpha = -\tau, \dots, -1, 0, 1, \dots, \tau\}$ be a linguistic term set, and H_S^1 and H_S^2 two HFLTSs. Let

$$H_S^1 = \cup_{s_{\delta_l^1} \in H_S^1} \{s_{\delta_l^1} \mid l = 1, \dots, \#H_S^1\} \quad (\#H_S^1 \text{ is the number of linguistic terms in } H_S^1) \text{ and } H_S^2 = \cup_{s_{\delta_l^2} \in H_S^2} \{s_{\delta_l^2} \mid l = 1, \dots, \#H_S^2\} \text{ where } \#H_S^1 = \#H_S^2 = L.$$

The Hamming distance is

$$d_{hd}(H_S^1, H_S^2) = \frac{1}{L} \sum_{l=1}^L \frac{|\delta_l^1 - \delta_l^2|}{2\tau + 1} \quad (1)$$

and the Euclidean distance is

$$d_{ed}(H_S^1, H_S^2) = \left(\frac{1}{L} \sum_{l=1}^L \left(\frac{|\delta_l^1 - \delta_l^2|}{2\tau + 1} \right)^2 \right)^{1/2} \quad (2)$$

Another kind distance measures are based on the Manhattan distance. Roselló, Sánchez [31] defined a space for computing this distance. The distance between HFLTSs is the shortest path connecting two vertices in the graph.

Definition 6. [31] Given an order-of-magnitude space $G_{S_n^*}$, the graph $G_{S_n^*}$ associated with S_n^* is the graph whose vertices are the basic labels of S_n^* or the connex union of basic labels, i.e. the set of vertices is $V(G_{S_n^*}) = \{x_{i,j} = [B_i, B_j] \mid B_i, B_j \in S_n^*, i \leq j\}$, and whose set of edges is $E(G_{S_n^*}) = \{x_{i,j} \sim x_{r,s} \mid (r = i \text{ and } s = j + 1) \text{ or } (s = j \text{ and } r = i + 1)\}$.

For example, consider the set of linguistic terms $S_n^* = \{B_1, B_2, \dots, B_5\}$. Each vertex is presented as $x_{i,j} = [B_i, B_j]$. The graph $G_{S_n^*}$ is given in Figure 1.

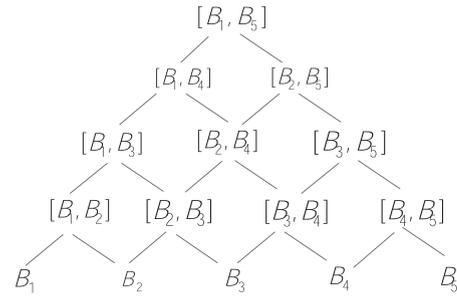


Figure 1 | An example of the graph representation.

According to the graph $G_{S_n^*}$, some interesting distance measures are proposed. For example, suppose $l: V(G_{S_n^*}) \times V(G_{S_n^*})$ is the geodesic distance in $V(G_{S_n^*})$. Roselló, Sánchez [31] gave a special situation where the weights of all the edges are equal, and then obtain the conclusion:

$$l([B_i, B_j], [B_r, B_s]) = k(|i - r| + |j - s|) \quad (3)$$

where all the edges have an equal weight k .

The definition of distance between two linguistic expressions is also given (Definition 7).

Definition 7. [31] The distance between two linguistic expressions $\mathcal{E} = [B_i, B_j]$ and $\mathcal{F} = [B_h, B_k]$ is defined as

$$d(\mathcal{E}, \mathcal{F}) = d_M(\psi(\mathcal{E}), \psi(\mathcal{F})) = |i - h| + |j - k| \quad (4)$$

The function ψ maps a linguistic expression $[B_i, B_j]$ as a point in the plane.

$$\psi[l_i, l_j] = (j - 1, i - 1) \quad (5)$$

3. A NOVEL DISTANCE MEASURE BETWEEN HFLTSS

3.1. Characteristics of HFLTSS

The previously presented H_s in Definition 4 appears in as an interval form and it is viewed as an information granule. Information granules offer a unique way of quantifying a diversity of sources of knowledge under consideration and expressing this aspect in the form of the level of granularity (specificity) [32]. We apply two criteria parameters (coverage criterion and specificity criterion [32]) to describe the characteristics of H_s .

Definition 8. [32] Characteristics of H_s are described by coverage index $cov(H_s)$ and specificity criterion $spe(H_s)$:

- i. A coverage criterion $cov(H_s)$ expresses to which extent H_s covers. It reflects the uncertainty degree of H_s .
- ii. A specificity criterion $spe(H_s)$ articulates the cumulative length of H_s . It reflects the position of H_s in the domain S .

Let $H_s = [H_{s^-}, H_{s^+}]$, and $H_{s^-} = B_i$, $H_{s^+} = B_j$. There are elements B_i, B_{i+1}, \dots, B_j between H_{s^-} and H_{s^+} . H_s 's granularity is defined as the cardinality of the set $(B_i, B_{i+1}, \dots, B_j)$, presented as $card(B_i, B_{i+1}, \dots, B_j)$. The coverage index is expressed as a ratio

$$cov(H_s) = \frac{card(B_i, B_{i+1}, \dots, B_j)}{card(S)} \quad (6)$$

The specificity index is expressed as

$$spe(H_s) = \sum_{k=i}^j \Delta^{-1}(B_k) / card(S) \quad (7)$$

where Δ^{-1} [3] is a function transforming a linguistic label B_k into a numerical value and $card(S)$ is the number of elements in the set S .

