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A Novel Method Based on Extended Uncertain 2-tuple Linguistic Muirhead Mean Operators to MAGDM under Uncertain 2-Tuple Linguistic Environment

Yi Liu[1,2,*], Jun Liu[3], Ya Qin[2], Yang Xu[1]

[1] *National-Local Joint Engineering Laboratory of System Credibility Automatic Verification, Southwest Jiaotong University, No. 111 North Section 1 of the second ring road, Chengdu, Sichuan 610031, P.R. China*
[2] *Data Recovery Key Laboratory of Sichuan Province, Neijiang Normal University, No. 705 of Dongtong Road, Neijiang, Sichuan 641000, P.R. China*
[3] *School of Computing, Ulster University at Jordanstown Campus, Shore Road, Newtownabbey, Co. Antrim, Northern Ireland BT37 0QB, UK*

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ABSTRACT

The present work is focused on multi-attribute group decision-making (MAGDM) problems with the uncertain 2-tuple linguistic information (ULI$_{2-tuple}$) based on new aggregation operators which can capture interrelationships of attributes by a parameter vector P. To begin with, we present some new uncertain 2-tuple linguistic MM aggregation (UL$_{2-tuple}$-MM) operators to handle MAGDM problems with ULI$_{2-tuple}$, including the uncertain 2-tuple linguistic Muirhead mean (UL$_{2-tuple}$-MM) operator, uncertain 2-tuple linguistic weighted Muirhead mean (UL$_{2-tuple}$-WMM) operator. In addition, we extend UL$_{2-tuple}$-WMM operator to a new aggregation operator named extended uncertain 2-tuple linguistic weighted Muirhead mean (EUL$_{2-tuple}$-WMM) operators in order to handle some decision-making problems with ULI$_{2-tuple}$ whose attribute values are expressed in ULI$_{2-tuple}$ and attribute weights are also 2-tuple linguistic information. Whilst, the some properties of these new aggregation operators are obtained and some special cases are discussed. Moreover, we propose a new method to solve the MAGDM problems with ULI$_{2-tuple}$. Finally, a numerical example is given to show the validity of the proposed method and the advantages of proposed method are also analysed.

1. INTRODUCTION

There are many complicated or ill-defined problems are not to be amenable for expressions in conventional quantitative ways in the real world, so it is not always adequate to represent such problems by only numerical based modelling. Therefore, the decision-makers (DMs) utilize linguistic descriptors to express their evaluations on the uncertain knowledge when they encounter such problems. Many researches have been carried out to model the problems by using the linguistic variables (LVs) and have applied successfully in different fields. In MADM problems, the linguistic decision information needs to be aggregated usually by some proper aggregation methods in order to obtain the order of the given decision alternatives and then to get the desirable one. Herrera *et al.* [1, 2] proposed 2-tuple linguistic representation model characterized by a linguistic term (LT) and a numeric value on basis of the concept of symbolic translation. The advantage of linguistic decision is that it can effectively avoid information distortion and losing. Some extensions of 2-tuple linguistic model have been developed, for example, hesitant 2-tuple linguistic information (LI$_{2-tuple}$) model [3–6], intuitionistic LI$_{2-tuple}$ model [7]. Whilst, a variety of decision-making methods based on 2-tuple linguistic model are also developed, for example, FLINSTONES [8], VIKOR method [9, 10], novel approach for FMEA [11], ELECTRE II [12], TOPSIS method [13], etc.

In the field of information fusion, information aggregation is an important research topic as it is a critical process of gathering relevant information from multiple sources. However, aggregation operator as a tool to aggregate relevant information has been focused and also used in many decision-making problems. In linguistic decision-making, many 2-tuple aggregation operators have been developed to aggregate information. We divide these 2-tuple linguistic aggregation operators (LA$_{2-tuple}$) into following five categories after reviewing related work: (1) LA$_{2-tuple}$ related to Choquet integral. For example, Merigo [14] presented the induced 2-tuple linguistic generalized ordered weighted averaging (2-TILGOWA) operator and generalized the 2-TILGOWA by using Choquet integrals. On this basis, Halouani *et al.* [15] defined 2-tuple choquet integral harmonic averaging (TCIHA), 2-tuple ordered choquet integral harmonic averaging (TOCIHA) and applied them to GDM. Ju *et al.* [16] proposed Trapezoid 2-tuple linguistic aggregation operator and new Shapley 2-tuple linguistic Choquet aggregation operators and applied to MADM; (2) LA$_{2-tuple}$ related to Harmonic operators [17, 18]; (3) Extended and induced LA$_{2-tuple}$. For instance, Wan proposed 2-tuple linguistic hybrid arithmetic aggregation operators [19], Hybrid geometric aggregation operators [20] and applied them to multi-attribute group decision-making (MAGDM) problems. Li *et al.* introduced

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[*] *Corresponding author. Email:* liuyiyl@126.com

the induced aggregation operators and distance measures under the 2-tuple linguistic environment and built MADM method in [21]. Wei established a new MAGDM method based on ET-WG operator and ET-OWG operator [22], some dependent $LA_{2-tuple}$ [23]; (4) 2-tuple linguistic power aggregation operators (2TLPA). For example, Xu *et al.* [24] studied the MAGDM method based on 2TLPA under linguistic environment, on basis of 2TLPA, Wu *et al.* [25] proposed some 2-tuple linguistic generalized power aggregation operators (2TLGPA); (5) others $LA_{2-tuple}$. For instance, Xu *et al.* [26] established decision methods based on proportional 2-tuple geometric weighted aggregation operators (PTWGA). In order to develop an approach for consensus problems in which expert preference information is expressed uncertain linguistic preference relations, Xu *et al.* [27] introduced uncertain 2-tuple linguistic variables ($ULVs_{2-tuple}$) and uncertain 2-tuple linguistic weighed averaging ($ULWA_{2-tuple}$) operator. As far as the interval-valued $LA_{2-tuple}$ are concerned, some new uncertain (or interval-valued) $LA_{2-tuple}$ were proposed in many literatures [28–34]. Whilst, some kinds of MAGDM methods based on these aggregation operators were also developed. As a famous aggregation operator, the advantages of Muirhead mean (MM) [35] have the two following aspects: (1) MM can capture the interrelationships of aggregation arguments and (2) MM is also a general operator because it contains other general operators when P takes different values. Many extensions of MM have been developed, for example, intuitionistic fuzzy MM operators [36], 2-tuple linguistic MM operators [37], hesitant fuzzy Maclaurin symmetric mean (HFMSM) [3].

This paper focuses on developing uncertain 2-tuple linguistic MM and then proposes a new method for MAGDM problems with uncertain 2-tuple linguistic information ($ULI_{2-tuple}$). The motivation of this work is based on the following facts:

1. Most of existing aggregation operators with 2-tuple linguistic information did not consider the weighted vector in the form of LVs or 2-tuples. In this proposal, not only the criteria of alternatives are evaluated in a linguistic manner rather than in precise numerical values, but also the weights of attributes (or criteria) are also assessed by a linguistic. It makes the DMs to express their decision more reasonable and also makes the assessment easier to be carried out.
2. Hesitant fuzzy linguistic term set (HFLTs) is an effective tool for dealing complex linguistic decision and some decision methods based on aggregation operators have been developed. But Zhang and Guo [38] pointed out that there are some drawbacks still exist although the results of the aggregation operators-based methods are in the form of HFL. Such as, the aggregation result based on Wei *et al*'s operators [39] is still an HFLTS, but there is some loss of information during the aggregation process; Zhang and Wu's approach [40], the aggregation result is a set of virtual LTs, which limits the interpretability of the aggregation results. However, in this proposal, the proposed method based on $ULT_{2-tuple}$ can availably abstain the loss and lack fidelity of information that occur formerly in the linguistic information processing.
3. Most of existing aggregation operators they cannot consider correlations among any amount of inputs. In this proposal, these proposed aggregation operators can capture interrelationships of multiple attributes by P and make aggregation process more flexible by the P.
4. The diversity and uncertainty of DMs assessment information can be well reflected and modeled using the 2-tuple LVs. It is much easier to solve the practical decision problems.

The rest of the paper is organized as follows: In Section 2, we review some definitions on linguistic term set (LTS), 2-tuple LV, which are used in the analysis throughout this paper. Section 3 is devoted to the new uncertain 2-tuple linguisticrepresentation model. Section 4 is focused on uncertain 2-tuple linguistic weighted Muirhead mean ($UL_{2-tuple}$-WMM) Operator along with their properties and some special cases. In Section 5, the concept of extended uncertain 2-tuple linguistic weighted Muirhead mean ($EUL_{2-tuple}$-WMM) operators in order to handle some decision-making problems with $ULI_{2-tuple}$ whose attribute values are expressed in $ULI_{2-tuple}$ and the attribute weight is also 2-tuple linguistic information. In Section 6, we construct a MAGDM approach based on $UL_{2-tuple}$-WMM and $EUL_{2-tuple}$-WMM operators proposed in Sections 4 and 5. Consequently, a practical example is provided in Section 7 to show the validity and advantages of the proposed method and some conclusions of this study are given in Section 8.

2. PRELIMINARIES

Some fundamental concepts of (uncertain) 2-tuple linguistic models and MM Operator are the basis of this work and so they are recapped firstly in this section.

2.1. Uncertain 2-Tuple Linguistic Representation Model

In what follows, we will use S to represent LTS $S = \left\{s_0, \cdots, s_g\right\}$ with odd cardinality $g+1$ if not otherwise specified in this work.

