International Journal of Computational Intelligence Systems
Vol. 12(2), 2019, pp. 580–596
DOI: https://doi.org/10.2991/ijcis.d.190424.001; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

A Novel Method Based on Fuzzy Tensor Technique for Interval-Valued Intuitionistic Fuzzy Decision-Making with High-Dimension Data

Shengyue Deng[1,2], Jianzhou Liu[1,*], Jintao Tan[2], Lixin Zhou[1,3]

[1] Department of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China
[2] School of Science, Hunan University of Technology, Zhuzhou, Hunan 412008, China
[3] Faculty of Science, Guilin University of Aerospace Technology, Guilin, Guangxi 541004, China

ARTICLE INFO

Article History

Received 29 Aug 2018
Accepted 17 Apr 2019

Keywords

Fuzzy tensor
Interval-valued intuitionistic fuzzy tensor
Generalized interval-valued intuitionistic fuzzy weighted averaging (GIIFWA) operator
Generalized interval-valued intuitionistic fuzzy weighted geometric (GIIFWG) operator
Multiple attribute group decision-making
Dynamic multiple attribute group decision-making

ABSTRACT

To solve the interval-valued intuitionistic fuzzy decision-making problems with high-dimension data, the fuzzy matrix is extended to the fuzzy tensor in this paper. Based on the constructed tensor definition, we propose the generalized interval-valued intuitionistic fuzzy weighted averaging (GIIFWA) and generalized interval-valued intuitionistic fuzzy weighted geometric (GIIFWG) operators. By exploring the properties of GIIFWA and GIIFWG operators, a new algorithm is presented to solve the interval-valued intuitionistic fuzzy multiple attribute group decision-making problem. Two typical application examples are also provided to demonstrate the efficiency and universal applicability of our proposed method.

1. INTRODUCTION

As an important branch of decision-making fields, the multiple attribute group decision-making has been paid a close attention in past decades. Normally, multiple attribute group decision-making problems is that multiple decision-makers select the optimal alternatives or ranking them from a set of feasible alternatives by the attribute weights and attribute values, for details refers to Xu and Cai [1]. However in some real applications such as Xu and Cai [1], Liu *et al.* [2], Wang *et al.* [3], Qin *et al.* [4], He [5], and Hashemi *et al.* [6], due to the undetermined decision-making environment, the multi-attribute group decision-making seems to be useless for decision-making. One alternative dealing with this difficulty is the fuzzy set, which was subsequently extended to intuitionistic fuzzy set by Atanassov [7] for applications in various decision-making areas, and Atanassov and Gargov [8] presented the concept and properties of interval-valued intuitionistic fuzzy set based on intuitionistic fuzzy set in 1989, which enriched intuitionistic fuzzy set theory. Especially in recent researches, multiple attribute group decision-making with incorporated interval-valued intuitionistic fuzzy sets has attracted great attentions and yielded plentiful results. For example, Xu [9] developed a method based on distance measure for group decision-making with interval-valued intuitionistic fuzzy matrices. Kabak and Ervural [10] devised a generic conceptual framework and a classification scheme for multiple attribute group decision-making methods. Yang *et al.* [11] proposed a new method based on dynamic intuitionistic normal fuzzy aggregation operators and VIKOR method with time sequence preference for the dynamic intuitionistic normal fuzzy multi-attribute decision-making problems. Liu [12] proposed the interval-valued intuitionistic fuzzy power Heronian aggregation operator and interval-valued intuitionistic fuzzy power weight Heronian aggregation operator for the multiple attribute group decision-making. Chen and Huang [13] proposed a new multi-attribute decision-making method by the interval-valued intuitionistic fuzzy weighted geometric average (IIFWGA) operator and the accuracy function of interval-valued intuitionistic fuzzy values. Wang and Chen [14] proposed an improved multiple attribute decision-making method by the score function S_{WC} of interval-valued

*Corresponding author: Email: liujz@xtu.edu.cn

intuitionistic fuzzy values and the linear programming methodology. Qiu and Li [15] employed the plant growth simulation algorithm (PGSA) to calculate the optimal preferences of the entire expert group and proposed a new method to solve the multi-attribute group decision-making problem.

However the above mentioned models which are based on matrix frame meet with difficulties in processing higher dimension data and might lose their efficiency. To tackle this problem, we introduce a new developed tensor model which is a generalization of matrix. The concepts of higher-order tensor eigenvalues and eigenvectors were introduced by [16] and [17]. Subsequently, the theory and algorithms of some special tensors and the spectra of tensors with their various applications have attracted wide attention [18–31]. For example, Ding and Wei [18,19] investigated the solutions of some structured multi-linear systems whose coefficient tensor is M-tensor. Qi [20] proved two new spectral properties and a maximum property of the largest H-eigenvalue in a symmetric nonnegative tensor system. Ni *et al.* [21] obtained an upper bound of different US-eigenvalues and the count of US-eigenpairs corresponding to all nonzero eigenvalues in the symmetric tensors. Ng *et al.* [22] proposed an iterative method to calculate the largest eigenvalue of an irreducible nonnegative tensor. Rajesh Kannan *et al.* [23] gained some properties of strong H-tensors and (general) H-tensors. Based on the diagonal product dominance and S diagonal product dominance of tensor, Wang *et al.* [24] established some new implementable attribute which can be used for identifying nonsingular H-tensor. By studying the general product of two n-dimensional tensors $\mathcal{A}$ and $\mathcal{B}$ with orders $m \geqslant 2$ and $k \geqslant 1$, Shao *et al.* [25,26] found that the product is a generalization of the usual matrix product and it satisfies the associative law. Bu *et al.* [27] gave some basic properties for the left (right) inverse, rank, and product of tensors. Pumplün [28] studied the tensor product of an associative and a nonassociative cyclic algebra. Giladi *et al.* [29] studied the volume ratio of the projective tensor products $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ with $1 \leqslant p \leqslant q \leqslant r \leqslant \infty$ and obtained asymptotic formulas that are sharp in almost all cases. Gutiérrez García *et al.* [30] employed tensor products of complete lattices into fuzzy set theory. Hilberdink [31] studied operators having (infinite) matrix representations and gave such operators infinite tensor products over the primes. Moreover, we have defined the concept of fuzzy tensor and established the general form of the fuzzy synthetic evaluation model for solving multiple attribute group decision-making problems [32].

Based on the research results we have achieved [32], we will propose two new generalized aggregation operators based on interval-valued intuitionistic fuzzy tensor for solving the interval-valued intuitionistic fuzzy multiple attribute group decision-making problem. Specifically, we will first establish the generalized interval-valued intuitionistic fuzzy weighted averaging (GIIFWA) and generalized interval-valued intuitionistic fuzzy weighted geometric (GIIFWG) operators. Then some properties about those new generalized aggregation operators are developed and a new algorithm is presented for the corresponding decision-making problems. Indeed as shown in numerical experiments, the proposed interval-valued intuitionistic fuzzy tensor model does provide a new way for solving multiple attribute group decision-making problems with high-dimension data.

The whole paper is arranged as follows: In Section 2, we introduce some concepts and properties of the fuzzy tensor and interval-valued intuitionistic fuzzy aggregation. Section 3 is devoted to the derivation of the GIIFWA and GIIFWG operators by the product of tensor and vector, and gives some properties of two new generalized aggregation operators. In Section 4, we present an algorithm for solving the interval-valued intuitionistic fuzzy multiple attribution group decision-making problems. In Section 5, two different application examples are shown for illustrating the proposed approach. A conclusion is finally drawn in Section 6.

2. PRELIMINARIES

This section provides basic preliminaries about the fuzzy tensor, interval-valued intuitionistic fuzzy set, and interval-valued intuitionistic fuzzy information aggregation theory.

