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A New Zero-Truncated Version of the Poisson Burr XII Distribution: Characterizations and Properties

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ABSTRACT

In this work, a new four-parameter zero-truncated Poisson Topp Leone Burr XII distribution is defined and studied. Various structural mathematical properties of the proposed model including ordinary and incomplete moments, residual and reversed residual life functions, generating functions, order statistics are investigated. Some useful characterizations are also presented.

1. INTRODUCTION

The Pearson system of frequency curves was introduced by Pearson [1] who worked out a set of four-parameter probability density functions (PDFs) as solutions to the following differential equation

$$f'(x)\big/ f(x) = P(x)/Q(x) = (x-a)\left(b_0 + b_1 x + b_2 x^2\right)^{-1}$$

where f is a density function and a, b_0, b_1, and b_2 Pearson family such as Gamma, Gaussian, Beta, and Student's t models. Analogously to the Pearson system, Burr [2] introduced another system of frequency curves that includes 12 types of cumulative distribution functions (CDFs) which yield a variety of density shapes, this system is obtained by considering CDFs satisfying a differential equation which has a solution, given by

$$G(x) = \left\{1 + \exp\left[-\int \tau(x)\,dx\right]\right\}^{-1},$$

where $\tau(x)$ is chosen such that $G(x)$ is a CDF on the real line and has 12 choices which made by Burr, resulted in 12 models which might be useful for modeling data, the principal aim in choosing one of these forms of distributions is to facilitate the mathematical analysis to which it is subjected, while attaining a reasonable approximation. A special attention has been devoted to one of these forms denoted by type XII (for more details see Burr [2–4], Burr and Cislak [5], Hatke [6], and Rodriguez [7]), whose CDF, $G(x)$, is given as

$$G_{\alpha,\beta}(x) = 1 - (1 + x^{\alpha})^{-\beta},$$

where both α and β are shape parameters. The location and scale parameters can easily be introduced to make $G_{\alpha,\beta}(x)$ a four-parameter distribution. The corresponding PDF is given by

$$g_{\alpha,\beta}(x) = \alpha\beta x^{\alpha-1}(1 + x^{\alpha})^{-\beta-1}.$$

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The Burr XII (BXII) (see Burr [2]) has many applications in different areas including reliability, acceptance sampling plans, and failure time modeling. Tadikamalla [8] studied the BXII model and its related models. Zimmer *et al.* [9] proposed a new three-parameter BXII distribution, this distribution, having the Weibull and the logistic as submodels, is a very popular distribution for modeling lifetime data and phenomenon with monotone failure rates. Shao [10] studied the maximum likelihood estimations for the three-parameter BXII model then Soliman [11] studied the estimation of parameters of life from progressively via censored data using Burr-XII model, Wu *et al.* [12] discussed the estimation problems for BXII model on the basis of progressive type **II** censoring under random removals where the number of units removed at each failure time has a discrete uniform model. Recently, Silva *et al.* [13] introduced the log-BXII regression models with censored data, Silva *et al.* [13] proposed a new location-scale regression model based on BXII model, Silva *et al.* [14] proposed a residual for the log-BXII regression distribution whose empirical model is close to normality, Afify *et al.* [15] studied the Weibull BXII distribution, Cordeiro *et al.* [16] proposed the double BXII model among others. For the other new extensions of the BXII see Altun *et al.* [17], Altun *et al.* [18], Paranaíba *et al.* [19], Yousof *et al.* [20], and Yousof *et al.* [21].

The rest of the paper is outlined as follows. In Section 2, we introduce the new model and its physical motivation. Section 3 presents some plots and the justification for introducing the new model. Some useful characterizations are presented in Section 4. In Section 5, we derive some statistical properties for the new model. Finally, we offer some concluding remarks in Section 6.