Let S be the LTS with odd cardinality $g+1$, for any label s_i, which represents a possible values for a LV and satisfy the following characteristics [1]:

1. $s_i > s_j$ if and only if $i > j$;
2. if $s_i \geqslant s_j$, then $max\left(s_i, s_j\right) = s_i$;
3. if $s_i \geqslant s_j$, then $min\left(s_i, s_j\right) = s_j$;
4. $Neg\left(s_i\right) = s_j$, such that $j = g - i$.

To compute with words without loss of information, the 2-tuple linguistic model based on the concept of symbolic translation was proposed in [1, 2, 41]. The model uses a 2-tuple (s_k, α) to represent LI, where $s_k \in S$, α denotes the value of symbolic translation, and $\alpha \in [-0.5, 0.5]$. 2-tuple linguistic model have been successfully applied to decision problems since it was proposed, but some situations can be characterized using 2-tuple linguistic model. For example, if a DM thinks that the profit of a project is “very good” or “at most medium,” then above linguistic model will fail to handle this situation. In order to solve this limitation, Xu [27] introduced uncertain linguistic variable (ULV) which is defined as follows:

Definition 1. [27] Let S be an LTS with cardinality $g+1$, then an ULV can be denoted by $\left[s^{-}, s^{+}\right]$, where $s^{-}, s^{+} \in S$, s^{-}, and s^{+} are the lower and upper limits of the ULV.

For example, in the above example, let LTS $S=\{s_0 =$ extremely poor, $s_1 =$ very poor, $s_2 =$ poor, $s_3 =$ slightly poor, $s_4 =$ Medium, $s_5 =$ slightly good, $s_6 =$ good, $s_7 =$ very good, $s_8 =$ extremely good$\}$, we can use the ULV $[s_0, s_4]$ to express this evaluation "at most medium." When $s^{-} = s^{+}$, the ULV will reduce to linguistic variable. Therefore, ULV is a kind of useful extension of LV.

In Definition 1, if $s^{-} = s^{+}$, then $\left[s^{-}, s^{+}\right]$ reduce to a LT s^{-}. Based on the 2-tuple linguistic model, Zhang and Guo [42] defined the 2-tuple linguistic variable ($LV_{2-tuple}$):

Definition 2. [42] Let S be an LTS with cardinality $g+1$, then the $LV_{2-tuple}$ can be denoted by $\left[\left(s^{-}, \alpha^{-}\right), \left(s^{+}, \alpha^{+}\right)\right]$, $s^{-} \leqslant s^{+}$, where $\left(s^{-}, \alpha^{-}\right), \left(s^{+}, \alpha^{+}\right) \in S \times [-0.5, 0.5)$.

2.2. MM Operator

The MM operator [35] is a general aggregation function and firstly proposed by Muirhead in 1902, it is defined as follows:

Definition 6. [35] Let $a_i\,(i=1,2,\cdots,n)$ be a set of nonnegative real numbers, $A=\{a_1, a_2, \cdots, a_n\}$ and $P=\left(p_1, p_2, \cdots, p_n\right) \in \mathbf{R}^n$ be a parameter vector, if

$$MM^{P}\left(a_1, \cdots, a_n\right)=\left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n} a_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \quad (1)$$

The we call MM^P the Muirhead mean (MM), where $S_n = \{\theta(j) \mid j=1,2,\cdots,n\}$ and $\theta(j)$ is any permutation of $(1,2,\cdots,n)$.

There are some special cases when the parameter vector assessed different values.

1. If $P=(1,0,\cdots,0)$, MM operator will reduce to arithmetic averaging operator

$$MM^{(1,0,\cdots,0)}\left(a_1, \cdots, a_n\right)=\frac{1}{n} \sum_{j=1}^{n} a_j. \quad (2)$$

2. If $P=(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,\ldots,0}^{n-k})$, PFLMM operator will reduce to Maclaurin symmetric mean (MSM) operator [46]

$$PFLMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,\ldots,0}^{n-k})}\left(a_1, \ldots, a_n\right)$$
$$=\left(\frac{\sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n} \prod_{j=1}^{k} a_{i_j}}{C_n^k}\right)^{\frac{1}{k}}; \quad (3)$$

3. If $P=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$, MM operator will reduce to geometric averaging operator

$$MM^{\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)}\left(a_1, \cdots, a_n\right)=\prod_{j=1}^{n} a_j^{\frac{1}{n}}. \quad (4)$$

We can see from the above discussion that and MM operator is a generalization of most existing aggregation operators, the interrelationships among multiple arguments are also considered in MM operator.

3. MODIFIED UNCERTAIN 2-TUPLE LINGUISTIC REPRESENTATION MODEL

Although the $LV_{2-tuple}$ was introduced, its representation model is not given. In this section, we introduce the uncertain 2-tuple linguistic representation model ($ULRM_{2-tuple}$) based on $LV_{2-tuple}$ and give a comparison rule of two uncertain 2-tuples on LTS with multi-granularity.

In order to describe the aggregation result with the LI, Wei [6] modified the translation functions by modifying the generalized 2-tuple linguistic model and translation functions and defined as follows:

Definition 3. [6] Let S be an LTS with granularity $g+1$ and $\beta \in [0,1]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta\colon [0,1] \rightarrow S \times [-0.5, 0.5)$$
$$\Delta(\beta)=\left(s_i, \alpha\right),$$

with

$$\begin{cases} s_i, & i=round(\beta g), \\ \alpha=\beta g-i, & \alpha \in [-0.5, 0.5). \end{cases}$$

where $round()$ is the usual round operation, $s_i \in S$ has the closest index label to β and α is the value of symbolic translation.

Definition 4. Let S be an LTS with granularity $g+1$ and $\left(s_i, \alpha\right)$ be a 2-tuple, where $s_i \in S$. There is a function Δ^{-1}, which can transform a 2-tuple into its equivalent numerical value $\beta \in [0,1]$. The transformation function can be defined as

$$\Delta^{-1}\colon S \times [-0.5/g, 0.5/g] \rightarrow [0,1]$$
$$\Delta^{-1}\left(s_i, \alpha\right)=(i+\alpha)/g=\beta.$$

Although the $LV_{2-tuple}$ was proposed by Xu [27], whose representation model do not been given. Motivated by interval-valued 2-tuple linguistic representation model [33], we put forward the $ULRM_{2-tuple}$ based on Definition 3 and Definition 4.

Definition 5. Let S be an LTS with granularity $g+1$. An interval-valued 2-tuple is composed of two LTs and two crisp numbers, denoted by $\left(s_i, \alpha_1\right), \left(s_j, \alpha_2\right)$, where $i \leqslant j$ and $\alpha_1 \leqslant \alpha_2$ if $i=j$. s_i, s_j represent the linguistic label of the LTS

S and α_1, α_2 represent the symbol translation. The uncertain 2-tuple that express the equivalent information to an interval value $[\beta_1, \beta_2]\,(\beta_1, \beta_2 \in [0,1], \beta_1 \leqslant \beta_2)$ is obtained by the following function:

$$\Delta([\beta_1, \beta_2]) = \left[(s_i, \alpha_1),\ (s_j, \alpha_2)\right],$$

with

$$\begin{cases} s_i, & i = round\left(\beta_1 g\right), \\ s_j, & j = round\left(\beta_2 g\right), \\ \alpha_1 = \beta_1 g - i, & \alpha_1 \in [-0.5, 0.5), \\ \alpha_2 = \beta_2 g - j, & \alpha_2 \in [-0.5, 0.5). \end{cases} \tag{5}$$

Conversely, there exist a function Δ^{-1} such that uncertain 2-tuple can be translated into an interval $[\beta_1, \beta_2]\,(\beta_1, \beta_2 \in [0,1], \beta_1 \leqslant \beta_2)$ as follows:

$$\begin{aligned} \Delta^{-1}\left[(s_i, \alpha_1), (s_j, \alpha_2)\right] &= \left[(\alpha_1 + i)/g, (\alpha_2 + j)/g\right] \\ &= [\beta_1, \beta_2]. \end{aligned} \tag{6}$$

If $s_i = s_j$ and $\alpha_1 = a_2$, Definition 5 reduce to Definition 3 and Definition 4. In the following sections, the translation functions Δ and Δ^{-1} defined by Equations 1 and 2 can help us to aggregate the LI. Comparisons of two uncertain 2-tuples can be carried out according to the following rules:

Let $S = \{s_0, s_1, \cdots, s_\tau\}$ be an LTS with granularity $g = \tau + 1$. For an uncertain 2-tuple $A = \left[(s_i, \alpha_1), (s_j, \alpha_2)\right]$ on the LTS S, the score function and accuracy function of A are introduced, respectively:

$$S^g(A) = \frac{1}{2}\left(\Delta^{-1}(s_i, \alpha_1) + \Delta^{-1}\left(s_j, \alpha_2\right)\right); \tag{7}$$

$$H^g(A) = \Delta^{-1}\left(s_j, \alpha_2\right) - \Delta^{-1}(s_i, \alpha_1). \tag{8}$$

It is obvious that $S(A) \in [0,1]$ and $H(A) \in [0,1]$. Now, the compare rule of two uncertain 2-tuple is listed as follows:

Let S_{g_1} and S_{g_2} be two LTSs with granularity g_1 and g_2, respectively. And A, B are two uncertain 2-tuples on S_{g_1}, S_{g_2}, respectively.

If $S^{g_1}(A) > S^{g_2}(B)$, then $A > B$;

If $S^{g_1}(A) < S^{g_2}(B)$, then $A < B$;

If $S^{g_1}(A) = S^{g_2}(B)$, then:

1. $H^{g_1}(A) > H^{g_2}(B)$, then $A > B$;
2. $H^{g_1}(A) < H^{g_2}(B)$, then $A < B$;
3. $H^{g_1}(A) = H^{g_2}(B)$, then $A = B$.