Let $\mathbf{R}$ be the real field and $\mathbf{F}$ and $\mathbf{IVIF}$ be the fuzzy set and interval-valued intuitionistic fuzzy set defined in universe $\mathbf{R}$, respectively. The $\mathbf{T_R}(m,n)$, $\mathbf{T_F}(m,n)$, and $\mathbf{T_{IVIF}}(m,n)$ denote the set of all mth-order n-dimension real tensors, fuzzy tensors, and interval-valued intuitionistic fuzzy tensors, respectively, and $[n] = \{1, 2, \cdots, n\}$. $\mathbf{F^n}$ and $\mathbf{IVIF^n}$ denote the n-dimensional fuzzy vector in the $\mathbf{F}$ and n-dimensional interval-valued intuitionistic fuzzy vector in the $\mathbf{IVIF}$, respectively.

Definition 2.1. [8] Let $\mathbf{X}$ be a finite nonempty set. Then

$$\tilde{A} = \{\langle x, \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x)\rangle | x \in \mathbf{X}\}$$

is called an interval-valued intuitionistic fuzzy set, where $\tilde{\mu}_{\tilde{A}}(x) \subset [0,1]$ and $\tilde{\nu}_{\tilde{A}}(x) \subset [0,1]$, $x \in \mathbf{X}$, with the condition:

$$\sup\ \tilde{\mu}_{\tilde{A}}(x) + \sup\ \tilde{\nu}_{\tilde{A}}(x) \leqslant 1, x \in \mathbf{X}$$

Note: For convenience, the interval-valued intuitionistic fuzzy numbers (IVIFNs) [33] can be denoted as $\tilde{A} = \left(\left[\mu_{\tilde{A}}^l(x), \mu_{\tilde{A}}^u(x)\right], \left[\nu_{\tilde{A}}^l(x), \nu_{\tilde{A}}^u(x)\right]\right)$ in this paper, where

$$\left[\mu_{\tilde{A}}^l, \mu_{\tilde{A}}^u\right] \subset [0,1], \left[\nu_{\tilde{A}}^l, \nu_{\tilde{A}}^u\right] \subset [0,1], \mu_{\tilde{A}}^u + \nu_{\tilde{A}}^u \leqslant 1.$$

and $\left[\mu_{\tilde{A}}^l, \mu_{\tilde{A}}^u\right]$ and $\left[\nu_{\tilde{A}}^l, \nu_{\tilde{A}}^u\right]$ represent the supported interval and opposed interval about an evaluation object, respectively.

Definition 2.2. [33] Let $\tilde{\alpha} = \left(\left[\mu_{\tilde{\alpha}}^l, \mu_{\tilde{\alpha}}^u\right], \left[\nu_{\tilde{\alpha}}^l, \nu_{\tilde{\alpha}}^u\right]\right)$, $\tilde{\alpha}_1 = \left(\left[\mu_{\tilde{\alpha}_1}^l, \mu_{\tilde{\alpha}_1}^u\right], \left[\nu_{\tilde{\alpha}_1}^l, \nu_{\tilde{\alpha}_1}^u\right]\right)$ and $\tilde{\alpha}_2 = \left(\left[\mu_{\tilde{\alpha}_2}^l, \mu_{\tilde{\alpha}_2}^u\right], \left[\nu_{\tilde{\alpha}_2}^l, \nu_{\tilde{\alpha}_2}^u\right]\right)$ be IVIFNs. Then

1. $\overline{\tilde{\alpha}} = \left(\left[\nu_{\tilde{\alpha}}^l, \nu_{\tilde{\alpha}}^u\right], \left[\mu_{\tilde{\alpha}}^l, \mu_{\tilde{\alpha}}^u\right]\right)$, where $\overline{\tilde{\alpha}}$ is the complement of $\tilde{\alpha}$.
2. $$\tilde{\alpha}_1 \wedge \tilde{\alpha}_2 = \left(\left[\min\left\{\mu_{\tilde{\alpha}_1}^l, \mu_{\tilde{\alpha}_2}^l\right\}, \min\left\{\mu_{\tilde{\alpha}_1}^u, \mu_{\tilde{\alpha}_2}^u\right\}\right], \left[\max\left\{\nu_{\tilde{\alpha}_1}^l, \nu_{\tilde{\alpha}_2}^l\right\}, \max\left\{\nu_{\tilde{\alpha}_1}^u, \nu_{\tilde{\alpha}_2}^u\right\}\right]\right);$$
3. $$\tilde{\alpha}_1 \vee \tilde{\alpha}_2 = \left(\left[\max\left\{\mu_{\tilde{\alpha}_1}^l, \mu_{\tilde{\alpha}_2}^l\right\}, \max\left\{\mu_{\tilde{\alpha}_1}^u, \mu_{\tilde{\alpha}_2}^u\right\}\right], \left[\min\left\{\nu_{\tilde{\alpha}_1}^l, \nu_{\tilde{\alpha}_2}^l\right\}, \min\left\{\nu_{\tilde{\alpha}_1}^u, \nu_{\tilde{\alpha}_2}^u\right\}\right]\right);$$
4. $\tilde{\alpha}_1 + \tilde{\alpha}_2 = \left(\left[\mu_{\tilde{\alpha}_1}^l + \mu_{\tilde{\alpha}_2}^l - \mu_{\tilde{\alpha}_1}^l\mu_{\tilde{\alpha}_2}^l, \mu_{\tilde{\alpha}_1}^u + \mu_{\tilde{\alpha}_2}^u - \mu_{\tilde{\alpha}_1}^u\mu_{\tilde{\alpha}_2}^u\right], \left[\nu_{\tilde{\alpha}_1}^l\nu_{\tilde{\alpha}_2}^l, \nu_{\tilde{\alpha}_1}^u\nu_{\tilde{\alpha}_2}^u\right]\right);.$

5. $\tilde{\alpha}_1 \cdot \tilde{\alpha}_2 = \left([\mu^l_{\tilde{\alpha}_1}\mu^l_{\tilde{\alpha}_2}, \mu^u_{\tilde{\alpha}_1}\mu^u_{\tilde{\alpha}_2}], [\nu^l_{\tilde{\alpha}_1} + \nu^l_{\tilde{\alpha}_2} - \nu^l_{\tilde{\alpha}_1}\nu^l_{\tilde{\alpha}_2}, \nu^u_{\tilde{\alpha}_1} + \nu^u_{\tilde{\alpha}_2} - \nu^u_{\tilde{\alpha}_1}\nu^u_{\tilde{\alpha}_2}]\right)$.
6. $\lambda\tilde{\alpha} = \left(\left[1-(1-\mu^l_{\tilde{\alpha}})^\lambda, 1-(1-\mu^u_{\tilde{\alpha}})^\lambda\right], \left[(\nu^l_{\tilde{\alpha}})^\lambda, (\nu^u_{\tilde{\alpha}})^\lambda\right]\right), \lambda > 0;$
7. $\tilde{\alpha}^\lambda = \left(\left[(\mu^l_{\tilde{\alpha}})^\lambda, (\mu^u_{\tilde{\alpha}})^\lambda\right], \left[1-(1-\nu^l_{\tilde{\alpha}})^\lambda, 1-(1-\nu^u_{\tilde{\alpha}})^\lambda\right]\right), \lambda > 0.$

Definition 2.3. [16] Let $\mathcal{A} \in \mathbf{T_R}(m, n_1 \times n_2 \times \cdots \times n_m)$, and its elements $a_{i_1 i_2 \cdots i_m} \in \mathbf{R}$ where $i_1 \in [n_1], i_2 \in [n_2], \cdots, i_m \in [n_m]$. Then $\mathcal{A}$ is called a mth-order tensor.