There are two decision scenarios. The first one is that decision makers use multi-granular linguistic term sets to discriminate the assessments with different precision levels. That is to say, decision makers are free to choose elements of different coverage criteria. The other one is that a decision maker chooses one linguistic term set and use several elements in this set to generate different HFLTSS to discriminate the assessments with different precision levels, just like HFLTSS in Definition 2. According to Definition 2, H_s is generated from $S = \{s_0, s_1, \dots, s_g\}$. S is an input space which is granulated to a collection of fuzzy sets (s_i) . S reveals the input space structure. S contains $card(S)$ elements. These elements' semantics meanings in S are the same. Their coverage criteria are the same according to our Definition 8. But their specificity criteria are different. In any linguistic approach, an important parameter to be determined is the granularity (cardinality) of uncertainty [33].

3.2. A Weighted HFLTSS Graph

The distance between two HFLTSS is defined as the geodesic distance in the graph $G_{S_n^*}$. Firstly, we give the definition of the graph $G_{S_n^*}$.

Definition 9. Given a weighted graph $G_{S_n^*} = (V, E, \emptyset)$, where V is a nonempty set of vertices; E is a set of edges; \emptyset is the weight set of edges.

Let us consider the vertices set. $S_n^* = \{B_1, \dots, B_n\}$ is a finite set of basic labels. S_n^* is the base of a graph, since every vertex is composed by the labels in S_n^* . The basic linguistic term set S_n^* has $card(S_n^*)$ granules. $card(S_n^*)$ is a desirable index of inherent diversity of a graph that we have to deal with. $V(G_{S_n^*})$ is presented as $x_{i,j}$ which is the label $[B_i, B_j]$ with $i \leq j$. A decision maker's assessment H_s is constructed by S_n^* . The granularity of H_s is presented as $card(B_i, B_{i+1}, \dots, B_j)$.

We group the vertices according to $cov(x_{i,j})$. The same group is at the same "cover". We use Δ_t to describe a group that represents the same coverage index. Δ_t will be: $\{\{B_i, B_j\} | B_i, B_j \in S_n^*, j = i + t\}$. For example, $\Delta_2 = \{\{B_1, B_3\}, \{B_2, B_4\}, \{B_3, B_5\}\}$ is a group at the same "cover", considering the set of linguistic terms $S_n^* = \{B_1, B_2, \dots, B_5\}$.

The graph $G_{S_n^*}$ can be seen as the combination of several levels of different coverage indexes from the basic labels set of S_n^* to the top level of one label set. A new definition of the graph $G_{S_n^*}$ is the union of all levers.

Definition 10. A graph $G_{S_n^*}$ is defined as the union of all levels

$$G_{S_n^*} = \bigcup_t S_n^* \cup (\Delta_t)$$

where $t = 1, 2, \dots, (n-1)$ and $S_n^* = \{B_1, \dots, B_n\}$.

For example, $G_{S_5^*} = S_5^* \cup (\Delta_1) \cup (\Delta_2) \cup (\Delta_3) \cup (\Delta_4)$.

In the graph $G_{S_n^*}$, the lowest layer represented the basic labels of S_n^* . The second layer represents the linguistic expressions created by two consecutive linguistic terms $[B_i, B_{i+1}]$, and so on up to the last layer. As a result, the higher a vertex is in, the more imprecise it becomes. When a decision maker is confident about his opinion on an alternative, he/she can assign an HFLTSS in the lower lever. However, if he/she is unconfident about his/her opinion, he/she might assign an HFLTSS in the higher lever. The coverage index of this interval, which is the vagueness information of H_s , also reflects the confidence of a decision maker.

Then, let us see the edges set $E(G_{S_n^*})$. The edge set is $E(G_{S_n^*}) = \{x_{i,j} \sim x_{r,s} | (r = i \text{ and } s = j + 1) \text{ or } (s = j \text{ and } r = i + 1)\}$. It gives the position relations of vertices.

Finally, we look at the weight set $\emptyset(G_{S_n^*})$. The weight set is $\emptyset(G_{S_n^*}) = \{w(x_{i,j} \sim x_{r,s}) \in \mathbb{R}^+ | x_{i,j} \sim x_{r,s} \in E(G_{S_n^*})\}$. We need to know not only the position relations but also qualitative relations of different vertices in order to separate different vertices. Through defining the edge weight of the graph $G_{S_n^*}$, we can know the qualitative relations of different vertices. That is the advantage of our approach. In the next (Section 3.3), we will give the weight function of edge set.

3.3. The Distance Measure Between HFLTSS

Note that each edge of a graph may have a specific weight $w(x_{i,j} \sim x_{r,s})$ given by a weight function $w : (G_{S_n^*}) \rightarrow \mathbb{R}^+$. Before

we present the definition of weight function for the graph $G_{S_n^*}$, we introduce some important concepts.