Example 1. Let $A = [(s_4, 0.1), (s_5, 0.2)]$ and $B = [(s_3, 0.2), (s_4, -0.1)]$ be two 2-tuples on LTSs S_7 and S_5, respectively. Since

$$S^7(A) = \frac{1}{2}\left(\Delta^{-1}(s_4, 0.1) + \Delta^{-1}(s_5, 0.2)\right) = 0.7417;$$

$$S^5(B) = \frac{1}{2}\left(\Delta^{-1}(s_3, 0.2) + \Delta^{-1}(s_4, -0.1)\right) = 0.8875,$$

we have $B > A$.

4. UL2$_{tuple}$-WMM OPERATOR

It is seen from the above discussion in Section 2.2 that and MM operator is a generalization of most existing aggregation operators, the interrelationships among multiple arguments are also considered in MM operator, but it can only process the crisp number. As 2-tuple linguistic model can avoid the information loss in the process of linguistic information processing, so it is necessary to extend traditional MM to uncertain linguistic environment in order to handle some decision-making problems with ULI$_{2-tuple}$. In this section, we will propose UL$_{2-tuple}$-MM and UL$_{2-tuple}$-WMM for the ULI$_{2-tuple}$, investigate some properties of the new operators and obtain some special cases of UL$_{2-tuple}$-MM operator when the parameter vector takes different values. Now, we extend the traditional MM operator to uncertain 2-tuple linguistic environment to solve more complex decision problems with ULI.

4.1. Uncertain 2-Tuple Linguistic Muirhead Mean Operator

In the following sections, the $\{[(r_i, \alpha_i), (l_i, \beta_i)]\,|i = 1, \cdots, n\}$ and $\left\{\left[\left(r_i', \alpha_i'\right), \left(l_i', \beta_i'\right)\right]|i = 1, \cdots, n\right\}$ are two n LVs$_{2-tuple}$ and always denoted by $\left\{\tilde{b}_1, \cdots, \tilde{b}_n\right\}$ and $\left\{\tilde{b}_1', \cdots, \tilde{b}_n'\right\}$, respectively, P always denote as a parameter vector $\left(p_1, p_2, \cdots, p_n\right) \in \mathbf{R}^n$ if not special specified.

Definition 7. Let $\left\{\tilde{b}_1, \cdots, \tilde{b}_n\right\}$ be the set of n LVs$_{2-tuple}$. Then the UL$_{2-tuple}$-MM is defined as follows:

$$\begin{aligned} &UL_{2-tuple} - MM^P\left(\tilde{b}_1, \cdots, \tilde{b}_n\right) = UL_{2-tuple} - MM^P\left([(r_1, \alpha_1), (l_1, \beta_1)], \cdots, [(r_n, \alpha_n), (l_n, \beta_n)]\right) \\ &= \Delta\left[\left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{\theta(j)}, \alpha_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(l_{\theta(j)}, \beta_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right], \end{aligned} \tag{9}$$

where $S_n = \{\theta(j)\,|j = 1, \cdots, n\}$ and $\theta(j)$ is any permutation of $(1, \cdots, n)$.

Example 2. Let $S = \{s_0, s_1, \cdots, s_6\}$ be an LTS and $\left\{\tilde{b}_1 = [(s_1, -0.2), (s_2, 0.1)], \tilde{b}_2 = [(s_3, 0.1), (s_4, 0.3)],\right\}$ $\left\{\tilde{b}_3 = [(s_5, -0.3), (s_6, -0.1)]\right\}$ be set of three LVs$_{2-tuple}$ and $P = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$. Let

$$UL_{2-tuple} - MM^P\left(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3\right) = \Delta[a, b].$$

According to Equation 9, we have

$$\begin{aligned}
a &= \left(\frac{1}{3!}\left(0.13^{\frac{1}{2}}\times 0.52^{\frac{1}{3}}\times 0.78^{\frac{1}{6}}+0.13^{\frac{1}{2}}\times 0.78^{\frac{1}{3}}\times 0.51^{\frac{1}{6}}+0.52^{\frac{1}{2}}\times 0.13^{\frac{1}{3}}\times 0.78^{\frac{1}{6}}\right.\right.\\
&\quad \left.\left.+0.52^{\frac{1}{2}}\times 0.78^{\frac{1}{3}}\times 0.13^{\frac{1}{6}}+0.78^{\frac{1}{2}}\times 0.13^{\frac{1}{3}}\times 0.52^{\frac{1}{6}}+0.78^{\frac{1}{2}}\times 0.52^{\frac{1}{3}}\times 0.13^{\frac{1}{6}}\right)\right)^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}}\\
&= 0.3869.\\
b &= \left(\frac{1}{3!}\left(0.35^{\frac{1}{2}}\times 0.72^{\frac{1}{3}}\times 0.98^{\frac{1}{6}}+0.35^{\frac{1}{2}}\times 0.98^{\frac{1}{3}}\times 0.72^{\frac{1}{6}}+0.72^{\frac{1}{2}}\times 0.35^{\frac{1}{3}}\times 0.98^{\frac{1}{6}}\right.\right.\\
&\quad \left.\left.+0.72^{\frac{1}{2}}\times 0.98^{\frac{1}{3}}\times 0.35^{\frac{1}{6}}+0.98^{\frac{1}{2}}\times 0.35^{\frac{1}{3}}\times 0.72^{\frac{1}{6}}+0.98^{\frac{1}{2}}\times 0.72^{\frac{1}{3}}\times 0.35^{\frac{1}{6}}\right)\right)^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}}\\
&= 0.6320.
\end{aligned}$$

Therefore,

$$\begin{aligned}
UL_{2\text{-}tuple}-MM^{P}\left(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3\right) &= \Delta\,[0.3869, 0.6320]\\
&= [(s_2, 0.3216),\ (s_4, -0.2078)].
\end{aligned}$$

The monotonicity of aggregate operators is an indispensable property in the study of aggregate operators which play a vital role in final decision-making results. Next, we can investigate the $UL_{2\text{-}tuple}-MM^{P}\left(\tilde{b}_1,\tilde{b}_2,\cdots,\tilde{b}_n\right)$ is idempotent, bounded, and monotonic.

Theorem 1. Let $\{\tilde{b}_1,\cdots,\tilde{b}_n\}$ be the set of n LVs$_{2\text{-}tuple}$. If $\tilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)] = \tilde{b} = [(r,\alpha),(l,\beta)]\,(i=1,2,\cdots,n)$, then

$$UL_{2\text{-}tuple}-MM^{P}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)=\tilde{b}.$$

Proof. Since $\tilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)] = \tilde{b} = [(r,\alpha),(l,\beta)]\,(i=1,2,\cdots,n)$, we have

$$\begin{aligned}
&UL_{2\text{-}tuple}-MM^{P}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)=UL_{2\text{-}TUPLE}-MM^{P}\left([(r_1,\alpha_1),(l_1,\beta_1)],\cdots,[(r_n,\alpha_n),(l_n,\beta_n)]\right)\\
&=\Delta\left[\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}(r,\alpha)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}(l,\beta)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\\
&=\Delta\left[\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\left(\Delta^{-1}(r,\alpha)\right)^{\sum_{j=1}^{n}p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\left(\Delta^{-1}(l,\beta)\right)^{\sum_{j=1}^{n}p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\\
&=\Delta\left[\left(\frac{1}{n!}\left(n!\left(\left(\Delta^{-1}(r,\alpha)\right)^{\sum_{j=1}^{n}p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(\frac{1}{n!}\left(n!\left(\left(\Delta^{-1}(l,\beta)\right)^{\sum_{j=1}^{n}p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\\
&=\Delta\left[\left(\left(\Delta^{-1}(r,\alpha)\right)^{\sum_{j=1}^{n}p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(\left(\Delta^{-1}(l,\beta)\right)^{\sum_{j=1}^{n}p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\\
&=\Delta\left[\left(\Delta^{-1}(r,\alpha),\Delta^{-1}(l,\beta)\right]=\tilde{b}.\right.
\end{aligned}$$

Theorem 2. (Monotonicity) Let $\{\tilde{b}_1,\cdots,\tilde{b}_n\}, \{\tilde{b}_i',\cdots,\tilde{b}_i'\}$ be the two sets of n $LVs_{2\text{-}tuple}$. If $(r_i,\alpha_i)\geqslant\left(r_i',\alpha_i'\right)$ and $(l_i,\beta_i)\geqslant\left(l_i',\beta_i'\right)$ for any $i\,(i=1,2,\cdots,n)$, then

$$\begin{aligned}
&UL_{2\text{-}tuple}-MM^{P}\left(b_1,\cdots,b_n\right)\\
&\geqslant UL_{2\text{-}tuple}-MM^{P}\left(b_1',\cdots,b_n'\right).
\end{aligned}$$

Proof. Since $(r_i,\alpha_i)\geqslant\left(r_i',\alpha_i'\right)$ and $(l_i,\beta_i)\geqslant\left(l_i',\beta_i'\right)$, we have

$$\begin{aligned}
\Delta^{-1}\left(r_{\theta(j)},\alpha_{\theta(j)}\right)\geqslant\Delta^{-1}\left(r_{\theta(j)}',\alpha_{\theta(j)}'\right),\Delta^{-1}\left(l_{\theta(j)},\beta_{\theta(j)}\right)\\
\geqslant\Delta^{-1}\left(l_{\theta(j)}',\beta_{\theta(j)}'\right).
\end{aligned}$$