Note: According to the **Definition 2.3**, we know that the matrix is the 2nd-order tensor.

Definition 2.4. [32] Let $\tilde{\mathcal{A}} \in \mathbf{T_F}(m, n_1 \times n_2 \times \cdots \times n_m)$, and its elements $a_{i_1 i_2 \cdots i_m} \in [0,1]$ where $i_1 \in [n_1], i_2 \in [n_2], \cdots, i_m \in [n_m]$, then $\tilde{\mathcal{A}}$ is called a mth-order fuzzy tensor.

Definition 2.5. Let $\tilde{\mathcal{A}}_{IVIF} = \left(a_{i_1 i_2 \cdots i_m}\right)_{n_1 \times n_2 \times \cdots \times n_m} \in \mathbf{T_{IVIF}}(m, n_1 \times n_2 \times \cdots \times n_m)$, and its elements $a_{i_1 i_2 \cdots i_m} = \left(\left[u^l_{i_1 i_2 \cdots i_m}, \mu^u_{i_1 i_2 \cdots i_m}\right], \left[\nu^l_{i_1 i_2 \cdots i_m}, \nu^u_{i_1 i_2 \cdots i_m}\right]\right)$ where $\left[\mu^l_{i_1 i_2 \cdots i_m}, \mu^u_{i_1 i_2 \cdots i_m}\right] \subset [0,1]$, $\left[\nu^l_{i_1 i_2 \cdots i_m}, \nu^u_{i_1 i_2 \cdots i_m}\right] \subset [0,1]$ satisfy the condition

$$\mu^u_{i_1 i_2 \cdots i_m} + \nu^u_{i_1 i_2 \cdots i_m} \leqslant 1,$$

and the interval $\left[\mu^l_{i_1 i_2 \cdots i_m}, \mu^u_{i_1 i_2 \cdots i_m}\right]$ and $\left[\nu^l_{i_1 i_2 \cdots i_m}, \nu^u_{i_1 i_2 \cdots i_m}\right]$ denote the supported interval and opposed interval about an evaluation object, respectively. Then $\tilde{\mathcal{A}}_{IVIF}$ is called a mth-order interval-valued intuitionistic fuzzy tensor.

Definition 2.6. [1] Let $\tilde{\alpha}_i\,(i = 1, 2, \cdots, n)$ be a collection of IVIFNs, and let IIFWA: $\mathbf{F^n_{IVIF}} \to \mathbf{F_{IVIF}}$. If

$$\mathrm{IIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \omega_1\tilde{\alpha}_1 + \omega_2\tilde{\alpha}_2 + \cdots + \omega_n\tilde{\alpha}_n$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_i\,(i = 1, 2, \cdots, n)$, with $\omega_i \in [0,1]\,(i = 1, 2, \cdots, n)$, and $\sum_{i=1}^{n} \omega_i = 1$, then the function IIFWA is called an interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator.

Definition 2.7. [1] Let IIFWG: $\mathbf{F^n_{IVIF}} \to \mathbf{F_{IVIF}}$. If

$$\mathrm{IIFWG}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \tilde{\alpha}_1^{\omega_1} \cdot \tilde{\alpha}_2^{\omega_2} \cdot \cdots \cdot \tilde{\alpha}_n^{\omega_n}$$

then the function IIFWG is called an interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator.

Definition 2.8. [33] Let $\tilde{\alpha} = \left([\mu^l_{\tilde{\alpha}}, \mu^u_{\tilde{\alpha}}], [\nu^l_{\tilde{\alpha}}, \nu^u_{\tilde{\alpha}}]\right)$ be an IVIFN. Then we call

$$s(\tilde{\alpha}) = \frac{1}{2}\left(\mu^l_{\tilde{\alpha}} - \nu^l_{\tilde{\alpha}} + \mu^u_{\tilde{\alpha}} - \nu^u_{\tilde{\alpha}}\right)$$

the score of $\tilde{\alpha}$, where s is the score function of $\tilde{\alpha}$, $s(\tilde{\alpha}) \in [-1, 1]$.

Definition 2.9. [33] The accuracy function of an IVIFN $\tilde{\alpha}$ is defined as

$$h(\tilde{\alpha}) = \frac{1}{2}\left(\mu^l_{\tilde{\alpha}} + \mu^u_{\tilde{\alpha}} + \nu^l_{\tilde{\alpha}} + \nu^u_{\tilde{\alpha}}\right)$$

where $h(\tilde{\alpha}) \in [0, 1]$.

Definition 2.10. [33] Let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be any two IVIFNs. Then

1. If $s(\tilde{\alpha}_1) < s(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.
2. If $s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2)$, then
   (a) If $h(\tilde{\alpha}_1) < h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.
   (b) If $h(\tilde{\alpha}_1) > h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.
   (c) If $h(\tilde{\alpha}_1) = h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \sim \tilde{\alpha}_2$.

Definition 2.11. [16] Suppose that $\mathcal{A} = \left(a_{i_1 i_2 \cdots i_m}\right)_{n_1 \times n_2 \times \cdots \times n_m} \in \mathbf{T_R}(m, n_1 \times n_2 \times \cdots \times n_m)$ is a mth-order tensor, and $X_j = \left(x^j_1, x^j_2, \cdots, x^j_n\right)^T \in \mathbf{R}^{n_j}\,(j \in [m-1])$ is a n_j-dimension vector, then the i_mth component of the vector $\mathcal{A} \circ X_1 \circ X_2 \cdots \circ X_{m-1}$ in $\mathbf{R}^{n_m}$ is defined as the following:

$$\left(\mathcal{A} \circ X_1 \circ X_2 \cdots \circ X_{m-1}\right)_{i_m} = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{m-1}=1}^{n_{m-1}} a_{i_1 i_2 \cdots i_m} x^1_{i_1} x^2_{i_2} \cdots x^{m-1}_{i_{m-1}}.$$

Definition 2.12. [34] Let U and V be universes and $\mathbf{F}(V)$ be the set of all fuzzy sets in V (power set).

- $f: U \to \mathbf{F}(V)$ is a mapping
- f is a fuzzy function iff

$$\mu_{f(u)}(v) = \mu_{\tilde{R}}(u, v), \forall (u, v) \in U \times V,$$

where $\mu_{\tilde{R}}(u, v)$ is the membership function of a fuzzy relation.

Note: The mapping f in **Definition 2.12** is also a fuzzy mapping.

Definition 2.13. Let $\tilde{\mathcal{A}}_{IVIF} = \left(a_{i_1 i_2 \cdots i_m}\right)_{n_1 \times n_2 \times \cdots \times n_m} \in \mathbf{T_{IVIF}}(m, n_1 \times n_2 \times \cdots \times n_m)$, and let the function GIIFWA: $\mathbf{F^{n_2 \times \cdots \times n_m}_{IVIF}} \to \mathbf{F^{n_1}_{IVIF}}$. If

$$\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right) = \sum_{i_2}^{n_2} \cdots \sum_{i_m}^{n_m} a_{i_1 i_2 \cdots i_m} \cdot x^2_{i_2} \cdot \cdots \cdot x^m_{i_m} \quad (1)$$

where $X_2 = \left(x^2_1, \cdots, x^2_{i_2}, \cdots, x^2_{n_2}\right)^T, \cdots, X_m = \left(x^m_1, \cdots, x^m_{i_m}, \cdots, x^m_{n_m}\right)^T$ are the weight vectors of $a_{:i_2:\cdots:}\,(i_2 = 1, 2, \cdots, n_2), \cdots, a_{:\cdots:i_m}\,(i_m = 1, 2, \cdots, n_m)$, respectively, and $\sum_{i_2=1}^{n_2} x^2_{i_2} = 1, x^2_{i_2} \geqslant 0; \cdots; \sum_{i_m=1}^{n_m} x^m_{i_m} = 1, x^m_{i_m} \geqslant 0$, then the function GIIFWA is called the GIIFWA operator.