Roselló, Prats [34] used an information function of qualitative labels to measure the consensus degree of two qualitative labels in the group decision. We modify this function to measure the distance between two linguistic intervals. The information I for the vertex $x_{i,j}$ is a positive continuous real function. It is used to build the weight function of each edge. The information I satisfies that if $x_{i,j}$ and $x_{h,k}$ are two vertices and $cov(x_{i,j}) \leq cov(x_{h,k})$, then $I(x_{i,j}) \geq I(x_{h,k})$.

$$I(x_{i,j}) = \ln \frac{1}{cov(x_{i,j})} \quad (8)$$

Definition 11. The weight function $w(G_{S_n^*}) \rightarrow \mathbb{R}^+$ of edge $E(G_{S_n^*}) = \{x_{i,j} \sim x_{r,s} \mid (r = i \text{ and } s = j + 1) \text{ or } (s = j \text{ and } r = i + 1)\}$ is

$$w(x_{i,j} \sim x_{r,s}) = 1 + k \cdot |I(x_{i,j}) - I(x_{r,s})| \quad (9)$$

where $k \in [0, +\infty]$ is the imprecise index.

Remarks. k reflects the importance of imprecise when deciding the weight of an edge. If two vertices are more different in coverage, then $|I(x_{i,j}) - I(x_{r,s})|$ is larger. Through adjusting the value of k , we can obtain different weights of the two vertices. Distance measures given by Falcó, García-Lapresta [27] and Roselló, Sánchez [31] can be viewed as a simplified form of Definition 11 where $k = 0$, which means that the weights of all edges are equal.

In our model, an edge is between two close neighbor levels. Since every vertex in a level is uniform distributed, we can get $cov([B_i, B_{i+1}]) = (t + 1) / card(S_n^*)$ for any vertex in Λ_t .

Hence, the weight of the edge between level Λ_t and Λ_{t+1} is

$$w(x_{i,j} \sim x_{r,s}) = 1 + k |\ln(t + 2) - \ln(t + 1)|$$

Corollary 1. If $k = 0$, the distance of all edges in the space $G_{S_n^*}$ are equal to 1.

The weight function (9) calculates the distance of two directly connected vertices. Now we introduce an approach to calculate the distance from arbitrary vertices v_0 to v_k . There are several possible reachable paths from vertex v_0 to v_k , and a path is denoted as a finite sequence $p = v_0 v_1 \dots v_k$, where $v_i \sim v_{i+1}$ is an edge. The path length of the path p is denoted as $\phi(p) = w(v_0 \sim v_1) + w(v_1 \sim v_2) + \dots + w(v_{k-1} \sim v_k)$. The shortest path length from v_0 to v_k is the minimum distance of all possible reachable paths from a starting vertex to a target vertex. We define the distance from vertex v_0 to v_k as

$$d(v_0 \sim v_k) = \min \{ \phi(p) : v_0 \rightsquigarrow v_k \}$$

where $p(v_0 \rightsquigarrow v_k)$ is a reachable path from v_0 to v_k . Dijkstra's algorithm [35] can be a good tool to find the shortest path from v_0 to v_k . Based on the refined Dijkstra's algorithm proposed in [36], we find the shortest paths from $x_{i,j}$ to $x_{r,s}$ in graph $G_{S_n^*}$.

Corollary 2. If the weights of all edges in the space $G_{S_n^*}$ are equal to 1, the geodesic distance of vertices $x_{i,j}$ and $x_{r,s}$ can be expressed through

$$d([B_i, B_j], [B_r, B_s]) = |i - r| + |j - s|$$

Proof. The shortest path is the union of two parts. $x_{a,j}$ and $x_{r,s}$ are in the same level. $x_{a,j}$ is between $x_{i,j}$ and $x_{r,s}$ in vertical direction.

Without loss of generality, we assume $r \geq i$ and $s \geq j$. See Figure 2.

The red lines in Figure 2 are the shortest path from $x_{i,j}$ to $x_{r,s}$. The path can be divided into two parts. One path is vertical and the other is horizontal. The vertical part is from $x_{i,j}$ to $x_{a,j}$, and the horizontal part is from $x_{a,j}$ to $x_{r,s}$. These two paths constitute the shortest path from $x_{i,j}$ to $x_{r,s}$.

The shortest path from $x_{i,j}$ to $x_{a,j}$ is a straight line. The distance of this line is

$$d([B_i, B_j], [B_a, B_j]) = |a - i| = a - i$$

The shortest path from $x_{a,j}$ to $x_{r,s}$ is like a zigzag line. The distance of this line is

$$d([B_a, B_j], [B_r, B_s]) = (s - j) + (r - a)$$

The whole distance from $x_{i,j}$ to $x_{r,s}$ is

$$d([B_i, B_j], [B_r, B_s]) = (a - i) + (s - j) + (r - a) = (r - i) + (s - j)$$

Other situations can be treated in a similar way. So we conclude:

$$d([B_i, B_j], [B_r, B_s]) = |i - r| + |j - s|$$

Remarks. We can conclude from Corollary 2 that in a special situation our distance measure coincides with Function (4). In other words, Function (4) is equal to our distance measure when $k = 0$. $k = 0$ means that we do not consider the coverage criterion of characteristics of H_s . If we consider the coverage criterion of HFLTSS, our distance measure shows its advantage. In fact, the coverage criterion of HFLTSS is just what many other distance measures will ignore. After a deeper analysis of the characteristics of HFLTSS, it is seen that the coverage criterion of HFLTSS reflects the hesitance of decision makers. The hesitance of decision makers reflects personality and knowledge background, which is important for FGDM. A distance measure without fully considering the characteristics of HFLTSS is not perfect.