And so

$$\left(\Delta^{-1}\left(r_{\theta(j)},\alpha_{\theta(j)}\right)\right)^{p_j}\geqslant\left(\Delta^{-1}\left(r_{\theta(j)}',\alpha_{\theta(j)}'\right)\right)^{p_j},$$

$$\left(\Delta^{-1}\left(l_{\theta(j)},\beta_{\theta(j)}\right)\right)^{p_j}\geqslant\left(\Delta^{-1}\left(l_{\theta(j)}',\beta_{\theta(j)}'\right)\right)^{p_j}.$$

And

$$\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{\theta(j)},\alpha_{\theta(j)}\right)\right)^{p_j} \geqslant \prod_{j=1}^{n}\left(\Delta^{-1}\left(r'_{\theta(j)},\alpha'_{\theta(j)}\right)\right)^{p_j},$$

$$\prod_{j=1}^{n}\left(\Delta^{-1}\left(l_{\theta(j)},\beta_{\theta(j)}\right)\right)^{p_j} \geqslant \prod_{j=1}^{n}\left(\Delta^{-1}\left(l'_{\theta(j)},\beta'_{\theta(j)}\right)\right)^{p_j}.$$

So, we obtain

$$\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{\theta(j)},\alpha_{\theta(j)}\right)\right)^{p_j}\right)$$

$$\geqslant \sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r'_{\theta(j)},\alpha'_{\theta(j)}\right)\right)^{p_j}\right),$$

$$\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(l_{\theta(j)},\beta_{\theta(j)}\right)\right)^{p_j}\right)$$

$$\geqslant \sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(l'_{\theta(j)},\beta'_{\theta(j)}\right)\right)^{p_j}\right).$$

And so

$$\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{\theta(j)},\alpha_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

$$\geqslant \left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r'_{\theta(j)},\alpha'_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

$$\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(l_{\theta(j)},\beta_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

$$\geqslant \left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(l'_{\theta(j)},\beta'_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

that is,

$$UL_{2-tuple}-MM^{P}\left(b_1,b_2,\cdots,b_n\right)$$

$$\geq UL_{2-tuple}-MM^{P}\left(b'_1,b'_2,\cdots,b'_n\right).$$

Theorem 3. ***(Boundeness)*** Let $\left\{\tilde{b}_1,\cdots,\tilde{b}_n\right\}$ be the set of n $LVs_{2-tuple}$, then

$$\Delta\left[min_i\left(r_i,\alpha_i\right),min_i\left(l_i,\beta_i\right)\right]$$

$$\leq h^{-} \leq UL_{2-tuple}-MM^{P}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)$$

$$\leq \Delta\left[max_i\left(r_i,\alpha_i\right),max_i\left(l_i,\beta_i\right)\right].$$

Proof. Since $min_i\left(r_i,\alpha_i\right) \leq max_i\left(r_i,\alpha_i\right)$ and $min_i\left(l_i,\beta_i\right) \leq max_i\left(l_i,\beta_i\right)$, it is easy to prove the Boundness of $UL_{2-tuple}$-MM. *It is* easy to follow from Equation 9 that commutativity of the operator holds, that is:

Theorem 4. ***(Commutativity)*** Let $\left\{\tilde{b}_1,\cdots,\tilde{b}_n\right\}$, $\left\{\tilde{b}'_1,\cdots,\tilde{b}'_n\right\}$ be the two sets of n $LVs_{2-tuple}$. *If* $\left\{\tilde{b}'_1,\cdots,\tilde{b}'_n\right\}$ is any permutation of $\left\{\tilde{b}_1,\cdots,\tilde{b}_n\right\}$, then

$$UL_{2-tuple}-MM^{P}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)$$

$$= UL_{2-tuple}-MM^{P}\left(\tilde{b}'_1,\cdots,\tilde{b}'_n\right).$$

Now, we develop some special cases of $UL_{2-tuple}$-MM operator considering the different parameter P.

1. If $P=(1,0,\cdots,0)$, $\mathrm{UL}_{2-tuple}$-MM operator reduce to uncertain 2-tuple linguistic average (UTLA) operator [33]

$$UL_{2-tuple}-MM^{(1,0,\cdots,0)}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)$$

$$= \Delta\left[\frac{1}{n}\sum_{j=1}^{n}\Delta^{-1}\left(r_j,\alpha_j\right),\frac{1}{n}\sum_{j=1}^{n}\Delta^{-1}\left(l_j,\beta_j\right)\right]. \tag{10}$$

2. If $P=(\lambda,0,\cdots,0)$, $\mathrm{UL}_{2-tuple}$-MM operator reduce to generalized uncertain 2-tuple linguistic average (GUTLA) operator [34]

$$UL_{2-tuple}-MM^{(1,0,\cdots,0)}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)$$

$$= \Delta\left[\left(\frac{1}{n}\sum_{j=1}^{n}\left(\Delta^{-1}\left(r_j,\alpha_j\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}},\left(\frac{1}{n}\sum_{j=1}^{n}\left(\Delta^{-1}\left(l_j,\beta_j\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right]. \tag{11}$$

3. If $P=(\overbrace{1,1,\cdots,}^{k}\overbrace{0,\cdots,0}^{n-k})$, $\mathrm{UL}_{2-tuple}$-MM operator reduce to $\mathrm{UL}_{2-tuple}$-MSM operator

$$UL_{2-tuple}-MSMP^{(\overbrace{1,1,\cdots,}^{k}\overbrace{0,\cdots,0}^{n-k})}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right) \tag{12}$$

$$=\Delta\left[\left(\frac{k!\,(n-k)!}{n!}\sum_{1\le i_1<\cdots<i_k\le n}\left(\prod_{j=1}^{n}\Delta^{-1}\left(r_{i_j},\alpha_{i_j}\right)\right)\right)^{\frac{1}{k}},\left(\frac{k!\,(n-k)!}{n!}\sum_{1\le i_1<\cdots<i_k\le n}\left(\prod_{j=1}^{n}\Delta^{-1}\left(r_{i_j},\beta_{i_j}\right)\right)\right)^{\frac{1}{k}}\right]$$

$$=\Delta\left[\left(\frac{1}{C_n^k}\sum_{1\le i_1<\cdots<i_k\le n}\left(\prod_{j=1}^{n}\Delta^{-1}\left(r_{i_j},\alpha_{i_j}\right)\right)\right)^{\frac{1}{k}},\left(\frac{1}{C_n^k}\sum_{1\le i_1<\cdots<i_k\le n}\left(\prod_{j=1}^{n}\Delta^{-1}\left(l_{i_j},\beta_{i_j}\right)\right)\right)^{\frac{1}{k}}\right].$$

4. If $P=(1,1,\cdots,1)$, $\mathrm{UL}_{2-tuple}$-MM operator reduce to uncertain 2-tuple linguistic geometric (UTLG) operator [34]

$$UL_{2-tuple}-MM^{(1,1,\cdots,1)}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)$$

$$=\Delta\left[\left(\prod_{j=1}^{n}\Delta^{-1}\left(r_j,\alpha_j\right)\right)^{\frac{1}{n}},\left(\prod_{j=1}^{n}\Delta^{-1}\left(l_j,\beta_j\right)\right)^{\frac{1}{n}}\right] \tag{13}$$

5. If $P=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, $\mathrm{UL}_{2-tuple}$-MM operator will reduce to uncertain 2-tuple linguistic geometric (UTLG) operator [34]

$$UL_{2-tuple}-MM^{\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)$$

$$=\Delta\left[\left(\prod_{j=1}^{n}\Delta^{-1}\left(r_j,\alpha_j\right)\right)^{\frac{1}{n}},\left(\prod_{j=1}^{n}\Delta^{-1}\left(l_j,\beta_j\right)\right)^{\frac{1}{n}}\right] \tag{14}$$

4.2. $\mathbf{UL2_{-tuple}}$-WMM Operators

Weights of attributes play a vital role in decision-making and will directly the results of decision-making results. In this Section, we introduce the $\mathrm{UL}_{2-tuple}$-MM aggregation operators which can not consider the weights of attributes, so it is very important to consider to weights of attributes in the process of decision-making. First of all, the $\mathrm{UL}_{2-tuple}$-WMM operator is introduced and defined as follows:

Definition 8. Let $\left\{\tilde{b}_1,\cdots,\tilde{b}_n\right\}$ be the set of n $\mathrm{LVs}_{2-tuple}$ and $(w_1,\cdots,w_n)$ be their associated weights with $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$. Then the $\mathrm{UL}_{2-tuple}$-WMM is defined as follows:

$$UL_{2-tuple}-WMM^{P}\left(\tilde{b}_1,\cdots,\tilde{b}_n\right)=UL_{2-tuple}-WMM^{P}\left(\left[\left(r_1,\alpha_1\right),\left(l_1,\beta_1\right)\right]\right),\cdots,\left(\left[\left(r_n,\alpha_n\right),\left(l_n,\beta_n\right)\right]\right) \tag{15}$$

$$=\Delta\left[\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(nw_{\theta(j)}\Delta^{-1}\left(r_{\theta(j)},\alpha_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(\frac{1}{n!}\left(\sum_{\theta\in S_n}\left(\prod_{j=1}^{n}\left(nw_{\theta(j)}\Delta^{-1}\left(l_{\theta(j)},\beta_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

where $S_n=\left\{\theta\left(j\right)|j=1,\cdots,n\right\}$ and $\theta\left(j\right)$ is any permutation of $(1,\cdots,n)$.