Definition 2.14. Suppose that the function GIIFWG: $\mathbf{F^{n_2 \times \cdots \times n_m}_{IVIF}} \to \mathbf{F^{n_1}_{IVIF}}$. If

$$\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right) = \prod_{i_2}^{n_2} \cdots \prod_{i_m}^{n_m} \left(a_{i_1 i_2 \cdots i_m}\right)^{x^2_{i_2} \cdot \cdots \cdot x^m_{i_m}} \quad (2)$$

then the function GIIFWG is called the GIIFWG operator.

3. GENERALIZED INTERVAL-VALUED INTUITIONISTIC FUZZY AGGREGATION OPERATORS BASED ON FUZZY TENSOR TECHNIQUE

Since the interval-valued intuitionistic fuzzy information aggregation is helpful for dealing with fuzzy multiple attribute decision-making problem, we will first develop, in this section, the GIIFWA and GIIFWG operators by the product of the mth-order fuzzy tensor with vector. Then both the GIIFWA and the GIIFWG operators are proved to having properties of idempotency and boundedness, which lays a theoretical foundation for the algorithm to solve the fuzzy multiple attribute group decision-making problems in next section.

Theorem 3.1. *Let* $\tilde{\mathcal{A}}_{IVIF} = \left(a_{i_1 i_2 \cdots i_m}\right)_{n_1 \times n_2 \times \cdots \times n_m} \in \boldsymbol{T_{IVIF}}(m, n_1 \times n_2 \times \cdots \times n_m)$ *be a mth-order interval-valued intuitionistic fuzzy tensor, and its elements* $a_{i_1 i_2 \cdots i_m} = \left(\left[\mu^l_{i_1 i_2 \cdots i_m}, \mu^u_{i_1 i_2 \cdots i_m}\right], \left[\nu^l_{i_1 i_2 \cdots i_m}, \nu^u_{i_1 i_2 \cdots i_m}\right]\right)$. *Then the aggregated value by using Equation (1) is*

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right)\\
&= \left(\left[1 - \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} \left(1 - \mu^l_{i_1 i_2 \cdots i_m}\right)^{x^2_{i_2} \cdots x^m_{i_m}},\right.\right.\\
&\left. 1 - \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} \left(1 - \mu^u_{i_1 i_2 \cdots i_m}\right)^{x^2_{i_2} \cdots x^m_{i_m}}\right],\\
&\left[\prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} \left(\nu^l_{i_1 i_2 \cdots i_m}\right)^{x^2_{i_2} \cdots x^m_{i_m}},\right.\\
&\left.\left.\prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} \left(\nu^u_{i_1 i_2 \cdots i_m}\right)^{x^2_{i_2} \cdots x^m_{i_m}}\right]\right)
\end{aligned}$$

where $X_2 = \left(x^2_1, \cdots, x^2_{i_2}, \cdots, x^2_{n_2}\right)^T, \cdots, X_m = \left(x^m_1, \cdots, x^m_{i_m}, \cdots, x^m_{n_m}\right)^T$ *are the weight vectors of* $a_{:i_2:\cdots:}\ (i_2 = 1, 2, \cdots, n_2), \cdots, a_{:\cdots:i_m}\ (i_m = 1, 2, \cdots, n_m)$, *respectively, and* $\sum_{i_2=1}^{n_2} x^2_{i_2} = 1, x^2_{i_2} \geqslant 0; \cdots; \sum_{i_m=1}^{n_m} x^m_{i_m} = 1, x^m_{i_m} \geqslant 0.$

Proof. We prove the **Theorem 3.1** by using mathematical induction on $n_2, \cdots, n_m$.

1. When $n_2 = \cdots = n_m = 1$, we have

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right) = a_{i_1 1 \cdots 1} x^2_1 \cdots x^m_1\\
&= \left(\left[1 - \left(1 - \mu^l_{i_1 1 \cdots 1}\right)^{x^2_1 \cdots x^m_1}, 1 - \left(1 - \mu^u_{i_1 1 \cdots 1}\right)^{x^2_1 \cdots x^m_1}\right],\right.\\
&\left.\left[\left(\nu^l_{i_1 1 \cdots 1}\right)^{x^2_1 \cdots x^m_1}, \left(\nu^u_{i_1 1 \cdots 1}\right)^{x^2_1 \cdots x^m_1}\right]\right).
\end{aligned}$$

2. Let $\mathbf{I}_1 = \{2, 3, \cdots, m\}$ and $\mathbf{I}_2 = \{n_2, n_3, \cdots, n_m\}$ be indicator sets. When at least one element in the indicator set $\mathbf{I}_2$ add to "1," then we consider the following cases:
   (a) When $j \in \mathbf{I}_1$ and $n_j = 2$, then we have

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right)\\
&= \sum_{i_j=1}^{2} a_{i_1 1 \cdots i_j \cdots 1} x^2_1 \cdots x^j_{i_j} \cdots x^m_1\\
&= \left(\left[1 - \prod_{i_j=1}^{2} \left(1 - \mu^l_{i_1 1 \cdots i_j \cdots 1}\right)^{x^2_1 \cdots x^j_{i_j} \cdots x^m_1},\right.\right.\\
&\left. 1 - \prod_{i_j=1}^{2} \left(1 - \mu^u_{i_1 1 \cdots i_j \cdots 1}\right)^{x^2_1 \cdots x^j_{i_j} \cdots x^m_1}\right],\\
&\left.\left[\prod_{i_j=1}^{2} \left(\nu^l_{i_1 1 \cdots i_j \cdots 1}\right)^{x^2_1 \cdots x^j_{i_j} \cdots x^m_1}, \prod_{i_j=1}^{2} \left(\nu^u_{i_1 1 \cdots i_j \cdots 1}\right)^{x^2_1 \cdots x^j_{i_j} \cdots x^m_1}\right]\right).
\end{aligned}$$

   (b) When $j1, j2 \in \mathbf{I}_1$ $(j1 \neq j2)$ and $n_{j1} = n_{j2} = 2$, then we have

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right)\\
&= \sum_{i_{j1}=1}^{2} \sum_{i_{j2}=1}^{2} a_{i_1 1 \cdots i_{j1} \cdots i_{j2} \cdots 1} x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{j2}_{i_{j2}} \cdots x^m_1\\
&= \left(\left[1 - \prod_{i_{j1}=1}^{2} \prod_{i_{j2}=1}^{2} \left(1 - \mu^l_{i_1 1 \cdots i_{j1} \cdots i_{j2} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{j2}_{i_{j2}} \cdots x^m_1},\right.\right.\\
&\left. 1 - \prod_{i_{j1}=1}^{2} \prod_{i_{j2}=1}^{2} \left(1 - \mu^u_{i_1 1 \cdots i_{j1} \cdots i_{j2} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{j2}_{i_{j2}} \cdots x^m_1}\right],\\
&\left[\prod_{i_{j1}=1}^{2} \prod_{i_{j2}=1}^{2} \left(\nu^l_{i_1 1 \cdots i_{j1} \cdots i_{j2} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{j2}_{i_{j2}} \cdots x^m_1},\right.\\
&\left.\left.\prod_{i_{j1}=1}^{2} \prod_{i_{j2}=1}^{2} \left(\nu^u_{i_1 1 \cdots i_{j1} \cdots i_{j2} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{j2}_{i_{j2}} \cdots x^m_1}\right]\right).\\
&\cdots,
\end{aligned}$$