3.4. Comparison Operators

The comparison of HFLTSSs represented by linguistic intervals is carried out according to an ordinary lexicographic order [27]. Lexicographic order is more intuitive for people.

Given $\mathcal{F}, \varepsilon \in G_{S_n^*}$, the binary relation \succsim_1 is defined as the first superior

$$\mathcal{F} \succsim_1 \varepsilon \Leftrightarrow d(\mathcal{F}, s_g) \leq d(\varepsilon, s_g)$$

The binary relation \succsim_2 is defined as the second superior

$$\mathcal{F} \succsim_2 \varepsilon \Leftrightarrow d(\mathcal{F}, [s_{g-1}, s_g]) \leq d(\varepsilon, [s_{g-1}, s_g])$$

This order seems natural: the closer the assessments are to the linguistic term s_g , the better the alternative is. The better alternative also has the longest geometric distance from s_0 . Then we give the proof.

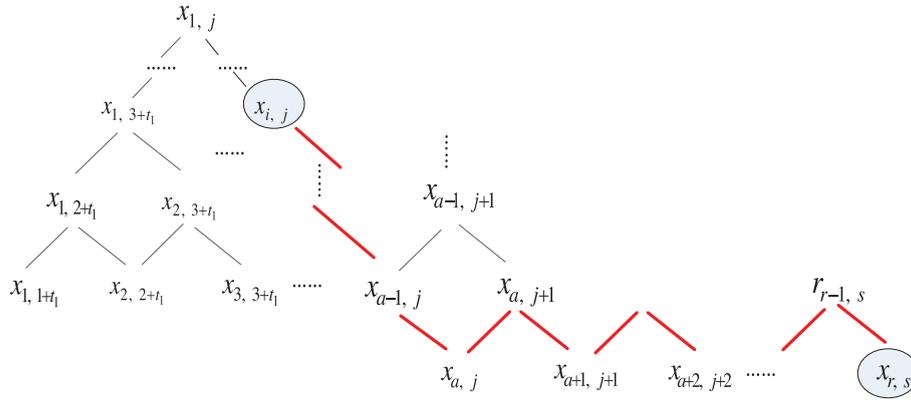


Figure 2 | An example to show the positions of vertices $x_{i,j}$ and $x_{r,s}$.

Lemma 1. For all $\mathcal{F}, \varepsilon \in G_{S_n^*}$, if $\mathcal{F} \succeq_1 \varepsilon$, it holds

$$\mathcal{F} \succeq_1 \varepsilon \Leftrightarrow d(\mathcal{F}, s_0) \geq d(\varepsilon, s_0)$$

Proof. $d(\mathcal{F}, s_g) = d([s_l, s_k], [s_g, s_g]) = |l-g| + |k-g| = 2g - (l+k)$

$$= 2g - (|l-0| + |k-0|)$$

$$= 2g - d([s_l, s_k], [s_0, s_0])$$

$$= 2g - d(\mathcal{F}, s_g)$$

$$\mathcal{F} \succeq_1 \varepsilon \Leftrightarrow d(\mathcal{F}, s_g) \leq d(\varepsilon, s_g) \Leftrightarrow d(\mathcal{F}, s_0) \geq d(\varepsilon, s_0)$$

Lemma 2. For all $\mathcal{F}, \varepsilon \in G_{S_n^*}$, if $\mathcal{F} \succeq_2 \varepsilon$, it holds

$$\mathcal{F} \succeq_2 \varepsilon \Leftrightarrow d(\mathcal{F}, [s_0, s_1]) \geq d(\varepsilon, [s_0, s_1])$$

Proof. $d(\mathcal{F}, [s_{g-1}, s_g]) = d([s_l, s_k], [s_{g-1}, s_g]) = |l-g+1| + |k-g| = 2g-1 - (l+k)$

$$= 2g - (|l-0| + |k-1|) = 2g - d([s_l, s_k], [s_0, s_1])$$

$$= 2g - d(\mathcal{F}, [s_0, s_1])$$

$$\mathcal{F} \succeq_2 \varepsilon \Leftrightarrow d(\mathcal{F}, [s_{g-1}, s_g]) \leq d(\varepsilon, [s_{g-1}, s_g])$$

$$\Leftrightarrow d(\mathcal{F}, [s_0, s_1]) \geq d(\varepsilon, [s_0, s_1])$$

4. APPLICATION TO FUZZY GROUP DECISION MAKING

4.1. The Description of FGDM

FGDM, which uses linguistic expresses as assessments, becomes an important research subject because of the fuzziness of objective things and hesitance of human thinking. Assume a set of decision makers $E = \{e_1, e_2, \dots, e_n\}$ and a set of alternatives $X = \{x_1, x_2, \dots, x_m\}$. The decision matrix is denoted by $A = (a_{ij})_{n \times m}$ where a_{ij} is the assessment for the alternative x_j with respect to decision maker e_i . The decision makers have a weighting vector $W = \{w_1, w_2, \dots, w_n\}$, where $0 \leq w_i \leq 1$ and $\sum_{i=1}^n w_i = 1$.