Example 3. Let $S=\{s_0,s_1,\cdots,s_6\}$ be an LTS and $\left\{\tilde{b}_1=[(s_1,-0.2),(s_2,0.1)],\tilde{b_2}=[(s_3,0.1),(s_4,0.3)],\tilde{b_3}=[(s_5,-0.3)]\right\}$ $\{(s_6,-0.1)\}$ be set of three $\mathrm{LVs}_{2-tuple}$ with weights vector (0.4, 0.3, 0.3) and $P=\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$. Let

$$UL_{2-tuple}-WMM^{P}\left(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3\right)=\Delta\left[a,b\right].$$

According to Equation 15, we have

$$\begin{aligned}a&=\left(\frac{1}{3!}\left(0.16^{\frac{1}{2}}\times0.47^{\frac{1}{3}}\times0.71^{\frac{1}{6}}+0.16^{\frac{1}{2}}\times0.71^{\frac{1}{3}}\times0.47^{\frac{1}{6}}+0.47^{\frac{1}{2}}\times0.16^{\frac{1}{3}}\times0.71^{\frac{1}{6}}\right.\right.\\&\quad\left.\left.+0.47^{\frac{1}{2}}\times0.71^{\frac{1}{3}}\times0.16^{\frac{1}{6}}+0.71^{\frac{1}{2}}\times0.16^{\frac{1}{3}}\times0.47^{\frac{1}{6}}+0.71^{\frac{1}{2}}\times0.47^{\frac{1}{3}}\times0.16^{\frac{1}{6}}\right)\right)^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}}\\&=0.3804.\\b&=\left(\frac{1}{3!}\left(0.42^{\frac{1}{2}}\times0.65^{\frac{1}{3}}\times0.89^{\frac{1}{6}}+0.42^{\frac{1}{2}}\times0.89^{\frac{1}{3}}\times0.65^{\frac{1}{6}}+0.65^{\frac{1}{2}}\times0.42^{\frac{1}{3}}\times0.89^{\frac{1}{6}}\right.\right.\\&\quad\left.\left.+0.65^{\frac{1}{2}}\times0.89^{\frac{1}{3}}\times0.42^{\frac{1}{6}}+0.89^{\frac{1}{2}}\times0.42^{\frac{1}{3}}\times0.65^{\frac{1}{6}}+0.89^{\frac{1}{2}}\times0.65^{\frac{1}{3}}\times0.42^{\frac{1}{6}}\right)\right)^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}}\\&=0.6236.\end{aligned}$$

Therefore,

$$\begin{aligned} &UL_{2-tuple} - WMM^{P}\left(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3\right) \\ &\quad = \Delta\left[0.3804, 0.6236\right] \\ &\quad = \left[\left(s_2, 0.2826\right), \left(s_4, -0.2581\right)\right]. \end{aligned}$$

If the weight vector $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$ in Definition 13, we have

$$\begin{aligned} &UL_{2-tuple} - WMM^{P}\left(\left[\left(r_1, \alpha_1\right), \left(l_1, \beta_1\right)\right], \cdots, \left[\left(r_n, \alpha_n\right), \left(l_n, \beta_n\right)\right]\right) \\ &= \Delta\left[\left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\left(n \times \frac{1}{n}\right)\Delta^{-1}\left(r_{\theta(j)}, \alpha_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\left(n \times \frac{1}{n}\right)\Delta^{-1}\left(l_{\theta(j)}, \beta_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right], \\ &= \Delta\left[\left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{\theta(j)}, \alpha_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(l_{\theta(j)}, \beta_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right], \\ &= UL_{2-tuple} - MM^{P}\left(\left[\left(r_1, \alpha_1\right), \left(l_1, \beta_1\right)\right], \cdots, \left[\left(r_n, \alpha_n\right), \left(l_n, \beta_n\right)\right]\right). \end{aligned}$$

That is,

Theorem 5. If $w = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$, $UL_{2-tuple}$-WMM operator reduce to $UL_{2-tuple}$-MM operator.

Similar to Theorem 2 and Theorem 3, we can prove $UL_{2-tuple}$ − WMM$^{P}\left(\tilde{b}_1, \cdots, \tilde{b}_n\right)$ are bounded, and monotonic.

Theorem 6. ***(Monotonicity)*** *Let* $\left\{\tilde{b}_1, \cdots, \tilde{b}_n\right\}$, $\left\{\tilde{b}'_1, \cdots, \tilde{b}'_n\right\}$ *be the two sets of* n $LVs_{2-tuple}$. *If* $\left(r_i, \alpha_i\right) \geq \left(r'_i, \alpha'_i\right)$ *and* $\left(l_i, \beta_i\right) \geq \left(l'_i, \beta'_i\right)$ *for any* $i\,(i = 1, 2, \cdots, n)$, *then*

$$\begin{aligned} &UL_{2-tuple} - WMM^{P}\left(\tilde{b}_1, \cdots, \tilde{b}_n\right) \\ &\geq UL_{2-tuple} - WMM^{P}\left(\tilde{b}'_1, \cdots, \tilde{b}'_n\right). \end{aligned}$$

Theorem 7. ***(Boundeness)*** *Let* $\left\{\tilde{b}_1, \cdots, \tilde{b}_n\right\}$ *be the set of* n $LVs_{2-tuple}$, *then*

$$\begin{aligned} &\Delta\left[min_i\left(r_i, \alpha_i\right), min_i\left(l_i, \beta_i\right)\right] \leq h^{-} \\ &\leq UL_{2-tuple} - WMM^{P}\left(\tilde{b}_1, \cdots, \tilde{b}_n\right) \\ &\leq \Delta\left[max_i\left(r_i, \alpha_i\right), max_i\left(l_i, \beta_i\right)\right]. \end{aligned}$$

Now, we will develop some special cases of $UL_{2-tuple}$-WMM operator when parameter P takes different values. Let $\left\{\tilde{b}_1, \cdots, \tilde{b}_n\right\}$ be a set of $LVs_{2-tuple}$, $w = \left(w_1, w_2, \cdots, w_n\right)$ be the weight vector of h_i with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

1. If $P = (1, 0, \cdots, 0)$, UL$_{2-tuple}$-WMM operator reduce to UL$_{2-tuple}$-WA operator [27]

$$\begin{aligned} &UL_{2-tuple} - WMM^{(1,0,\cdots,0)}\left(\tilde{b}_1, \cdots, \tilde{b}_n\right) \\ &= \Delta\left[\left(\sum_{j=1}^{n} w_j\Delta^{-1}\left(r_j, \alpha_j\right)\right), \left(\sum_{j=1}^{n} w_j\Delta^{-1}\left(l_j, \delta_j\right)\right)\right]. \end{aligned} \tag{16}$$

2. If $P = (\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{n-k})$, UL$_{2-tuple-}$WMM operator reduce to uncertain 2-tuple linguistic weighted Maclaurin symmetric mean (UL$_{2-tuple-}$WMSM) operator

$$\begin{aligned} &UL_{2-tuple} - MSM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}\left(\tilde{b}_1, \cdots, \tilde{b}_n\right) \\ &= \Delta\left[\left(\frac{1}{C_n^k}\sum_{1 \leq i_1 < \cdots < i_k \leq n}\left(\prod_{j=1}^{n}\left(nw_{\theta(j)}\Delta^{-1}\left(r_{i_j}, \alpha_{i_j}\right)\right)\right)\right)^{\frac{1}{k}}, \left(\frac{1}{C_n^k}\sum_{1 \leq i_1 < \cdots < i_k \leq n}\left(\prod_{j=1}^{n}\left(nw_{\theta(j)}\Delta^{-1}\left(l_{i_j}, \beta_{i_j}\right)\right)\right)\right)^{\frac{1}{k}}\right]. \end{aligned} \tag{17}$$

Remark 1. It seen from the above discussions that UL$_{2-tuple}$-WMM operator can consider correlations among any amount of inputs by a parameter P, and it is a generalization of some existing operators.

5. EUL$_{2-tuple}$-WMM OPERATORS

Herrera *et al.* [2] extended the 2-tuple linguistic averaging operators to address the decision problems whose attribute values and the attribute weight are given in 2-tuple linguistic information. Motivated by this idea, we extend UL$_{2-tuple}$-WMM operator to uncertain 2-tuple linguistic weighted Muirhead mean (EUL$_{2-tuple}$-WMM) operator in order to handle some decision problems with ULI$_{2-tuple}$ whose attribute values are expressed in ULI$_{2-tuple}$ and attribute weights are also represented by 2-tuple linguistic information.

Definition 9. Let $\{\tilde{b}_1, \cdots, \tilde{b}_n\}$ be the set of n LVs$_{2-tuple}$ and $W = ((w_1, \gamma_1), \cdots, (w_n, \gamma_n))$ be their associated 2-tuple linguistic weight vector. Then EUL$_{2-tuple}$-WMM is defined as follows:

$$EUL_{2\text{-}tuple} - WMM^P\left(\tilde{b}_1, \cdots, \tilde{b}_n\right) = UL_{2\text{-}tuple} - WMM^P\left(\left[\left(r_1, \alpha_1\right), \left(l_1, \beta_1\right)\right]\right), \cdots, \left(\left[\left(r_n, \alpha_n\right), \left(l_n, \beta_n\right)\right]\right) \tag{18}$$

$$= \Delta\left[\left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\frac{n\Delta^{-1}\left(w_{\theta(j)}, \gamma_{\theta(j)}\right)}{\sum_{j=1}^{n}\Delta^{-1}\left(w_j, \gamma_j\right)}\Delta^{-1}\left(r_{\theta(j)}, \alpha_{\theta(j)}\right)\right)^{pj}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}},\right.$$

$$\left.\left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n}\left(\frac{n\Delta^{-1}\left(w_{\theta(j)}, \gamma_{\theta(j)}\right)}{\sum_{j=1}^{n}\Delta^{-1}\left(w_j, \gamma_j\right)}\Delta^{-1}\left(l_{\theta(j)}, \beta_{\theta(j)}\right)\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right],$$

where $S_n = \{\theta(j) | j = 1, \cdots, n\}$ and $\theta(j)$ is any permutation of $(1, \cdots, n)$.