   (c) When $j1, j2, \cdots, jl \in \mathbf{I}_1$ $(j1 \neq j2 \neq \cdots \neq jl)$ and $n_{j1} = n_{j2} = \cdots = n_{jl} = 2$, then we have

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right)\\
&= \sum_{i_{j1}=1}^{2} \cdots \sum_{i_{jl}=1}^{2} a_{i_1 1 \cdots i_{j1} \cdots i_{jl} \cdots 1} x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{jl}_{i_{jl}} \cdots x^m_1\\
&= \left(\left[1 - \prod_{i_{j1}=1}^{2} \cdots \prod_{i_{jl}=1}^{2} \left(1 - \mu^l_{i_1 1 \cdots i_{j1} \cdots i_{jl} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{jl}_{i_{jl}} \cdots x^m_1},\right.\right.\\
&\left. 1 - \prod_{i_{j1}=1}^{2} \cdots \prod_{i_{jl}=1}^{2} \left(1 - \mu^u_{i_1 1 \cdots i_{j1} \cdots i_{jl} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{jl}_{i_{jl}} \cdots x^m_1}\right],\\
&\left[\prod_{i_{j1}=1}^{2} \cdots \prod_{i_{jl}=1}^{2} \left(\nu^l_{i_1 1 \cdots i_{j1} \cdots i_{jl} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{jl}_{i_{jl}} \cdots x^m_1},\right.\\
&\left.\left.\prod_{i_{j1}=1}^{2} \cdots \prod_{i_{jl}=1}^{2} \left(\nu^u_{i_1 1 \cdots i_{j1} \cdots i_{jl} \cdots 1}\right)^{x^2_1 \cdots x^{j1}_{i_{j1}} \cdots x^{jl}_{i_{jl}} \cdots x^m_1}\right]\right).\\
&\cdots,
\end{aligned}$$

   (d) When all the elements in the indicator $\mathbf{I}_2$ add to "1," that is, $n_2 = \cdots = n_m = 2$, then we have

$$\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right)$$

$$
\begin{aligned}
&= \sum_{i_2=1}^{2}\cdots\sum_{i_m=1}^{2} a_{i_1 i_2\cdots i_m} x_{i_2}^2\cdots x_{i_m}^m \\
&= \left(\left[1-\prod_{i_2=1}^{2}\cdots\prod_{i_m=1}^{2}\left(1-\mu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m},\right.\right. \\
&\qquad \left. 1-\prod_{i_2=1}^{2}\cdots\prod_{i_m=1}^{2}\left(1-\mu^u_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}\right], \\
&\qquad \left[\prod_{i_2=1}^{2}\cdots\prod_{i_m=1}^{2}\left(\nu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m},\right. \\
&\qquad \left.\left.\prod_{i_2=1}^{2}\cdots\prod_{i_m=1}^{2}\left(\nu^u_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}\right]\right).
\end{aligned}
$$

Therefore, according to the above analysis, when at least one element in the indicator set $\mathbf{I}_2$ add to "1," the **Theorem 3.1** holds.

3. Suppose that $n_2 = K_2, n_3 = K_3, \cdots, n_m = K_m$, the **Theorem 3.1** holds, that is,

$$
\begin{aligned}
&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right) \\
&= \sum_{i_2=1}^{K_2}\cdots\sum_{i_m=1}^{K_m} a_{i_1 i_2\cdots i_m} x_{i_2}^2\cdots x_{i_m}^m \\
&= \left(\left[1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_m=1}^{K_m}\left(1-\mu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m},\right.\right. \\
&\qquad \left. 1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_m=1}^{K_m}\left(1-\mu^u_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}\right], \\
&\qquad \left[\prod_{i_2=1}^{K_2}\cdots\prod_{i_m=1}^{K_m}\left(\nu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m},\right. \\
&\qquad \left.\left.\prod_{i_2=1}^{K_2}\cdots\prod_{i_m=1}^{K_m}\left(\nu^u_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}\right]\right).
\end{aligned}
$$

Let $\mathbf{I}_3 = \{K_2, K_3, \cdots, K_m\}$ be an indicator set. When at least one element in the indicator set $\mathbf{I}_3$ add to "1," then we consider the following cases:

(a) When $j \in \mathbf{I}_1$ and $n_j = K_j + 1$, then we have

$$
\begin{aligned}
&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right) \\
&= \sum_{i_2=1}^{K_2}\cdots\sum_{i_j=1}^{K_j+1}\cdots\sum_{i_m=1}^{K_m} a_{i_1 i_2\cdots i_j\cdots i_m} x_1^2\cdots x_{i_j}^j\cdots x_1^m \\
&= \left(\left[1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_j=1}^{K_j+1}\cdots\prod_{i_m=1}^{K_m}\left(1-\mu^l_{i_1 i_2\cdots i_j\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_j}^j\cdots x_{i_m}^m},\right.\right. \\
&\qquad \left. 1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_j=1}^{K_j+1}\cdots\prod_{i_m=1}^{K_m}\left(1-\mu^u_{i_1 i_2\cdots i_j\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_j}^j\cdots x_{i_m}^m}\right], \\
&\qquad \left[\prod_{i_2=1}^{K_2}\cdots\prod_{i_j=1}^{K_j+1}\cdots\prod_{i_m=1}^{K_m}\left(\nu^l_{i_1 i_2\cdots i_j\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_j}^j\cdots x_{i_m}^m},\right. \\
&\qquad \left.\left.\prod_{i_2=1}^{K_2}\cdots\prod_{i_j=1}^{K_j+1}\cdots\prod_{i_m=1}^{K_m}\left(\nu^u_{i_1 i_2\cdots i_j\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_j}^j\cdots x_{i_m}^m}\right]\right).
\end{aligned}
$$

(b) When $j1, j2 \in \mathbf{I}_1$ $(j1 \neq j2)$ and $n_{j1} = K_{j1} + 1$, $n_{j2} = K_{j2} + 1$, then we have

$$
\begin{aligned}
&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right) \\
&= \sum_{i_2=1}^{K_2}\cdots\sum_{i_{j1}=1}^{K_{j1}+1}\cdots\sum_{i_{j2}=1}^{K_{j2}+1}\cdots\sum_{i_m=1}^{K_m} a_{i_1 i_2\cdots i_{j1}\cdots i_{j2}\cdots i_m} x_{i_2}^2 \\
&\qquad \cdots x_{i_{j1}}^{j1}\cdots x_{i_{j2}}^{j2}\cdots x_{i_m}^m
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right) \\
&= \sum_{i_2=1}^{K_2}\cdots\sum_{i_{j1}=1}^{K_{j1}+1}\cdots\sum_{i_{j2}=1}^{K_{j2}+1}\cdots\sum_{i_m=1}^{K_m} a_{i_1 i_2\cdots i_{j1}\cdots i_{j2}\cdots i_m} x_{i_2}^2 \\
&\qquad \cdots x_{i_{j1}}^{j1}\cdots x_{i_{j2}}^{j2}\cdots x_{i_m}^m \\
&= \left(\left[1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1}+1}\cdots\prod_{i_{j2}=1}^{K_{j2}+1}\cdots\right.\right. \\
&\qquad \prod_{i_m=1}^{K_m}\left(1-\mu^l_{i_1 i_2\cdots i_{j1}\cdots i_{j2}\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_{j1}}^{j1}\cdots x_{i_{j2}}^{j2}\cdots x_{i_m}^m}, \\
&\qquad 1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1}+1}\cdots\prod_{i_{j2}=1}^{K_{j2}+1}\cdots \\
&\qquad \left.\prod_{i_m=1}^{K_m}\left(1-\mu^u_{i_1 i_2\cdots i_{j1}\cdots i_{j2}\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_{j1}}^{j1}\cdots x_{i_{j2}}^{j2}\cdots x_{i_m}^m}\right], \\
&\qquad \left[\prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1}+1}\cdots\prod_{i_{j2}=1}^{K_{j2}+1}\cdots\right. \\
&\qquad \prod_{i_m=1}^{K_m}\left(\nu^l_{i_1 i_2\cdots i_{j1}\cdots i_{j2}\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_{j1}}^{j1}\cdots x_{i_{j2}}^{j2}\cdots x_{i_m}^m}, \\
&\qquad \prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1}+1}\cdots\prod_{i_{j2}=1}^{K_{j2}+1}\cdots \\
&\qquad \left.\left.\prod_{i_m=1}^{K_m}\left(\nu^u_{i_1 i_2\cdots i_{j1}\cdots i_{j2}\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_{j1}}^{j1}\cdots x_{i_{j2}}^{j2}\cdots x_{i_m}^m}\right]\right). \\
&\cdots,
\end{aligned}
$$