Decision makers use comparative linguistic expressions to express their assessments. The key problem in our decision process is in the aggregation phase. We need to aggregate $A_j = \{a_{1j}, a_{2j}, \dots, a_{nj}\}$ to obtain $\mathcal{O} = \{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m\}$ where \mathcal{O}_j is the global opinion of alternative x_j .

In this section, we will introduce an approach based on distance measures to solve FGDM problems.

4.2. The Aggregation Approach Based on Distance Measure

The basic assumptions of the aggregation approach are as following:

- The result of aggregating a set of linguistic intervals is also a linguistic interval.
- The linguistic interval should reflect all information of the set of linguistic intervals as accurately as possible.

In the aggregation phase, we find a linguistic expression which has the minimal distances from all assessments $A_j = \{a_{1j}, a_{2j}, \dots, a_{nj}\}$ as the global assessment \mathcal{O}_j . This aggregation result can satisfy the above two assumptions. We call the linguistic express “substitution” because the linguistic expression is a best substitution to replace a set of assessments $\{a_{1j}, a_{2j}, \dots, a_{nj}\}$.

Now we give the Algorithm 1 to find a linguistic expression which has the minimal distances with $U = \{u_1, u_2, \dots, u_n\}$.

Algorithm 1:

Step 1. Construct $G_{S_n^*} = (V, E, \emptyset)$, where $u_i \in V$.

Step 2. Calculate distance for u_i .

For a starting node $u_i \in V$, use Dijkstra’s shortest path algorithm to calculate the distance $d(u_i \sim v)$ for all $v \in V$.

Step 3. Find the substitution $x_{i,j}$.

We present the model based on the weighted averaging operator to find a linguistic expression having the minimal distances with $\{u_1, u_2, \dots, u_n\}$

$$\text{Min} \sum_{\substack{u_i \in U \\ x_{i,j} \in V}} d(u_i, x_{i,j}) \cdot y_{i,j} \cdot w_i$$

$$\text{s.t.} \begin{cases} y_{i,j} = 0 \text{ or } 1 \\ \sum_{\substack{u_i \in U \\ x_{i,j} \in V}} y_{i,j} = 1 \\ \sum_{i=1}^n w_i = 1 \end{cases}$$

In the model, $y_{i,j} = 1$ means that vertex $x_{i,j}$ is selected as the best substitution. The condition is that we can only select one node. $\sum_{\substack{u_i \in U \\ x_{i,j} \in V}} y_{i,j} = 1$ makes the condition satisfied. The linguistic interval $x_{i,j}$ is the linguistic interval that can reflect all information of the set assessment.

We can extend this model to meet the needs of some operators. Now we give an extension for the hesitant fuzzy LOWA (HFLOWA) operator.

$$\text{Min} \sum_{\substack{u_{\sigma_i} \in U \\ x_{i,j} \in V}} d(u_{\sigma_i}, x_{i,j}) \cdot y_{i,j} \cdot w_i$$

$$\text{s.t.} \begin{cases} y_{i,j} = 0, 1 \\ \sum_{\substack{u_i \in U \\ x_{i,j} \in V}} y_{i,j} = 1 \\ \sum_{i=1}^n w_i = 1 \end{cases}$$

where $W = (w_1, \dots, w_n)^T$ is an associated weighting vector, and $\{u_{\sigma_1}, u_{\sigma_2}, \dots, u_{\sigma_n}\}$ is a permutation of $\{u_1, u_2, \dots, u_n\}$ such that $u_{\sigma_i} > u_{\sigma_j}$ for all $i < j$.

A common extension operator is given as follows.

$$\text{Min} \sum_{\substack{u_{\sigma_i} \in U \\ x_{i,j} \in V}} \mathcal{O}\{d(u_{\sigma_i}, x_{i,j}), y_{i,j} \cdot w_i\}$$

$$\text{s.t.} \begin{cases} y_{i,j} = 0, 1 \\ \sum_{\substack{u_i \in U \\ x_{i,j} \in V}} y_{i,j} = 1 \\ \sum_{i=1}^n w_i = 1 \end{cases}$$

where $W = (w_1, \dots, w_n)^T$ is an associated weighting vector and $\mathcal{O}\{\cdot\}$ is an aggregation operator.

Since our models reach the result that has the minimal distances with $\{a_{1j}, a_{2j}, \dots, a_{nj}\}$, the proximity degree of the aggregation result must be better than the one calculated by any other aggregation operators. We use proximity measures [37] to evaluate the distance

between the experts' individual opinions and the group or collective opinion. Proximity degree is a number between 0 and 1; the closer it is to 1, the better the aggregation result is.

Definition 12. Proximity degree function is defined by

$$\delta_j = 1 - \frac{\sum_{i=1}^n d(x_{ij}, \mathcal{O}_j) \cdot w_i}{\text{Max}_{v_x \in V} (d(v_x, \mathcal{O}_j))} \tag{10}$$

In function (10), we find the v_x in $G_{S_n}^*$ that has the longest distance from \mathcal{O}_j . The effect of $\text{Max}_{v_x \in V} (d(v_x, \mathcal{O}_j))$ in function (10) is to normalize the proximity degree for being between 0 and 1.