2. THE NEW MODEL AND ITS PHYSICAL MOTIVATION

The CDF and the PDF of the Topp Leone BXII (TLBXII) distributions (Reyad and Othman [22]) are given by

$$H_{b,\alpha,\beta}(x) = \left[1-(1+x^{\alpha})^{-2\beta}\right]^{b} \tag{1}$$

and

$$h_{b,\alpha,\beta}(x) = 2b\alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-2\beta-1}\left[1-(1+x^{\alpha})^{-2\beta}\right]^{b-1} \tag{2}$$

respectively. Suppose $Z_1, Z_2, ..., Z_N$ be independent identically random variable (iid rv) with common CDF of the TLBXII model and N be a rv with probability mass function (PMF)

$$P(N=n) = a^{n}/\left[(e^{a}-1)\,n!\right]|_{(n=1,2,...,\ a>0)}$$

and define

$$M_N = \max\{Z_1, Z_2, ..., Z_N\}$$

then

$$F(x) = \sum_{n=0}^{\infty} p\left(M_N \le x | N=n\right) p(N=n) \tag{3}$$

As described in Ramos *et al.* [23], Eq. (3) can be expressed as

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{e^{a}-1}\frac{a^{n}}{n!}\left(\left(1-(1+x^{\alpha})^{-2\beta}\right)^{b}\right)^{n} \tag{4}$$

Using Eqs. (2) and (4), we can write

$$F(x) = \left(1-\exp\left\{-a\left[1-(1+x^{\alpha})^{-2\beta}\right]^{b}\right\}\right)\Big/(1-e^{-a}). \tag{5}$$

Equation (5) is the CDF of the zero-truncated Poisson Topp Leone BXII (ZTPTLBXII) model. Henceforward $f(x) = f_{a,b,\alpha,\beta}(x)$ and $F(x) = F_{a,b,\alpha,\beta}(x)$. The corresponding PDF of Eq. (5) reduces to

$$f(x) = \frac{2ab\alpha\beta}{(1-e^{-a})}x^{\alpha-1}(1+x^{\alpha})^{-2\beta-1}\left[1-(1+x^{\alpha})^{-2\beta}\right]^{b-1}\underbrace{\exp\left\{-a\left[1-(1+x^{\alpha})^{-2\beta}\right]^{b}\right\}}_{A}. \tag{6}$$

Now we can provide a useful linear representation for the ZTPTLBXII density function in Eq. (6). Expanding the quantity A in power series, we can write

$$f(x) = \sum_{i=0}^{\infty} \frac{2(-1)^{i}a^{i+1}b\alpha\beta}{i!(1-e^{-a})}x^{\alpha-1}(1+x^{\alpha})^{-2\beta-1}\left[1-(1+x^{\alpha})^{-2\beta}\right]^{b(i+1)-1}. \tag{7}$$

Consider the power series

$$\left(1-\zeta\right)^{\tau-1}=\sum_{r=0}^{\infty}(-1)^r\,\Gamma\left(\tau\right)\zeta^{\,r}/\left[j!\,\Gamma\left(\tau-j\right)\right]=\sum_{r=0}^{\infty}(-1)^r\binom{\tau-1}{r}\zeta^{\,r}, \tag{8}$$

which holds for $|\zeta|<1$ and $\tau>0$ real non-integer. Using the power series in Eq. (8) and after some algebra the PDF of the ZTPTLBXII model in Eq. (7) can be expressed as

$$\begin{aligned}
f(x) &= \sum_{i=0}^{\infty}\frac{2(-1)^i a^{i+1}b\alpha\beta}{i!\,(1-e^{-a})}x^{\alpha-1}\left(1+x^{\alpha}\right)^{-2\beta-1}\left[1-\left(1+x^{\alpha}\right)^{-2\beta}\right]^{b(i+1)-1}\\
&= \sum_{i,r=0}^{\infty}\frac{2(-1)^{i+r}a^{i+1}b}{i!\,(1-e^{-a})}\alpha\beta x^{\alpha-1}\left(1+x^{\alpha}\right)^{-2\beta-1}\left(1+x^{\alpha}\right)^{-2r\beta}\binom{b\,(i+1)-1}{r}\\
&= \sum_{i,r=0}^{\infty}\frac{2(-1)^{i+r}a^{i+1}b}{i!\,(1-e^{-a})}\binom{b\,(i+1)-1}{r}\alpha\beta x^{\alpha-1}\left(1+x^{\alpha}\right)^{-2\beta(1+r)-1}\\
&= \sum_{i,r=0}^{\infty}\frac{2(-1)^{i+r}a^{i+1}b}{i!\,(1-e^{-a})\left[2\,(1+r)\right]}\binom{b\,(i+1)-1}{r}\alpha\left[2\,(1+r)\,\beta\right]x^{\alpha-1}\left(1+x^{\alpha}\right)^{-2\beta(1+r)-1}\\
&= \sum_{r=0}^{\infty}\underbrace{\sum_{i=0}^{\infty}\frac{(-1)^{i+r}a^{i+1}b}{i!\,(1-e^{-a})\,(1+r)}\binom{b\,(i+1)-1}{r}}_{v_r}\underbrace{\alpha\left[2\,(1+r)\,\beta\right]x^{\alpha-1}\left(1+x^{\alpha}\right)^{-2(1+r)\beta-1}}_{g_{\alpha,2(1+r)\beta}(x)}\\
&= \sum_{r=0}^{\infty}v_r g_{\alpha,2(1+r)\beta}\left(x\right),
\end{aligned}$$