(c) When $j1, j2, \cdots, jl \in \mathbf{I}_1$ $(j1 \neq j2 \neq \cdots \neq jl)$ and $n_{j1} = K_{j1} + 1, n_{j2} = K_{j2} + 1, \cdots, n_{jl} = K_{jl} + 1$, then we have

$$
\begin{aligned}
&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right) \\
&= \sum_{i_2=1}^{K_2}\cdots\sum_{i_{j1}=1}^{K_{j1}+1}\cdots\sum_{i_{jl}=1}^{K_{jl}+1}\cdots\sum_{i_m=1}^{K_m} a_{i_1 i_2\cdots i_{j1}\cdots i_{j2}\cdots i_m} x_{i_2}^2 \\
&\qquad \cdots x_{i_{j1}}^{j1}\cdots x_{i_{jl}}^{jl}\cdots x_{i_m}^m \\
&= \left(\left[1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1}+1}\cdots\prod_{i_{jl}=1}^{K_{jl}+1}\cdots\right.\right. \\
&\qquad \prod_{i_m=1}^{K_m}\left(1-\mu^l_{i_1 i_2\cdots i_{j1}\cdots i_{jl}\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_{j1}}^{j1}\cdots x_{i_{jl}}^{jl}\cdots x_{i_m}^m}, \\
&\qquad 1-\prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1}+1}\cdots\prod_{i_{jl}=1}^{K_{jl}+1}\cdots \\
&\qquad \left.\prod_{i_m=1}^{K_m}\left(1-\mu^u_{i_1 i_2\cdots i_{j1}\cdots i_{jl}\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_{j1}}^{j1}\cdots x_{i_{jl}}^{jl}\cdots x_{i_m}^m}\right], \\
&\qquad \left[\prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1+1}}\cdots\prod_{i_{j1}=1}^{K_{j1+1}}\cdots\prod_{i_m=1}^{K_m}\right.
\end{aligned}
$$

$$\left(\nu^{l}_{i_1 i_2\cdots i_{j1}\cdots i_{jl}\cdots i_{jm}}\right)^{x^2_{i_2}\cdots x^{j1}_{j1}\cdots x^{jl}_{jl}\cdots x^m_{in}},$$

$$\prod_{i_2=1}^{K_2}\cdots\prod_{i_{j1}=1}^{K_{j1+1}}\cdots\prod_{i_{j1}=1}^{K_{jl+1}}\cdots\prod_{i_m=1}^{K_m}\left(\nu^{u}_{i_1 i_2\cdots i_{j1}\cdots i_{jl}\cdots i_{jm}}\right)^{x^2_{i_2}\cdots x^{j1}_{j1}\cdots x^{jl}_{jl}\cdots x^m_{im}}\Bigg]\Bigg)$$

$\cdots,$

(d) When all the elements in the indicator $\mathbf{I}_3$ add to "1," that is, $n_2 = K_2 + 1, n_3 = K_3 + 1, \cdots, n_m = K_m + 1$, then we have

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\\
&=\sum_{i_2=1}^{K_2+1}\cdots\sum_{i_m=1}^{K_m+1} a_{i_1 i_2\cdots i_m} x^2_{i_2}\cdots x^m_{i_m}\\
&=\Bigg(\Bigg[1-\prod_{i_2=1}^{K_2+1}\cdots\prod_{i_m=1}^{K_m+1}\left(1-\mu^l_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\\
&\quad 1-\prod_{i_2=1}^{K_2+1}\cdots\prod_{i_m=1}^{K_m+1}\left(1-\mu^u_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\Bigg],\\
&\quad\Bigg[\prod_{i_2=1}^{K_2+1}\cdots\prod_{i_m=1}^{K_m+1}\left(\nu^l_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\\
&\quad\prod_{i_2=1}^{K_2+1}\cdots\prod_{i_m=1}^{K_m+1}\left(\nu^u_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\Bigg]\Bigg).
\end{aligned}$$

Therefore, for any $n_2, n_3, \cdots, n_m$, the **Theorem 3.1** holds from (1), (2), and (3). This completes the proof of **Theorem 3.1**.

Corollary 3.1. [1] Let $\tilde{\mathcal{A}}_{IVIF} \in \mathbf{T}_{\mathbf{IVIF}}(2, n\times m)$ be an interval-valued intuitionistic fuzzy matrix, and $\tilde{\mathcal{A}}_{IVIF} = (a_{ij})_{n\times m}$ where $a_{ij} = \left(\left[\mu^l_{ij}, \mu^u_{ij}\right], \left[\nu^l_{ij}, \nu^u_{ij}\right]\right)$, then their aggregated value by using the GIIFWA operator is also an IVIFN and

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X\right)\\
&=\left(\left[1-\prod\nolimits_{j=1}^{m}\left(1-\mu^l_{ij}\right)^{x_j}, 1-\prod\nolimits_{j=1}^{m}\left(1-\mu^u_{ij}\right)^{x_j}\right],\right.\\
&\quad\left.\left[\prod\nolimits_{j=1}^{m}\left(\nu^l_{ij}\right)^{x_j}, \prod\nolimits_{j=1}^{m}\left(\nu^u_{ij}\right)^{x_j}\right]\right)
\end{aligned}$$

where $X = (x_1, \cdots, x_j, \cdots, x_m)^T$ is the weight vector of $a_{:j}(j = 1, 2, \cdots, m)$, with $x_j \in [0,1]$ and $\sum_{j=1}^{m} x_j = 1$.

Remark 3.1.

Clearly, the **Theorem 3.1** is the extension of **Corollary 3.1** which is the **Theorem 2.3.1** in Xu [1].

Theorem 3.2. *Let* $\tilde{\mathcal{A}}_{IVIF} = \left(a_{i_1 i_2\cdots i_m}\right)_{n_1\times n_2\times\cdots\times n_m} \in \boldsymbol{T}_{\boldsymbol{IVIF}}(m, n_1\times n_2\times\cdots\times n_m)$ *where its elements* $a_{i_1 i_2\cdots i_m} = \left(\left[\mu^l_{i_1 i_2\cdots i_m}, \mu^u_{i_1 i_2\cdots i_m}\right], \left[\nu^l_{i_1 i_2\cdots i_m}, \nu^u_{i_1 i_2\cdots i_m}\right]\right)$. *Then the aggregated value by using Equation (2) is*

$$\begin{aligned}
&GIIFWG\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\\
&=\Bigg(\Bigg[\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\mu^l_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\\
&\quad\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\mu^u_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\Bigg],\\
&\quad\Bigg[1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\nu^l_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\\
&\quad 1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\nu^u_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\Bigg]\Bigg)
\end{aligned}$$

Proof. The proof of the **Theorem 3.2** is similar to the proof of **Theorem 3.1**.