Our aggregation approach not only solves the problems of aggregating decision makers assessments, but also gets the result with good proximity degree. So we think it is especially suitable for FGDM.

4.3. The Process of FGDM with HFLTSS

Based on the proposed algorithm of finding a substitute expression of HFLTSS, the proposed FGDM approach is presented as follows:

- **Step 1.** Transform comparative linguistic assessment a_{ij} into HFLTSS. Use the transformation function E_{GH} [10] to convert comparative linguistic expressions into HFLTSS. The HFLTSS is presented as $env(H_s) = [H_s^-, H_s^+]$.
- **Step 2.** Construct a graph for decision makers.

For all decision makers, all assessments are the vertices in graph $G_{S_n}^*$ which is described as a weighted graph $G_{S_n}^* = (V, E, \mathcal{O})$, where $S_n^* = \{B_1, \dots, B_n\}$ is the basic label set.

$$\text{Min} \sum_{\substack{a_{ij} \in A_j \\ x_{h,l} \in V(G_{S_n}^*)}} d(a_{ij}, x_{h,l}) \cdot y_{h,l} \cdot w_i$$

$$\text{s.t.} \begin{cases} y_{h,l} = 0 \text{ or } 1 \\ \sum_{x_{h,l} \in V(G_{S_n}^*)} y_{h,l} = 1 \\ \sum_{i=1}^n W_i = 1 \end{cases}$$

- **Step 3.** Aggregate assessments of decision makers.

Given $j = 1$, the assessment set according to alternative x_j is $A_j = \{a_{1j}, a_{2j}, \dots, a_{nj}\}$. We use the following model to obtain \mathcal{O}_j .

- \mathcal{O}_j is the global opinion of alternative x_j

$$j = j + 1.$$

Repeat Steps 2 and 3 until $j = m$.

We obtain $\mathcal{O} = \{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m\}$ where \mathcal{O}_j is the global opinion of alternative x_j .

• **Step 4.** Exploitation phase.

Establish a rank ordering among $\{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m\}$ using the comparison operators. Firstly, rank the opinions with the binary relation \succeq_1 . Calculate $d(\mathcal{O}_j, s_g)$ and rank the set $\{d(\mathcal{O}_1, s_g), \dots, d(\mathcal{O}_m, s_g)\}$. If the first superior can give a clear ranking order, the exploitation phase is over. If not, rank the opinions with the binary relation \succeq_2 . Repeat the process until a clear ranking order is obtained.

4.4. Discussion and Comparative Analyses

Now we highlight some advantages of our approach with respect to others. We focus on two aspects:

1. Accuracy of the distance measure

In our approach, the distance measure is based on the Manhattan distance. Although most distances or similar measures are based on the Hamming distance and Euclidean distance, we think these kinds of distance measures cannot reflect the diversity of the linguistic intervals. The linguistic intervals cover different numbers of granule in S . Distance function in definition 5 adds linguistic terms in H_S^1 to ensure $card(H_S^1) = card(H_S^2) = L$. This approach can solve the problem where H_S^1 has less granules than H_S^2 . But the granules in H_S^1 , which are added in, are not original ones in S_n^* . The granules in a same linguistic term set have the same semantics meaning. So, the granules in H_S^1 has different semantics meanings from the granules in H_S^2 . Directly computing on two granules with different semantics is not suitable. We call this problem information distortion. Applying Hamming distance to measure information granules with different coverage indexes needs further studies.

2. Complexity of the aggregation model

It is universally acknowledged that the aggregation result of a set of hesitant fuzzy information should have the minimums difference with the set of hesitant fuzzy information. Different aggregation models realize this aim from different aspects. Rodriguez, Martinez [9] defined two aggregation operators, min_upper and max_lower , which carry out the aggregation by using HFLTS. These aggregators do not require that the cardinality of H_S^i is equal. Liao, Xu [8] proposed a family of aggregators based on distance measures (Definition 5). The possible information distortion problems are proposed in the first aspect. In our aggregation phase, we find a linguistic express which has the minimal distances when all assessments are the global assessments. Through adjusting k of Function (9), we can recognize nuances among alternatives in FGDM.

5. ILLUSTRATIVE EXAMPLES

We use two examples to illustrate the process of our aggregation approach.

Example 1¹. Consider two alternatives $X = \{x_1, x_2\}$, a group decision makers $E = \{e_1, e_2, e_3\}$, weights $\{w_1, w_2, w_3\}$, and the assessment matrix $(a_{ij})_{3 \times 2}$, as given in Table 1.

Table 1 | Decision makers assessments of example 1.

	x_1	x_2	w_i
e_1	l_4	l_3	1/3
e_2	$[l_3, l_4]$	$[l_3, l_4]$	1/3
e_3	$[l_2, l_4]$	$[l_3, l_5]$	1/3

The graph $G_{S_n^*}$ that describes all assessments is constructed, where $S_n^* = \{l_1, l_2, \dots, l_5\}$, and $l_1 = \text{VERY BAD}$, $l_2 = \text{BAD}$, $l_3 = \text{NORMAL}$, $l_4 = \text{GOOD}$, $l_5 = \text{VERY GOOD}$.