where

$$g_{\alpha,2(1+r)\beta}\left(x\right)=2\alpha\left(1+r\right)\beta x^{\alpha-1}\left(1+x^{\alpha}\right)^{-2(1+r)\beta-1},$$

is the BXII density with parameters α and $2\,(1+r)\,\beta$.

$$v_r=\sum_{i=0}^{\infty}\frac{(-1)^{i+r}a^{i+1}b}{i!\,(1-e^{-a})\,(1+r)}\binom{b\,(i+1)-1}{r}$$

Via integrating Eq. (9), we obtain the same mixture representation

$$F\left(x\right)=\sum_{r=0}^{\infty}v_r G_{\alpha,\,2(1+r)\beta}\left(x\right), \tag{9}$$

where $G_{\alpha,\,2(1+r)\beta}\left(x\right)$ is the CDF of the BXII density with parameters α and $2\,(1+r)\,\beta$. The new model has a wide application in many types of data such as Guinea pigs (Bjerkedal [24]) and many other data types.

3. PLOTS AND JUSTIFICATION

In this section, we provide some graphical plots of the PDF and hazard rate function (HRF) of the ZTPTLBXII model to show its flexibility. Figure 1(a) displays some plots of the ZTPTLBXII density for some parameter values a, b, α, and β. Plots of the HRF of the ZTPTLBXII model for selected parameter values are given in Fig. 1(b), where the HRF can be upside down bathtub (unimodal) and decreasing.

The justification for introducing the ZTPTLBXII lifetime model is based on the wider use of the BXII model. As well as we are motivated to introduce the ZTPTLBXII lifetime model because it exhibits the unimodal hazard rate as illustrated in Fig. 1(b). It is shown above that the ZTPTLBXII lifetime model can be viewed as a linear mixture of the BXII densities as illustrated in Eqs. (9) and (10).

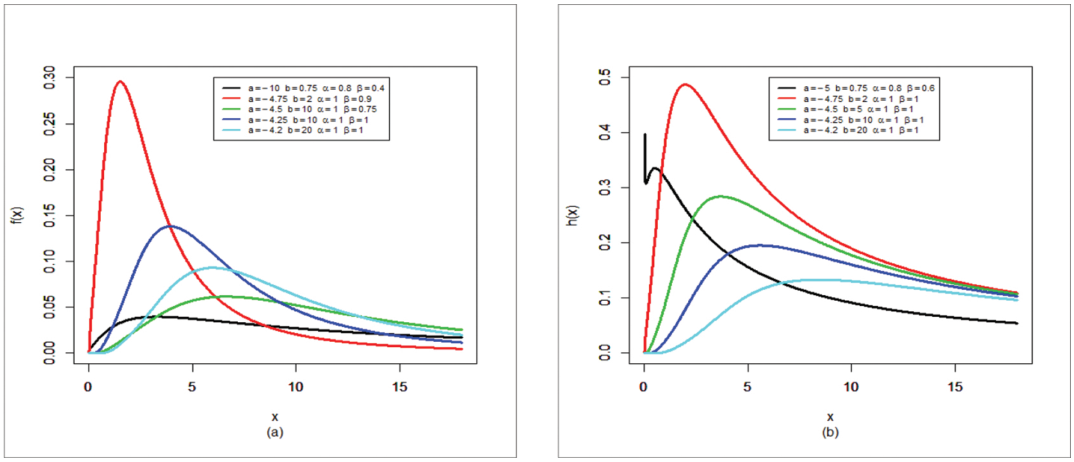

Figure 1 Plots of the zero-truncated Poisson Topp Leone BXII (ZTPTLBXII PDF) (right panel) and HRF (left panel).