Corollary 3.2. [1] Suppose that $\tilde{\mathcal{A}}_{IVIF} \in \mathbf{T}_{\mathbf{IVIF}}(2, n\times m)$ is an interval-valued intuitionistic fuzzy matrix, and $\tilde{\mathcal{A}}_{IVIF} = \left(a_{ij}\right)_{n\times m}$ where $a_{ij} = \left(\left[\mu^l_{ij}, \mu^u_{ij}\right], \left[\nu^l_{ij}, \nu^u_{ij}\right]\right)$, then their aggregated value by using the GIIFWG operator is also an IVIFN, and

$$\begin{aligned}
&\text{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X\right) = \left(\left[\prod_{j=1}^{m}(\mu^l_{ij})^{x_j}, \prod_{j=1}^{m}(\mu^u_{ij})^{x_j}\right],\right.\\
&\left.\left[1-\prod_{j=1}^{m}(1-\nu^l_{ij})^{x_j}, 1-\prod_{j=1}^{m}(1-\nu^u_{ij})^{x_j}\right]\right)
\end{aligned}$$

where $X = \left(x_1, \cdots, x_j, \cdots, x_m\right)^T$ is the exponential weight vector of $a_{:j}\left(j = 1, 2, \cdots, m\right)$, with $x_j \in [0,1]$ and $\sum_{j=1}^{m} x_j = 1$.

Remark 3.2.

The **Theorem 3.2** is the general form of **Corollary 3.2** which is the **Theorem 2.3.2** in Xu [1].

Theorem 3.3. *The operational results in **Theorems 3.1** and **3.2** are* n_1*-dimension **IVIF** vectors.*

Proof. By the **Theorems 3.1** and **3.2**, we have

$$\begin{aligned}
&\text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\\
&=\Bigg(\Bigg[1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^l_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\\
&\quad 1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^u_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\Bigg],\\
&\quad\Bigg[\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^l_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\\
&\quad\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^u_{i_1 i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\Bigg]\Bigg)
\end{aligned}$$

and

$$\begin{aligned}&\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\\&=\left(\left[\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\mu^l_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\right.\right.\\&\left.\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\mu^u_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\right],\\&\left[1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\nu^l_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\right.\\&\left.\left.1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\nu^u_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\right]\right)\end{aligned}$$

and $i_1\in[n_1]$, then both $\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)$ and $\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\in\mathbf{IVIF}^{n_1}$.

Therefore, the operational results in the **Theorems 3.1** and **3.2** are n_1-dimension **IVIF** vectors.

Theorem 3.4. *Let* $\tilde{\mathcal{A}}_{IVIF}=\left(a_{i_1i_2\cdots i_m}\right)_{n_1\times n_2\times\cdots\times n_m}\in\boldsymbol{T_{IVIF}}(m,n_1\times n_2\times\cdots\times n_m)$ *be a mth-order interval-valued intuitionistic fuzzy tensor, and* $X_2=\left(x^2_1,\cdots,x^2_{i_2},\cdots,x^2_{n_2}\right)^T$, $\cdots$, $X_m=\left(x^m_1,\cdots,x^m_{i_m},\cdots,x^m_{n_m}\right)^T$ *are the weight vectors of* $a_{:i_2:\cdots:}$ $(i_2=1,2,\cdots,n_2)$, $\cdots$, $a_{:\cdots:i_m}$ $(i_m=1,2,\cdots,n_m)$, *respectively, that is,* $\sum_{i_2=1}^{n_2}x^2_{i_2}=1$, $x^2_{i_2}\geqslant 0$; $\cdots$; $\sum_{i_m=1}^{n_m}x^m_{i_m}=1$, $x^m_{i_m}\geqslant 0$. *Then* $\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)$ *and* $\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)$ *are fuzzy mappings.*

Proof. $\tilde{\mathcal{A}}_{IVIF}\in\mathbf{T_{IVIF}}(m,n_1\times n_2\times\cdots\times n_m)$ is a mth-order interval-valued intuitionistic fuzzy tensor.

According to the **Definition 2.5**, we have

$$\tilde{\mathcal{A}}_{IVIF}=\left(\left(\left[\mu^l_{i_1i_2\cdots i_m},\mu^u_{i_1i_2\cdots i_m}\right],\left[\nu^l_{i_1i_2\cdots i_m},\nu^u_{i_1i_2\cdots i_m}\right]\right)\right)_{n_1\times n_2\times\cdots\times n_m}$$

for arbitrary $\left[\mu^l_{i_1i_2\cdots i_m},\mu^u_{i_1i_2\cdots i_m}\right]\subset[0,1]$, $\left[\nu^l_{i_1i_2\cdots i_m},\nu^u_{i_1i_2\cdots i_m}\right]\subset[0,1]$ and $\mu^u_{i_1i_2\cdots i_m}+\nu^u_{i_1i_2\cdots i_m}\leqslant 1$.

Owing to $X_2=\left(x^2_1,\cdots,x^2_{i_2},\cdots,x^2_{n_2}\right)^T$, $\cdots$, $X_m=\left(x^m_1,\cdots,x^m_{i_m},\cdots,x^m_{n_m}\right)^T$ are the weight vectors of $a_{:i_2:\cdots:}$ $(i_2=1,2,\cdots,n_2)$, $\cdots$, $a_{:\cdots:i_m}$ $(i_m=1,2,\cdots,n_m)$, respectively, that is, $\forall x^2_{i_2}\in[0,1]$, $\cdots$, $\forall x^m_{i_m}\in[0,1]$. Then we obtain $X_2\in[0,1]^{n_2}$, $\cdots$, $X_m\in[0,1]^{n_m}$.

On the basis of the **Theorem 3.3**, we get $\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\in\mathbf{IVIF}^{n_1}$ and $\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\in\mathbf{IVIF}^{n_1}$.

Thus $\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)$ and $\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)$ are fuzzy mappings from $[0,1]^{n_2\times n_3\times\cdots\times n_m}$ to $\mathbf{IVIF}^{n_1}$ by the **Definition 2.12**.

Theorem 3.5. *Let* $\tilde{\mathcal{A}}_{IVIF}=(a_{i_1i_2\cdots i_m})_{n_1\times n_2\times\cdots\times n_m}\in\boldsymbol{T_{IVIF}}(m,n_1\times n_2\times\cdots\times n_m)$ *be a mth-order interval-valued intuitionistic fuzzy tensor, where* $a_{i_1i_2\cdots i_m}=\left(\left[\mu^l_{i_1i_2\cdots i_m},\mu^u_{i_1i_2\cdots i_m}\right],\left[\nu^l_{i_1i_2\cdots i_m},\nu^u_{i_1i_2\cdots i_m}\right]\right)$. *And* $X_2=\left(x^2_1,\cdots,x^2_{i_2},\cdots,x^2_{n_2}\right)^T$, $\cdots$, $X_m=(x^m_1,\cdots,x^m_{i_m},\cdots,x^m_{n_m})^T$ *are the weight vectors of* $a_{:i_2:\cdots:}$ $(i_2=1,2,\cdots,n_2)$, $\cdots$, $a_{:\cdots:i_m}$ $(i_m=1,2,\cdots,n_m)$, *respectively, and* $\sum_{i_2=1}^{n_2}x^2_{i_2}=1$, $x^2_{i_2}\geqslant 0$; $\cdots$; $\sum_{i_m=1}^{n_m}x^m_{i_m}=1$, $x^m_{i_m}\geqslant 0$. *Then we have the following properties of GIIFWA operator:*