Similar to Theorem 6 and Theorem 7, we can prove $EUL_{2-tuple} - WMM^P\left(\tilde{b}_1, \cdots, \tilde{b}_n\right)$ is bounded and monotonic.

Theorem 8. ***(Monotonicity)*** Let $\{\tilde{b}_1, \cdots, \tilde{b}_n\}$, $\{\tilde{b}'_1, \cdots, \tilde{b}'_n\}$ be the two sets of n $LVs_{2-tuple}$, $W = ((w_1, \gamma_1), \cdots, (w_n, \gamma_n))$ be their associated 2-tuple linguistic weight vector. If $(r_i, \alpha_i) \geq \left(r'_i, \alpha'_i\right)$ and $(l_i, \beta_i) \geq \left(l'_i, \beta'_i\right)$ for any i $(i = 1, \cdots, n)$, then

$$EUL_{2\text{-}tuple} - WMM^P\left(b_1, b_2, \cdots, b_n\right)$$
$$\geq EUL_{2\text{-}tuple} - WMM^P\left(b'_1, b'_2, \cdots, b'_n\right).$$

Theorem 9. ***(Boundeness)*** Let $\{\tilde{b}_1, \cdots, \tilde{b}_n\}$ be the set of n $LVs_{2-tuple}$, and $W = ((w_1, \gamma_1), \cdots, (w_n, \gamma_n))$ be their associated linguistic weight vector. then

$$\Delta\left[min_i\left(r_i, \alpha_i\right), min_i\left(l_i, \beta_i\right)\right]$$
$$\leq EUL_{2\text{-}tuple} - WMM^P\left(\tilde{b}_1, \cdots, \tilde{b}_n\right)$$
$$\leq \Delta\left[max_i\left(r_i, \alpha_i\right), max_i\left(l_i, \beta_i\right)\right].$$

Let $\{\tilde{b}_1, \cdots, \tilde{b}_n\}$ be a set of $LVs_{2-tuple}$, $w = (w_1, w_2, \cdots, w_n)$ be the *2-tuple* linguistic weight vector of h_i $(i = 1, 2, \cdots, n)$. Next, we will obtain some special cases of $EUL_{2-tuple}$-WMM operator when the parameter takes different values.

1. If $P = (1, 0, \cdots, 0)$, EUL$_{2-tuple}$-WMM operator reduce to uncertain 2-tuple linguistic weighted averaging operator [9]

$$EUL_{2\text{-}tuple} - WMM^{(1,0,\cdots,0)}\left(\tilde{b}_1, \cdots, \tilde{b}_n\right)$$
$$= \Delta\left[\sum_{j=1}^{n}\left(\frac{\Delta^{-1}\left(w_j, \gamma_j\right)}{\sum_{j=1}^{n}\Delta^{-1}\left(w_j, \gamma_j\right)}\Delta^{-1}\left(r_j, \alpha_j\right)\right),\right.$$
$$\left.\sum_{j=1}^{n}\left(\frac{\Delta^{-1}\left(w_j, \gamma_j\right)}{\sum_{j=1}^{n}\Delta^{-1}\left(w_j, \gamma_j\right)}\Delta^{-1}\left(l_j, \beta_j\right)\right)\right], \tag{19}$$

2. If $P = (\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{n-k})$, EUL$_{2-tuple-}$WMM operator reduce to uncertain 2-tuple linguistic weighted Maclaurin symmetric mean (UL$_{2-tuple-}$WMSM) operator

$$UL_{2\text{-}tuple} - MSM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}\left(\tilde{b}_1, \cdots, \tilde{b}_n\right) = \Delta\left[\left(\frac{1}{C_n^k}\sum_{1 \leq i_1 < \cdots < i_k \leq n}\left(\prod_{j=1}^{n}\left(\frac{n\Delta^{-1}\left(w_{\theta(j)}, \gamma_{\theta(j)}\right)}{\sum_{j=1}^{n}\Delta^{-1}\left(w_j, \gamma_j\right)}\Delta^{-1}\left(r_{i_j}, \alpha_{i_j}\right)\right)\right)\right)^{\frac{1}{k}},\right. \tag{20}$$

$$\left.\left(\frac{1}{C_n^k}\sum_{1 \leq i_1 < \cdots < i_k \leq n}\left(\prod_{j=1}^{n}\left(\frac{n\Delta^{-1}\left(w_{\theta(j)}, \gamma_{\theta(j)}\right)}{\sum_{j=1}^{n}\Delta^{-1}\left(w_j, \gamma_j\right)}\Delta^{-1}\left(l_{i_j}, \beta_{i_j}\right)\right)\right)\right)^{\frac{1}{k}}\right].$$

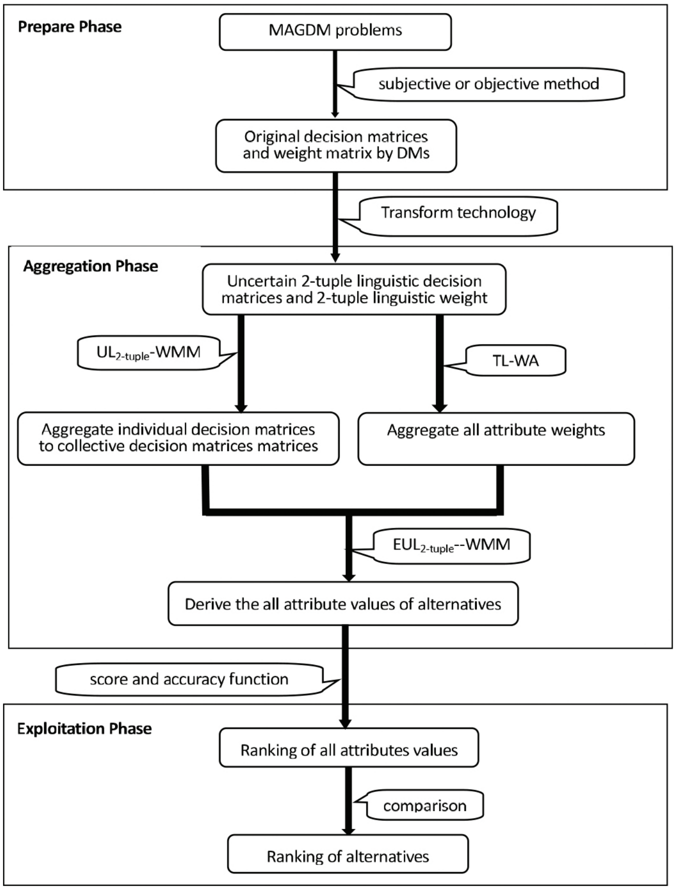

Figure 1 The decision process of proposed multi-attribute group decision making (MAGDM) method.

6. AN APPROACH TO MAGDM WITH ULI$_{2-tuple}$

In this section, a MAGDM method with ULI$_{2-tuple}$ will be developed based on the proposed UL$_{2-tuple}$-WMM and EUL$_{2-tuple}$-WMM operator.

Assuming that there are l DMs $DM_1, \cdots, DM_l$ in a MAGDM problem, the set of m alternatives expressed by $A = \{A_1, A_2, \cdots, A_m\}$ and the set of attributes (or criteria) $C = \{C_1, C_2, \cdots, C_n\}$. l DMs $DM_1, \cdots, DM_l$ are given a weight vector $(\lambda_1, \cdots, \lambda_l)$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, the weight of DM reflects his or her relative importance in the group decision-making process. Let $D_k = (r_{ij}^k)_{m\times n}$ $(k = 1, 2, \cdots, l)$ be the linguistic decision matrix of the kth DM, where r_{ij}^k is the linguistic information provided by the kth DM DM_k on the assessment of A_i w. r. t C_j. Let $W_k = \left(w_1^k, w_2^k, \cdots, w_n^k\right)$ be the linguistic weighted vector given by the DM DM_k, where w_i^k is a LT assigned to attribute C_i by DM DM_k.

In what follows, we use UL$_{2-tuple}$-WMM and EUL$_{2-tuple}$-WMM operator to develop an method to solve MAGDM problems with ULI$_{2-tuple}$. In order to obtain the best alternative(s), the following steps are involved and the decision process is shown as Figure 1:

Step 1. Transform linguistic decision matrix $D_k = (r_{ij}^k)_{m\times n}$ into uncertain 2-tuple linguistic decision matrix (ULDM$_{2-tuple}$) $D_k = (\tilde{r}_{ij}^{k})_{m\times n} = \left(\left[\left(s_{ij}^k, 0\right), \left(t_{ij}^k, 0\right)\right]\right)_{m\times n}$, where $s_{ij}^k \leq t_{ij}^k$. Whilst, transform the linguistic weighted vector $W_k = \left(w_1^k, w_2^k, \cdots, w_n^k\right)$ into 2-tuple linguistic weight vector $W_k = \left(\left(w_1^k, 0\right), \left(w_2^k, 0\right), \cdots, \left(w_n^k, 0\right)\right)$.