4. CHARACTERIZATIONS

We will need the following two Lemmas for the characterization of the distribution:

Assumption A. X is an absolutely continuous rv with CDF $F(x)$ and PDF $f(x)$. We assume $E(X)$ exists and $f(x)$ is differentiable. We assume further

$$\alpha = \sup\{x \mid f(x) > 0\} \text{ and } \beta\{x \mid f(x) < 1\}.$$

Lemma 1. If

$$E\left(X \mid X \leq x\right) = g(x) f(x) / F(x),$$

where $g(x)$ is a continuous differentiable function in (α, β), then $f(x) = c \exp\left[\int \frac{x - g'(x)}{g(x)} dx\right]$, c is determined by the condition $\int_{\alpha}^{\beta} f(x)\, dx = 1$.

Proof.

$$g(x) = \frac{\int_{\alpha}^{x} u f(u)\, du}{f(x)}, \text{ thus } \int_{\alpha}^{x} u f(u)\, du = f(x)\, g(x)$$

Differentiating both sides of the above equation, we obtain $x f(x) = f'(x)\, g(x) + f(x)\, g'(x)$ on simplification, we get

$$\left[f'(x) / f(x)\right] = \left[x - g'(x)\right] / g(x).$$

On integrating both sides of the above equation, we obtain

$$f(x) = c \exp\left[\int \frac{x - g'(x)}{g(x)} dx\right],$$

where c is determined by the condition

$$\int_{\alpha}^{\beta} f(x)\, dx = 1.$$

Lemma 2. Under the assumption A, if

$$E\left(X\,|\,X \geq x\right) = h\left(x\right) f(x) \big/ \left[1 - F\left(x\right)\right],$$

where $h\left(x\right)$ is a continuous differentiable function in (α, β) , then

$$f(x) = c\exp\left[\int = \frac{x + h'\left(x\right)}{h\left(x\right)} dx\right],$$

where c is determined by the condition $\int_{\alpha}^{\beta} f(x)\,dx = 1.$

Proof.

$$h\left(x\right) = \frac{\int_{x}^{\infty} uf(u)\,du}{f(x)}, \text{ thus } \int_{x}^{\infty} uf(u)\,du = f(x)\,h\left(x\right)$$

Differentiating both sides of the above equation, we obtain $-xf(x) = f'\left(x\right) h\left(x\right) + f(x)\,h'\left(x\right)$ on simplification, we obtain

$$\left[f'\left(x\right) \big/ f(x)\right] = -\left[x + h'\left(x\right)\right] \big/ h\left(x\right).$$

On integrating both sides of the above equation, we obtain

$$f(x) = c\exp\left[\int -\frac{x + h'\left(x\right)}{h\left(x\right)} dx\right],$$

where c is determined by the condition

$$\int_{\alpha}^{\beta} f(x)\,dx = 1.$$

Theorem 1. Suppose that the random variable X satisfies the conditions of the assumption A with $\alpha = 0$ and $\beta = \infty$. Then, $E\left(X|X \leq x\right) = g\left(x\right) \tau\left(x\right)$, where $\tau\left(x\right) = \frac{f(x)}{F\left(x\right)}$ and

$$g\left(x\right) = \frac{\sum_{r=0}^{\infty} \nu_r \left[2\left(1 + r\right)\beta\right] B\left(x^{\alpha}, 1 + \frac{1}{\alpha}, 2\left(1 + r\right)\beta - \frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty} \nu_r g_{\alpha, 2(1+r)\beta}\left(x\right)},$$