We assume that the global opinion $\mathcal{O}_i = [l_h, l_k]$ satisfies

$$\min \sum_{i=1}^3 d(x_{i1}, \mathcal{O}_1) w_i$$

$$\min \sum_{i=1}^3 d(x_{i2}, \mathcal{O}_2) w_i$$

When $k = 0$, the weight of any edge is

$$w(x_{i,j} \sim x_{r,s}) = 1 + k | \ln(t + 2) - \ln(t + 1) | = 1.$$

Using Lingo software, we obtain $\mathcal{O}_1 = \mathcal{O}_2 = [l_3, l_4]$.

The global opinions of the two alternatives are the same.

Since $\mathcal{O}_1 = \mathcal{O}_2$, we can get alternatives order $x_1 \sim x_2$.

Then we calculate the group consensus degree

$$\delta_1 = 1 - \frac{\sum_{i=1}^n d(x_{ij}, \mathcal{O}_j) \cdot w_i}{Max_{v_x \in V} (d(v_x, \mathcal{O}_j))} = 1 - \frac{1 + 0 + 1}{5 \cdot 3} = 0.867$$

$$\delta_2 = 0.867$$

v_x is l_1 in order to get $Max_{v_x \in V} (d(v_x, \mathcal{O}_j))$.

Since $\delta_1 = \delta_2$, the proximity degrees of the two aggregation results are the same.

Example 2. Consider two alternatives $X = \{x_1, x_2\}$, a group decision makers $E = \{e_1, e_2, e_3\}$, weights $\{w_1, w_2, w_3\}$, and the assessment matrix $(a_{ij})_{3 \times 2}$, as given in Table 2.

When $k = 1$, the weight of any edge is

$$w(x_{i,j} \sim x_{r,s}) = 1 + | \ln(t + 2) - \ln(t + 1) |$$

$x_{i,j} \sim x_{r,s}$ satisfies $(r = i \text{ and } s = j + 1)$ or $(s = j \text{ and } r = i + 1)$

¹This example is a revised version of Example 3 in [26].

Table 2 | Decision makers assessments example 2.

	x_1	x_2	w_i
e_1	l_2	l_3	1/3
e_2	$[l_3, l_4]$	$[l_4, l_5]$	1/3
e_3	$[l_2, l_5]$	$[l_1, l_4]$	1/3

The edge connecting the basic S_n^* and level Λ_1 is

$$w(l_i \sim [l_i, l_{i+1}]) = 1 + |I(l_i) - I([l_{i-1}, l_i])| \\ = 1 + |\lg(1/5) - \lg(2/5)| \\ = 1.30$$

The edge connecting the level Λ_1 and Λ_2 is

$$w([l_i, l_{i+1}] \sim [l_i, l_{i+2}]) = 1 + |I([l_i, l_{i+1}]) - I([l_i, l_{i+2}])| \\ = 1 + |\lg(2/5) - \lg(3/5)| \\ = 1.18$$

The edge connecting the level Λ_2 and Λ_3 is

$$w([l_i, l_{i+2}] \sim [l_i, l_{i+3}]) = 1 + |I([l_i, l_{i+2}]) - I([l_i, l_{i+3}])| \\ = 1 + |\lg(3/5) - \lg(4/5)| \\ = 1.12$$

The edge connecting the level Λ_3 and Λ_4 is

$$w([l_i, l_{i+3}] \sim [l_i, l_{i+4}]) = 1 + |I([l_i, l_{i+3}]) - I([l_i, l_{i+4}])| \\ = 1 + |\lg(4/5) - \lg(5/5)| \\ = 1.10$$

Construct model to find the substitution \mathcal{O}_j .

$$\text{Min} \sum_{\substack{a_{ij} \in A_j \\ x_{h,l} \in G_{S_n^*}}} d(a_{ij}, x_{h,l}) \cdot y_{h,l} \cdot w_k$$

$$\text{s.t.} \begin{cases} y_{h,l} = 0 \text{ or } 1 \\ \sum_{x_{h,l} \in V G_{S_n^*}} y_{h,l} = 1 \\ \sum_{k=1}^p W_k = 1 \end{cases}$$

\mathcal{O}_j is the global opinion of alternative x_j

$$\mathcal{O}_1 = [l_2, l_4]$$

$$\mathcal{O}_2 = [l_3, l_4]$$

Since $\mathcal{O}_1 <_1 \mathcal{O}_2$, we can get alternatives order $x_1 < x_2$.

Then we calculate the group consensus degree

$$\delta_1 = 1 - \frac{\sum_{i=1}^n d(x_{ij}, \mathcal{O}_j) \cdot w_i}{\text{Max}_{v_x \in V} (d(v_x, \mathcal{O}_j))} \\ = 1 - \frac{1.12 + 1.18 + 2.48}{4.72 \times 3} \\ = 0.662$$

v_x is l_1 or l_5 for achieving $\text{Max}_{v_x \in V} (d(v_x, \mathcal{O}_j))$.

$$\delta_2 = 1 - \frac{\sum_{i=1}^n d(x_{ij}, \mathcal{O}_j) \cdot w_i}{\text{Max}_{v_x \in V} (d(v_x, \mathcal{O}_j))} = 1 - \frac{2.3 + 2.36 + 1.3}{5.9 \times 3} = 0.663$$

v_x is l_1 for achieving $\text{Max}_{v_x \in V} (d(v_x, \mathcal{O}_j))$.