There are three cases should be paid attention to in the process of transforming the original linguistic decision matrix into ULDM$_{2-tuple}$. Now, we take an example to show the three cases: Let $S = \{s_0 =$ extremely poor (EP), $s_1 =$ very poor (VP), $s_2 =$ poor (P), $s_3 =$ slightly poor (SP), $s_4 =$ Medium (M), $s_5 =$ slightly good (SG), $s_6 =$ good (G), $s_7 =$ very good (VG), $s_8 =$ extremely good (EG)$\}$. We can transform the linguistic decision matrix into ULDM$_{2-tuple}$ in the following ways:

1. A certain grade such as good, which can be expressed as $[(s_6, 0), (s_6, 0)]$;
2. A interval such as fair-good, which can be expressed as $[(s_4, 0), (s_6, 0)]$.
3. If DM do not provide any assessment of an alternative, then the situation can be expressed as $[(s_0, 0), (s_8, 0)]$.

Step 2. Aggregate all individual decision matrix D_k $(k = 1, 2, \cdots, l)$ to collective matrix D based on the UL$_{2-tuple}$-WA operator

$$\tilde{r}_{ij} = UL_{2-tuple} - WA\left(\tilde{r}_{ij}^1, \tilde{r}_{ij}^2, \cdots, \tilde{r}_{ij}^l\right). \tag{21}$$

Step 3. Aggregate all attribute weights provided by l DMs based on the TL-WA operator

$$\left(w_j, \varepsilon_j\right) = \Delta\left[\sum_{k=1}^{l} \lambda_k \Delta^{-1}\left(w_j^k, 0\right)\right]. \tag{22}$$

Step 4. Use the EUL$_{2-tuple}$-WMM operator to derive the all attribute (criteria) values $\tilde{r}_{ij}$ $\left(j = 1, 2, \ldots, m\right)$ of the alternative A_i, i. e.

$$\tilde{r}_i = EUL_{2-tuple} - WMM^P\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}\right). \tag{23}$$

Step 5. Calculate the score values and accuracy values of $\tilde{r}_i$ of all collective overall values

Step 6. Arrange all alternatives A_i $(i = 1, \cdots, m)$. The bigger the $S(A_i)$, the better the A_i.

Step 7. End.

7. NUMERICAL EXAMPLE AND COMPARATIVE ANALYSIS

7.1. Numerical Example

In this section, an illustrative example on an evaluation on enterprise technology innovation management was cited and adapted from [43] to show the application of the proposed MAGDM method. Technological innovation is not only directly related to the survival and development of an enterprise, but also affect the economic development of a region or even a country. As we all know, the management of an enterprise's technological innovation activities is an important manifestation of its technological innovation capability. In evaluating the technological innovation capability of enterprises, the following evaluation index system should be considered:

1. G_1: Innovation system construction, attitude to innovation failure, and incentives for innovation by the enterprise distribution system;
2. G_2: Establishment and implementation of technological innovation strategy, the formation and maintenance of enterprise innovation culture;
3. G_3: The feasibility of research and development project feasibility report;
4. G_4: The completeness of the monitoring and evaluation system and innovation awareness of leaders and staff.

Now there are three DMs DM_1, DM_2, DM_3 (weight vector $(0.3, 0.4, 0.3)$) assess the technical innovation management of 5 large enterprises $A_i\,(i = 1, 2, \cdots, 5)$ by questionnaires survey and discussion. The three DMs employ the linguistic terms set $S = \{s_0 = \text{EP}, s_1 = \text{VP}, s_2 = \text{P}, s_3 = \text{SP}, s_4 = \text{M}, s_5 = \text{SG}, s_6 = \text{G}, s_7 = \text{VG}, s_8 = \text{EG}\}$ which is given in Section 6 to evaluate the five enterprises w. r. t the above evaluation criteria. The relative importance of the criteria was rated by the three DMs with a set of five LT $W = \{w_0 = \text{very unimportant(VU)}, w_1 = \text{unimportant (U)}, w_2 = \text{medium (M)}, w_3 = \text{important (I)}, w_4 = \text{very important (VI)}$. The assessment of the five enterprises on each criteria and criteria weights provided by the three DMs are presented in Tables 1 and 2. Now we determine the best technology innovation management enterprise.

Table 1 The decision matrices given by DMs DMi (i = 1, 2, 3).

		G_1	G_2	G_3	G_4
DM_1	A_1	VG	M	P	VG
	A_2	EG	VG	P-M	SP
	A_3	G-VG	G	P-M	G-VG
	A_4	SP-SG	SG-G	VP-EG	SG-G
	A_5	M-G	G-VG	M	P-SP
DM_2	A_1	G	M	SG-G	G-VG
	A_2	EG	VG	P-M	SP
	A_3	SG-VG	G	M-SG	M-G
	A_4	VP-EG	SG-VG	M-G	G-VG
	A_5	M-G	G-VG	P-M	SP-M
DM_3	A_1	SG-G	M-G	SP-M	VG
	A_2	EG	G-VG	P-M	SP-M
	A_3	G-VG	SG-G	SP-M	SG-VG
	A_4	SP-SG	VP-EG	M-SG	SG-G
	A_5	SG-G	G-VG	M-SG	P-SP

DM, decision-maker.

Table 2 The linguistic weights of criteria.

	G_1	G_2	G_3	G_4
DM_1	VI	M	M	I
DM_2	VI	I	M	I
DM_3	I	M	I	I

DM, decision-maker.

Now, we utilize the proposed method based on $\text{UL}_{2-tuple}$-WMM and $\text{EUL}_{2-tuple}$-WMM operator to drive the collective overall value, we obtain following:

Step 1. Transform original linguistic decision matrix into $\text{ULDM}_{2-tuple}$ $D_k = (\tilde{r}^k_{ij})_{m\times n} = \left(\left[\left(s^k_{ij}, 0\right), \left(t^k_{ij}, 0\right)\right]\right)_{m\times n}$ and shown in Table 3. Whilst, transform the linguistic weighted vector $W_k = \left(w^k_1, w^k_2, \cdots, w^k_n\right)$ into 2-tuple linguistic weight vector $W_k = \left(\left(w^k_1, 0\right), \left(w^k_2, 0\right), \cdots, \left(w^k_n, 0\right)\right)$ and shown in Table 4.

Step 2. Aggregate all individual decision matrix $D_k\,(k = 1, 2, 3)$ to D based on the $\text{UL}_{2-tuple}$-WA operator and shown in Table 5.

Step 3. Aggregate all attribute weights provided by l DMs based on the TL-WA operator and shown in Table 6,

Step 4–6. Utilize the $\text{EUL}_{2-tuple}$-WMM operator to derive the all attribute (criteria) values of the alternative $A_i\,(i = 1, 2, \cdots, 5)$. For convenience, parameters $p = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, the aggregation results and ranking of alternatives shown in Table 7.

From Table 7, the desirable alternative is A_4.

7.2. Decision-Making Results Analysis

In this section, we will analyse the decision-making result through the different parameter vector P in our proposed methods based on HFWMM operators. The ranking results are shown in Table 8.

We explain the following aspects to explain the influence of P on the decision-making results:

1. We see from the Section 3 that many uncertain $\text{LA}_{2-tuple}$ are the special cases of $\text{UL}_{2-tuple}$-MM and $\text{EUL}_{2-tuple}$-WMM operators, so our method is more general. Specially, when $P = (\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, 0 \cdots, 0}^{k})$, the $\text{EUL}_{2-tuple}$-WMM operator will become uncertain 2-tuple linguistic weighted Maclaurin mean, which is also family aggregation operators when the parameter k takes different value.
2. It follows from Table 8 that the aggregation results obtained by $\text{EUL}_{2-tuple}$-WMM operators are almost remain unchanged in this example though the parameter vector P change, this phenomenon also illustrates $\text{EUL}_{2-tuple}$-WMM operators have goodrobust property.
3. Parameter vector P can capture interrelationship between the individual arguments. Different P can be regarded as the DMs' risk preference.

7.3. Comparisons and Discussions

In order to show the validity of the proposed methods, we compare our proposed methods with other existing methods including the interval-valued 2-tuple VIKOR method. The results are shown in Table 9, which indicates that four methods have the same desirable alternative, which further verifies the validity of the method proposed with $\text{EUL}_{2-tuple}$-WMM operator.

In the following, some comparisons are made and comparison results are listed in Table 10.

where MA means multiple attributes and PV means parameter vector. 2

IVTWA and IVTGA are two very useful aggregation operator in decision problems with interval-valued $LI_{2-tuple}$. We can see from Section 3 that IVTWA and HFGA are special cases of UL $_{2-tuple}$-MM operator. Compared with the IVTWA and HFGA, in which there are three limitations: (1) the method based on IVTWA and HFGA think that the input arguments are not dependent; (2) the method based on IVTWA and HFGA doesn't consider the interrelationship among input arguments; (3) the method based on IVTWA and HFGA only solve such a kind of decision problems in which the relative weights of attributes are evaluated in precis numerical values. Compared with the interval-valued 2-tuple linguistic VIKOR and other existing MAGDM methods, the mainly advantages of the proposed method are focused on the following aspects:

1. The proposed method based on $ULT_{2-tuple}$ can availably abstain the loss and lack fidelity of information that occur formerly in the linguistic information processing.
2. Not only the criteria of alternatives are evaluated in a linguistic manner rather than in precise numerical values, but also the weights of attributes (or criteria) are also assessed by a linguistic. It makes the DMs to express their decision more reasonable and also makes the assessment easier to be carried out.
3. The main advantage of these aggregation operators are that they can capture interrelationships of multiple attributes by *P* and make aggregation process more flexible by the *P*.
4. The diversity and uncertainty of DMs assessment information can be well reflected and modeled using the $LVs_{2-tuple}$. It is much easier to solve the practical decision problems.

8. CONCLUSIONS

In recent years, aggregation operators play a vital role in decision-making and more and more aggregation operators under different

Table 3 | $ULDM_{2-tuple}$ of three DMs DMi (i = 1, 2, 3).