if and only if

$$f(x) = \sum_{r=0}^{\infty} \nu_r g_{\alpha, 2(1+r)\beta}\left(x\right).$$

Proof.
Suppose that

$$f(x) = \sum_{r=0}^{\infty} \nu_r g_{\alpha, 2(1+r)\beta}\left(x\right),$$

then

$$\begin{aligned} f(x)\,g\left(x\right) &= \sum_{r=0}^{\infty} \nu_r \int_{0}^{x} u g_{\alpha, 2(1+r)\beta}(u) du \\ &= \sum_{r=0}^{\infty} \nu_r \int_{0}^{x} 2\alpha\left(1 + r\right)\beta u^{\alpha}\left(1 + u^{\alpha}\right)^{-2(1+r)\beta - 1} du \\ &= \sum_{r=0}^{\infty} \nu_r \int_{0}^{x^{\alpha}} 2\left(1 + r\right)\beta\left(1 + t\right)^{\beta 2(1+r)\beta - 1} t^{\frac{1}{\alpha}} dt \\ &= \sum_{r=0}^{\infty} \nu_r \left[2\left(1 + r\right)\beta\right] B\left(1 + \frac{1}{\alpha}, 2\left(1 + r\right)\beta - \frac{1}{\alpha}\right). \end{aligned}$$

Thus

$$g(x)=\frac{\sum_{r=0}^{\infty}\nu_r\left[2(1+r)\beta\right]B\left(x^{\alpha},1+\frac{1}{\alpha},2(1+r)\beta-\frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x)}.$$

Suppose

$$g(x)=\frac{\sum_{r=0}^{\infty}\nu_r 2(1+r)\beta B\left(x^{\alpha},1+\frac{1}{\alpha},2(1+r)\beta-\frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x)},$$

then

$$\begin{aligned} g'(x) &= x-\frac{\sum_{r=0}^{\infty}\nu_r\left[2(1+r)\beta\right]B\left(x^{\alpha},1+\frac{1}{\alpha},2(1+r)\beta-\frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x)} \\ &\quad\times\frac{\sum_{r=0}^{\infty}\nu_r 2\alpha(1+r)\beta\frac{x^{\alpha-2}\left\{\alpha-1-\left[2\alpha\beta(1+r)+1\right]x^{\alpha}\right\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x)} \\ &= x-g(x)\frac{\sum_{r=0}^{\infty}\nu_r\alpha 2(1+r)\beta\frac{x^{\alpha-2}\left\{\alpha-1-\left[2\alpha\beta(1+r)+1\right]x^{\alpha}\right\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x)} \end{aligned}$$

Thus

$$\frac{x-g'(x)}{g(x)}=\frac{\sum_{r=0}^{\infty}\nu_r 2\alpha(1+r)\beta\frac{x^{\alpha-2}\{\alpha-1-[2\alpha\beta(1+r)+1]x^{\alpha}\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x)}.$$

By Lemma 1, we have

$$\frac{f'(x)}{f(x)}=\frac{\sum_{r=0}^{\infty}\nu_r\alpha(1+r)\beta\frac{x^{\alpha-2}\{\alpha-1-[2\alpha\beta(1+r)+1]x^{\alpha}\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x)}.$$

Integrating both sides of the above equation we obtain

$$f(x)=c\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x),$$

where c is constant. Using the condition $\int_0^{\infty}f(x)\,dx=1$, we obtain

$$f(x)=\sum_{r=0}^{\infty}\nu_r g_{\alpha,2(1+r)\beta}(x).$$

Theorem 2. Suppose that the random variable X satisfies the conditions of the assumption A with $\alpha = 0$ and $\beta = \infty$. Then, $E\left(X \mid X \geq x\right) = h(x)\, r(x)$, where $r(x) = \dfrac{f(x)}{F(x)}$ and

$$g(x) = \frac{\sum_{r=0}^{\infty} \nu_r [2(1+r)\beta] B_c\left(x^{\alpha}, 1+\frac{1}{\alpha}, \beta - \frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty} \nu_r g_{\alpha, 2(1+r)\beta}(x)},$$

if and only if

$$f(x) = \sum_{r=0}^{\infty} \nu_r g_{\alpha,(1+r)\beta}(x),$$

where

$$B(a_1, a_2) = \int_0^{\infty} t^{a_1 - 1} (1+t)^{-(a_1 + a_2)}\, dt$$

and

$$B(z, a_1, a_2) = \int_0^{z} t^{a_1 - 1} (1+t)^{-(a_1 + a_2)}\, dt$$

are the beta and the incomplete beta functions of the second type, respectively.