Since $\delta_1 < \delta_2$, the proximity degree of \mathcal{O}_2 is better than \mathcal{O}_1 .

Remarks: We apply $I(x_{i,j})$ to build the weight function of each edge in Example 2. If the two vertices are more different in coverage, then $|I(x_{i,j}) - I(x_{r,s})|$ is larger. This result is interesting to readers since weight of any edge in Example 1 ignores the difference of two vertices in coverage, which will lead to information loss. So the approach in Example 1 cannot tell the difference between alternatives. It gets the conclusion $x_1 \sim x_2$. While our approach maintains all information and gets the conclusion $x_1 < x_2$, one of the advantages of our approach to be highlighted is the accuracy of the distance measure. The information I for the vertex $x_{i,j}$ used to build the weight function of each edge is just for the accuracy of the distance measure.

Roselló, Sánchez [31] computed the distances from optimal labels F and solved group decision-making under multi-granular linguistic assessments. With their method, we obtain the following result of Example 2:

$$d(x_1, F) = d(l_2, l_5) \cdot w_1 + d([l_3, l_4], l_5) \cdot w_2 + d([l_2, l_5], l_5) \cdot w_3 \\ w_3 = \frac{7.2 + 3.66 + 3.6}{3} = 4.82$$

$$d(x_2, F) = d(l_3, l_5) \cdot w_1 + d([l_4, l_5], l_5) \cdot w_2 + d([l_1, l_4], l_5) \cdot w_3 \\ w_3 = \frac{4.96 + 1.3 + 5.8}{3} = 4.02$$

$$d(x_1, F) > d(x_2, F) \Rightarrow x_1 < x_2$$

This result is the same with ours.

Liao *et al.* [8] extended the shorter HFLTS to obtain the same length and compute the distances to rank alternatives in multi-criteria decision making. With their method, the solution for the problem in Example 2 is presented below:

Firstly, one can extend these preferences into the same length and get matrix

$$(a_{ij})_{3 \times 2} = \begin{pmatrix} [l_2, l_2, l_2, l_2] & [l_3, l_3, l_4, l_4] & [l_2, l_3, l_4, l_5] \\ [l_3, l_3, l_3, l_3] & [l_4, l_4, l_5, l_5] & [l_1, l_2, l_3, l_4] \end{pmatrix}$$

The hesitant fuzzy linguistic positive ideal solution is $m^+ = \{\{l_5\}, \{l_5\}\}$ and negative ideal solution is $m^- = \{\{l_0\}, \{l_0\}\}$. The satisfaction degree $\eta(x_i)$ for each alternatives can be calculated using Equation (11).

$$\eta(x_i) = \frac{(1 - \theta) d(x_i, m^-)}{\theta d(x_i, m^+) + (1 - \theta) d(x_i, m^-)} \quad (11)$$

where $\theta = 0.5$ and $d(x_i, m)$ is Hamming distance between x_1 and m , $\text{card}(x_i) = \text{card}(m) = L$, $\tau = \text{card}(\mathbb{S}_n^*)$.

$$d(x_i, m) = \frac{1}{L} \sum_{l=1}^L \frac{|\delta_l^1 - \delta_l^2|}{2\tau + 1} \quad (12)$$

Finally, we get

$$\eta(x_1) = 0.6$$

$$\eta(x_2) = 0.667$$

$$\eta(x_1) < \eta(x_2) \Rightarrow x_1 < x_2$$

This result is the same with ours.

Hence, consistent ranking results are reached between our approach and that of Roselló *et al.* and Liao *et al.* for the problem in Example 2. However, the advantages of our approach with respect to others are highlighted in Section 4.4.

6. CONCLUSION

This newly developed distance measure can complement the existing computational models and is particularly suitable for solving FGDM problems in a context-free grammar. HFLTSS have diversity not only in center points but also in the coverage (degree of uncertainty). The most difficult problem when dealing with HFLTSS is aggregation because the coverage indices of HFLTSS are different. A proper operator is the key. We propose a new approach to get the aggregation result. The idea is the result should have the minimal distances with the HFLTSS. This idea is reasonable and consistent with the definition of proximity measure in GDM. This distance measure applied into FGDM is very suitable. And our numerical examples illustrate the process and effects of our method.

However, our distance measure assumes edges connecting the same two levels are of equal length. In fact, this condition is based on the assumption that the basic linguistic term set \mathbb{S}_n^* is balanced linguistic term set where terms are uniformly and symmetrically distributed. So, future research will include the use of the proposed methodology to deal with unbalanced linguistic term sets and explore its application to FGDM problems with unbalanced linguistic term sets. Moreover, the grammar G_H , which is used to generate comparative linguistic expressions, also has some potential improvements. More freedom can be given to generate linguistic expressions, which are closer to human language. Future research will also include the use of the proposed distance measure to treat other kind decision models, such as TOPSIS method.

CONFLICT OF INTEREST

This section is to certify that we have no potential conflict of interest.

This article does not contain any studies with human participants or animals performed by any of the authors.

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