		A_1	A_2	A_3	A_4	A_5
DM_1	G_1	$[(s_7,0),(s_7,0)]$	$[(s_8,0),(s_8,0)]$	$[(s_6,0),(s_7,0)]$	$[(s_3,0),(s_5,0)]$	$[(s_4,0),(s_6,0)]$
	G_2	$[(s_4,0),(s_4,0)]$	$[(s_7,0),(s_7,0)]$	$[(s_6,0),(s_6,0)]$	$[(s_5,0),(s_6,0)]$	$[(s_6,0),(s_7,0)]$
	G_3	$[(s_2,0),(s_2,0)]$	$[(s_2,0),(s_4,0)]$	$[(s_2,0),(s_4,0)]$	$[(s_1,0),(s_8,0)]$	$[(s_4,0),(s_4,0)]$
	G_4	$[(s_7,0),(s_7,0)]$	$[(s_3,0),(s_3,0)]$	$[(s_6,0),(s_7,0)]$	$[(s_5,0),(s_6,0)]$	$[(s_2,0),(s_3,0)]$
DM_2	G_1	$[(s_6,0),(s_6,0)]$	$[(s_8,0),(s_8,0)]$	$[(s_5,0),(s_7,0)]$	$[(s_1,0),(s_8,0)]$	$[(s_4,0),(s_6,0)]$
	G_2	$[(s_4,0),(s_4,0)]$	$[(s_7,0),(s_7,0)]$	$[(s_6,0),(s_6,0)]$	$[(s_5,0),(s_7,0)]$	$[(s_6,0),(s_7,0)]$
	G_3	$[(s_5,0),(s_6,0)]$	$[(s_2,0),(s_4,0)]$	$[(s_4,0),(s_5,0)]$	$[(s_4,0),(s_6,0)]$	$[(s_2,0),(s_4,0)]$
	G_4	$[(s_6,0),(s_7,0)]$	$[(s_3,0),(s_3,0)]$	$[(s_4,0),(s_6,0)]$	$[(s_6,0),(s_7,0)]$	$[(s_3,0),(s_4,0)]$
DM_3	G_1	$[(s_5,0),(s_6,0)]$	$[(s_7,0),(s_8,0)]$	$[(s_6,0),(s_7,0)]$	$[(s_3,0),(s_5,0)]$	$[(s_5,0),(s_6,0)]$
	G_2	$[(s_4,0),(s_6,0)]$	$[(s_7,0),(s_7,0)]$	$[(s_5,0),(s_6,0)]$	$[(s_1,0),(s_8,0)]$	$[(s_6,0),(s_7,0)]$
	G_3	$[(s_3,0),(s_4,0)]$	$[(s_2,0),(s_4,0)]$	$[(s_3,0),(s_4,0)]$	$[(s_4,0),(s_5,0)]$	$[(s_4,0),(s_5,0)]$
	G_4	$[(s_7,0),(s_7,0)]$	$[(s_3,0),(s_4,0)]$	$[(s_5,0),(s_7,0)]$	$[(s_5,0),(s_6,0)]$	$[(s_2,0),(s_3,0)]$

DM, decision-maker; ULDM-2, uncertain 2-tuple linguistic decision matrix.

Table 4 | The 2-tuple linguistic weights of criteria.

	DM_1	DM_2	DM_3
G_1	$(e_4,0)$	$(e_4,0)$	$(e_3,0)$
G_2	$(e_2,0)$	$(e_3,0)$	$(e_2,0)$
G_3	$(e_2,0)$	$(e_2,0)$	$(e_3,0)$
G_4	$(e_3,0)$	$(e_3,0)$	$(e_3,0)$

DM, decision-maker.

Table 5 | Collective decision matrix.

	A_1	A_2	A_3	A_4	A_5
G_1	$\Delta[0.525, 0.7875]$	$\Delta[0.9625, 1]$	$\Delta[0.7, 0.875]$	$\Delta[0.275, 0.775]$	$\Delta[0.5375, 0.75]$
G_2	$\Delta[0.5, 0.575]$	$\Delta[0.875, 0.875]$	$\Delta[0.7125, 0.75]$	$\Delta[0.475, 0.875]$	$\Delta[0.75, 0.875]$
G_3	$\Delta[0.4375, 0.525]$	$\Delta[0.25, 0.5]$	$\Delta[0.3875, 0.55]$	$\Delta[0.3875, 0.7875]$	$\Delta[0.4, 0.5375]$
G_4	$\Delta[0.825, 0.875]$	$\Delta[0.6125, 0.825]$	$\Delta[0.6125, 0.8]$	$\Delta[0.675, 0.8]$	$\Delta[0.3, 0.425]$

Table 6 | 2-tuple linguistic weight.

G_1	G_2	G_3	G_4
$\Delta(0.925)$	$\Delta(0.6)$	$\Delta(0.575)$	$\Delta(0.75)$
$(w_4, -0.3)$	$(w_2, 0.4)$	$(w_2, 0.3)$	$(w_3, 0)$

Table 7 The ranking of alternatives.

	Aggregation Results	Score Values	Ranking
A_1	$[(s_4, 0.357), (s_5, 0.3037)]$	0.6292	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
A_2	$[(s_4, 0.163), (s_5, 0.1187)]$	0.5801	
A_3	$[(s_5, -0.3934), (s_6, -0.1986)]$	0.6504	
A_4	$[(s_3, 0.3765), (s_6, 0.3439)]$	0.6079	
A_5	$[(s_4, -0.3168), (s_5, -0.1133)]$	0.5356	

Table 8 Ranking results by using different parameter vector P in EUL2–tuple-WMM operator.

Parameter Vector P	Score Values	Ranking Results
$(1, 0, 0, 0)$	$S(A_1) = 0.6834, S(A_2) = 0.6819,$ $S(A_3) = 0.6932, S(A_4) = 0.6251, S(A_3) = 0.5669$	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
$(1, 1, 0, 0)$	$S(A_1) = 0.6658, S(A_2) = 0.6455,$ $S(A_3) = 0.6798, S(A_4) = 0.6185, S(A_3) = 0.5583$	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
$(1, 1, 1, 0)$	$S(A_1) = 0.6469, S(A_2) = 0.6114,$ $S(A_3) = 0.6657, S(A_4) = 0.6129, S(A_3) = 0.5465$	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
$(1, 1, 1, 1)$	$S(A_1) = 0.6292, S(A_2) = 0.5801,$ $S(A_3) = 0.6504, S(A_4) = 0.6079, S(A_3) = 0.5356$	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$S(A_1) = 0.6292, S(A_2) = 0.5801,$ $S(A_3) = 0.6504, S(A_4) = 0.6079, S(A_3) = 0.5356$	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
$(2, 0, 0, 0)$	$S(A_1) = 0.7337, S(A_2) = 0.7801,$ $S(A_3) = 0.7321, S(A_4) = 0.6442, S(A_3) = 0.6037$	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
$(3, 0, 0, 0)$	$S(A_1) = 0.7751, S(A_2) = 0.8634,$ $S(A_3) = 0.7658, S(A_4) = 0.6639, S(A_3) = 0.6339$	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$

WMM, weighted Muirhead mean; EUL2, extended uncertain 2-tuple linguistic.

Table 9 Ranking results by using different methods.

Aggregation Operator	Parameter Vector	Ranking Results
VIKOR	No	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
EUL$_{2-tuple}$ -WMSM	YES/when $p = (1, 1, 1, 0)$	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
EUL$_{2-tuple}$ -WMM in this paper	YES/when $p = (1, 1, 1, 1)$	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$

WMM, weighted Muirhead mean; EUL2, extended uncertain 2-tuple linguistic.

Table 10 The comparison of different methods.

Methods	Captures Interrelationship of MAs	Makes Method Flexible by PV
VIKOR	×	×
EUL$_{2-tuple}$ -WMSM	√	√
EUL$_{2-tuple}$ -WMM in this paper	√	√

WMM, weighted Muirhead mean; EUL2, extended uncertain 2-tuple linguistic,

environment have been developed. But they still have some limitations in solving some practical problems. Some traditional Maclaurin Symmetric Mean (MSM) operator fails in dealing with the linguistic information. In this paper, the MAGDM problems with the ULI$_{2-tuple}$ are investigated based on new aggregation operator which can captures interrelationships of attributes by a parameter vector P. First of all, we presented some UL$_{2-tuple}$-MM operators to handle MAGDM problems with LI$_{2-tuple}$, including the UL$_{2-tuple}$-MM operator, UL$_{2-tuple}$-WMM operator. In addition, we extend UL$_{2-tuple}$-WMM operator to EUL$_{2-tuple}$-WMM operators in order to handle some decision-making problems with ULI$_{2-tuple}$ whose attribute values are expressed in ULI$_{2-tuple}$ and attribute weight is 2-tuple linguistic information. Whilst, the some properties of these new aggregation operator were obtained. Moreover, we presented a new method to solve the MAGDM problems with ULI$_{2-tuple}$. Finally, we give an illustrative example to indicate the availability of the new methods and some comparisons are also obtained. In further research, it is necessary to solve the real decision-making problems by applying these operators.

The multigranular fuzzy linguistic modeling allows the use of several LTSs in fuzzy linguistic modeling and has been frequently used in GDM field due to its capability of allowing each expert to express his/her preferences using his/her own LTS. Zhang *et al.* [44] defined a new linguistic computational model to deal with multigranular linguistic distribution assessments for its application to large-scale MAGDM problems with linguistic information. In our future research,by means of academic thought of TODIM method based on unbalanced HFLTs [45], we will study the TODIM method for large-scale MAGDM problems with multigranular linguistic information or unbalance unbalanced linguistic information.

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