Proof.

Suppose that $f(x) = \sum_{r=0}^{\infty} \nu_r g_{\alpha,2(1+r)\beta}(x)$, then

$$\begin{aligned} f(x)\, g(x) &= \sum_{r=0}^{\infty} \nu_r \int_x^{\infty} u g_{\alpha,2(1+r)\beta}(u)\, du \\ &= \sum_{r=0}^{\infty} \nu_r \int_x^{\infty} 2\alpha(1+r)\beta u^{\alpha} (1+u^{\alpha})^{-2(1+r)\beta - 1}\, du \\ &= \sum_{r=0}^{\infty} \nu_r \int_x^{\infty} 2(1+r)\beta (1+t)^{-2(1+r)\beta - 1}\, t^{\frac{1}{\alpha}}\, dt \\ &= \sum_{r=0}^{\infty} \nu_r [2(1+r)\beta] B_c\left(1 + \frac{1}{\alpha}, 2(1+r)\beta - \frac{1}{\alpha}\right). \end{aligned}$$

Thus

$$g(x) = \frac{\sum_{r=0}^{\infty} \nu_r [2(1+r)\beta]\, B_c\left(1+\frac{1}{\alpha}, 2(1+r)\beta - \frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty} \nu_r g_{\alpha,\, 2(1+r)\beta}(x)}.$$

Suppose

$$g(x) = \frac{\sum_{r=0}^{\infty} \nu_r [2(1+r)\beta]\, B_c\left(1+\frac{1}{\alpha}, 2(1+r)\beta - \frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty} \nu_r g_{\alpha,\, 2(1+r)\beta}(x)},$$

then

$$g'(x) = -x - \frac{\sum_{r=0}^{\infty} \nu_r [2(1+r)\beta] B\left(x^{\alpha}, 1+\frac{1}{\alpha}, 2(1+r)\beta - \frac{1}{\alpha}\right)}{\sum_{r=0}^{\infty} \nu_r g_{\alpha,\,2(1+r)\beta}(x)}$$

$$\times \frac{\sum_{r=0}^{\infty} \nu_r 2\alpha (1+r)\beta \frac{x^{\alpha-2}\{\alpha-1-[2\alpha\beta(1+r)+1]x^{\alpha}\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty} \nu_r g_{\alpha,2(1+r)\beta}(x)}$$

$$= -x - g(x) \frac{\sum_{r=0}^{\infty} \nu_r \alpha 2 (1+r)\beta \frac{x^{\alpha-2}\{\alpha-1-[2\alpha\beta(1+r)+1]x^{\alpha}\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty} \nu_r g_{\alpha,2(1+r)\beta}(x)}$$

Thus

$$\frac{x - g'(x)}{g(x)} = \frac{\sum_{r=0}^{\infty} \nu_r 2\alpha (1+r)\beta \frac{x^{\alpha-2}\{\alpha-1-[2\alpha\beta(1+r)+1]x^{\alpha}\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty} \nu_r g_{\alpha,(1+r)\beta}(x)}.$$

By Lemma 2, we have

$$\frac{f'(x)}{f(x)} = \frac{\sum_{r=0}^{\infty} \nu_r \alpha (1+r)\beta \frac{x^{\alpha-2}\{\alpha-1-[2\alpha\beta(1+r)+1]x^{\alpha}\}}{(1+x^{\alpha})^{2(1+r)\beta+2}}}{\sum_{r=0}^{\infty} \nu_r g_{\alpha,\,2(1+r)\beta}(x)}.$$

Integrating both sides of the above equation we obtain

$$f(x) = c \sum_{r=0}^{\infty} \nu_r g_{\alpha,\,2(1+r)\beta}(x),$$

where c is constant. Using the condition

$$\int_0^{\infty} f(x)\,dx = 1,$$

we obtain $f(x) = \sum_{r=0}^{\infty} \nu_r g_{\alpha,\,2(1+r)\beta}(x)$.

5. MATHEMATICAL PROPERTIES

5.1. Moments and Incomplete Moments

The $r^{(th)}$ ordinary moment of X is given by

$$\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\,dx.$$

Then we obtain

$$\mu_r' = \sum_{r=0}^{\infty} \nu_r \int_0^{\infty} x^n g_{\alpha,\,2(1+r)\beta}(x) = \sum_{r=0}^{\infty} \nu_r [2(1+r)\beta] B\left(1+\frac{n}{\alpha}, 2(1+r)\beta - \frac{n}{\alpha}\right). \tag{10}$$

By setting $n=1$ in Eq. (11), we have the mean of X. The $s^{(th)}$ incomplete moment, say $\mathbf{I}_s(t)$, of X can be expressed from (9) as $\mathbf{I}_s(t) = \int_{-\infty}^{t} x^s f(x)\, dx$. Then

$$\mathbf{I}_s(t) = \sum_{r=0}^{\infty} v_r \int_{-\infty}^{t} x^s g_{\alpha,\,2(1+r)\beta}(x)\, dx = \sum_{r=0}^{\infty} v_r \left[2(1+r)\beta\right] B\left(t^{\alpha}; 1+\frac{s}{\alpha}, 2(1+r)\beta-\frac{s}{\alpha}\right), \tag{11}$$

The general equation for the first incomplete, $I_1(t)$, can be derived from Eq. (12) as

$$\mathbf{I}_1(t) = \sum_{r=0}^{\infty} v_r \int_{-\infty}^{t} x g_{\alpha,\,2(1+r)\beta}(x)\, dx = \sum_{r=0}^{\infty} v_r \left[2(1+r)\beta\right] B\left(t^{\alpha}; 1+\frac{1}{\alpha}, 2(1+r)\beta-\frac{1}{\alpha}\right).$$

5.2. Moment Generating Function

The moment generating function (MGF) of X, say $M_X(t) = \mathbf{E}[\exp(tX)]$, can be obtained from Eq. (9) as $M_X(t) = \sum_{r=0}^{\infty} v_r M_{2(1+r)\beta}(t)$, where $M_{2(1+r)\beta}(t)$ is the mgf of the BXII distribution with parameters α, $2(1+r)\beta$, then we have

$$\begin{aligned} M_X(t) &= \sum_{r=0}^{\infty} v_r M_{r+1}(t) \\ &= 2\alpha\beta \sum_{r=0}^{\infty} (1+r) v_r \int_0^{\infty} \exp(tx) x^{\alpha-1} (1+x^{\alpha})^{-2(1+r)\beta-1} dx \\ &= 2\alpha\beta \sum_{r=0}^{\infty} (1+r) v_r \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^{\infty} x^{\alpha+k-1} (1+x^{\alpha})^{-2(1+r)\beta-1} dx. \end{aligned}$$

Let $u = x^{\alpha}$, then

$$\begin{aligned} M_X(t) &= 2\beta \sum_{r=0}^{\infty} (1+r) v_r \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^{\infty} u^{\frac{k}{\alpha}} (1+u)^{-2(1+r)\beta-1} du \\ &= 2\beta \sum_{r=0}^{\infty} (1+r) v_r \sum_{k=0}^{\infty} \frac{t^k}{k!} B\left(1+\frac{n}{\alpha}, 2(1+r)\beta-\frac{n}{\alpha}\right), \end{aligned}$$

which also means that the $r^{(th)}$ ordinary moment of X is

$$\mu_r' = 2\beta \sum_{r=0}^{\infty} v_r (1+r) B\left(1+\frac{n}{\alpha}, 2(1+r)\beta-\frac{n}{\alpha}\right).$$

5.3. Order Statistics

Let $X_1, \ldots, X_n$ be a random sample from the ZTPTLBXII model of distributions and let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. The PDF of $i^{(th)}$ order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \left[\mathrm{B}(i, n-i+1)\right]^{-1} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F^{j+i-1}(x), \tag{12}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting Eqs. (5) and (6) in Eq. (13) and using a power series expansion, we get that

$$f(x) F(x)^{j+i-1} = \sum_{h=0}^{\infty} v_h g_{\alpha,(1+h)\beta}(x),$$

where

$$\begin{aligned} v_h = & \sum_{w,m,k=0}^{\infty} \frac{2^{b(w+1)-m} b a^{w+1} (-1)^{w+m+k+h} (k+1)^w}{w!\, h!\, (1+h)(1-e^{-a})^{j+i}} \binom{j+i-1}{k} \binom{b(w+1)-1}{m} \\ & \times \left[\binom{1+\beta(w+1)+m}{h} - \binom{2+\beta(w+1)+m}{h}\right] \end{aligned}$$

The PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}\left[\mathrm{B}(i,n-i+1)\right]^{-1}\sum_{h=0}^{\infty}v_h g_{\alpha,(1+h)\beta}(x).$$

For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^{q}\right)=\sum_{j=0}^{n-i}\frac{(-1)^{j}\binom{n-i}{j}}{\mathrm{B}(i,n-i+1)}\sum_{h=0}^{\infty}v_h(1+h)\beta B\left(1+\frac{q}{\alpha},(1+h)\beta-\frac{q}{\alpha}\right). \tag{13}$$

5.4. Quantile Spread Ordering

The quantile spread (QS) of the rv $T\sim$ ZTPTLBXII (a,b,α,β) having CDF (5) is given by

$$QS_T(\nu)|_{[\nu\in(0.5,1)]}=\left[F^{-1}(\nu)\right]-\left[F^{-1}(1-\nu)\right],$$

which implies

$$QS_T(\nu)=\left[S^{-1}(1-\nu)\right]-\left[S^{-1}(\nu)\right],$$

where

$$F^{-1}(\nu)=S^{-1}(1-\nu)\text{ and }S(t)=1-F(t)$$

is the survival function. The QS of a distribution describes how the probability mass is placed symmetrically about its median and hence can be used to formalize concepts such as peakedness and tail weight traditionally associated with kurtosis. So, it allows us to separate concepts of kurtosis and peakedness for asymmetric models. Let T_1 and T_2 be two rvs following the ZTPTLBXII (a,b,α,β) model with QS_{T_1} and QS_{T_2}. Then T_1 is called smaller than T_2 in QS order, denoted as $T_1\leq_{(QS)}T_2$, if

$$QS_{T_1}(\nu)|_{[\nu\in(0.5,1)]}\leq QS_{T_2}(\nu).$$

Following properties of the QS order can be obtained:

- The order $\leq_{(QS)}$ is a location-free

$$T_1\leq_{(QS)}T_2\text{ if }(T_1+\mathbf{C})\leq_{(QS)}T_2|_{(C\in R)}.$$

- The order $\leq_{(QS)}$ is dilative

$$T_1\leq_{(QS)}\mathbf{C}T_1\text{ whenever }\mathbf{C}\geq 1\text{ and }T_2\leq_{(QS)}\mathbf{C}T_2|_{(C\geq 1)}.$$

- Let F_{T_1} and F_{T_2} be symmetric, then

$$T_1\leq_{(QS)}T_2\text{ if, and only if }F_{T_1}^{-1}(\tau)\leq F_{T_2}^{-1}(\nu)|_{[\nu\in(0.5,1)]}.$$

- The order $\leq_{(QS)}$ implies ordering of the mean absolute deviation around the median, say $\gamma_{(Median)}(T_i)|_{(i=1,2)}$,

$$\gamma_{(Median)}(T_1)=E\left[|T_1-Median(T_1)|\right]$$

and

$$\gamma_{(Median)}(T_2)=E\left[|T_2-Median(T_2)|\right],$$

where

$$T_1\leq_{(QS)}T_2\text{ implies }\gamma_{(Median)}(T_1)\leq_{(QS)}\gamma_{(Median)}(T_2).$$

Finally

$$T_1\leq_{(QS)}T_2\text{ if, and only if }-T_1\leq_{(QS)}-T_2.$$

6. CONCLUSIONS

In this paper, a new four-parameter ZTPTLBXII distribution is defined and studied. The new model has a strong physical motivation. Various structural mathematical properties of the proposed model including ordinary and incomplete moments, residual and reversed residual life functions, generating functions, order statistics are investigated also the QS ordering is defined for formalizing concepts such as peakedness and tail weight traditionally associated with kurtosis on the new model. Some useful characterizations are also presented.

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