1. (Idempotency). If all the elements of $\tilde{\mathcal{A}}_{IVIF}$ are equal, that is, $a_{i_1i_2\cdots i_m}=\alpha$, $i_1\in[n_1]$, $i_2\in[n_2]$, $\cdots$, $i_m\in[n_m]$, then

$$\begin{aligned}&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\\&=(\alpha,\alpha,\cdots,\alpha)^T\in\mathbf{IVIF}^{n_1}\end{aligned}$$

2. (Boundedness). Let

$$\begin{aligned}\alpha^-&=\left(\left[\min_{i_1,i_2,\cdots,i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\},\min_{i_1,i_2,\cdots,i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}\right],\right.\\&\left.\left[\max_{i_1,i_2,\cdots,i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\},\max_{i_1,i_2,\cdots,i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right]\right),\\\alpha^+&=\left(\left[\max_{i_1,i_2,\cdots,i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\},\max_{i_1,i_2,\cdots,i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}\right],\right.\\&\left.\left[\min_{i_1,i_2,\cdots,i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\},\min_{i_1,i_2,\cdots,i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right]\right)\end{aligned}$$

and $(\alpha^-,\cdots,\alpha^-)^T,(\alpha^+,\cdots,\alpha^+)^T\in\mathbf{IVIF}^{n_1}$. Then, for any X_2, $X_3,\cdots,X_m$, we have

$$\begin{aligned}(\alpha^-,\cdots,\alpha^-)^T&\leqslant\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\\&\leqslant\left(\alpha^+,\cdots,\alpha^+\right)^T.\end{aligned}$$

Proof.

1. Let $\alpha=([\mu^l,\mu^u],[\nu^l,\nu^u])$. By the **Theorem 3.1** and $a_{i_1i_2\cdots i_m}=\alpha\,(i_1\in[n_1],i_2\in[n_2],\cdots,i_m\in[n_m])$, we have the i_1th component of GIIFWA operator and

$$\begin{aligned}&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)_{i_1}\\&=\left(\left[1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^l_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}},\right.\right.\\&\left.1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^u_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\right],\\&\left[\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^l_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\right.\\&\left.\left.\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^u_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\right]\right)_{i_1}\\&=\left(\left[1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^l\right)^{x^2_{i_2}\cdots x^m_{i_m}},\right.\right.\\&\left.1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^u\right)^{x^2_{i_2}\cdots x^m_{i_m}}\right],\\&\left[\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^l\right)^{x^2_{i_2}\cdots x^m_{i_m}},\right.\\&\left.\left.\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^u\right)^{x^2_{i_2}\cdots x^m_{i_m}}\right]\right)_{i_1}\end{aligned}$$

$$= \left(\left[1-\left(1-\mu^l\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}, 1-(1-\mu^u)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m} \right], \left[(\nu^l)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}, (\nu^u)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m} \right] \right)_{i_1}$$
$$= \left(\left[1-\left(1-\mu^l\right), 1-(1-\mu^u)\right], [\nu^l, \nu^u]\right)_{i_1}$$
$$= \left([\mu^l, \mu^u], [\nu^l, \nu^u]\right)_{i_1}$$
$$= \alpha$$

and $i_1 \in [n_1]$, then we get

$$\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right) = (\alpha, \alpha, \cdots, \alpha)^T \in \mathbf{IVIF}^{n_1}.$$

2. Since for any $i_1, i_2, \cdots, i_m$, we have $\min_{i_1,i_2,\cdots,i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\} \leqslant \mu^l_{i_1 i_2\cdots i_m} \leqslant \max_{i_1,i_2,\cdots,i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\}$, $\min_{i_1,i_2,\cdots,i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\} \leqslant \mu^u_{i_1 i_2\cdots i_m} \leqslant \max_{i_1,i_2,\cdots,i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\}$, $\min_{i_1,i_2,\cdots,i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\} \leqslant \nu^l_{i_1 i_2\cdots i_m} \leqslant \max_{i_1,i_2,\cdots,i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\}$ and $\min_{i_1,i_2,\cdots,i_m}\left\{\nu^u_{i_1 i_2\cdots i_m}\right\} \leqslant \nu^u_{i_1 i_2\cdots i_m} \leqslant \max_{i_1,i_2,\cdots,i_m}\left\{\nu^u_{i_1 i_2\cdots i_m}\right\}$. Then

$$1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$\geqslant 1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\min_{i_1 i_2\cdots i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$= 1-\left(1-\min_{i_1 i_2\cdots i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \min_{i_1 i_2\cdots i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\},$$

$$1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^u_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$\geqslant 1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\min_{i_1 i_2\cdots i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$= 1-\left(1-\min_{i_1 i_2\cdots i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \min_{i_1 i_2\cdots i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\},$$

$$\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$\geqslant \prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\min_{i_1 i_2\cdots i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \left(\min_{i_1 i_2\cdots i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \min_{i_1 i_2\cdots i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\}$$

and

$$\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^u_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$\geqslant \prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\min_{i_1 i_2\cdots i_m}\left\{\nu^u_{i_1 i_2\cdots i_m}\right\}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \left(\min_{i_1 i_2\cdots i_m}\left\{\nu^u_{i_1 i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \min_{i_1 i_2\cdots i_m}\left\{\nu^u_{i_1 i_2\cdots i_m}\right\}.$$

Similarly, we get

$$1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$\leqslant 1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\max_{i_1 i_2\cdots i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$= 1-\left(1-\max_{i_1 i_2\cdots i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \max_{i_1 i_2\cdots i_m}\left\{\mu^l_{i_1 i_2\cdots i_m}\right\},$$

$$1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\mu^u_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$\leqslant 1-\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(1-\max_{i_1 i_2\cdots i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$= 1-\left(1-\max_{i_1 i_2\cdots i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \max_{i_1 i_2\cdots i_m}\left\{\mu^u_{i_1 i_2\cdots i_m}\right\},$$

$$\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^l_{i_1 i_2\cdots i_m}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$\leqslant \prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\max_{i_1 i_2\cdots i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\}\right)^{x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \left(\max_{i_1 i_2\cdots i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} x_{i_2}^2\cdots x_{i_m}^m}$$
$$= \max_{i_1 i_2\cdots i_m}\left\{\nu^l_{i_1 i_2\cdots i_m}\right\}$$

and

$$\begin{aligned}&\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\nu^u_{i_1i_2\cdots i_m}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\\&\leqslant\prod_{i_2=1}^{n_2}\cdots\prod_{i_m=1}^{n_m}\left(\min_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right)^{x^2_{i_2}\cdots x^m_{i_m}}\\&=\left(\max_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right)^{\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m}x^2_{i_2}\cdots x^m_{i_m}}\\&=\max_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}.\end{aligned}$$

Without loss of generality, for $\forall i_1\in[n_1]$, let

$$\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)_{i_1}=\alpha$$

where $\alpha=\left(\left[\mu^l,\mu^u\right],\left[\nu^l,\nu^u\right]\right)$. By the **Definitions 2.8** and **2.10**, we get

$$\begin{aligned}s(\alpha)&=\frac{1}{2}\left(\mu^l-\nu^l+\mu^u-\nu^u\right)\\&\leqslant\frac{1}{2}\left(\max_{i_1i_2\cdots i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\}-\min_{i_1i_2\cdots i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\}\right.\\&\quad\left.+\max_{i_1i_2\cdots i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}-\min_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right)\\&=s\left(\alpha^+\right)\end{aligned}$$

and

$$\begin{aligned}s(\alpha)&=\frac{1}{2}\left(\mu^l-\nu^l+\mu^u-\nu^u\right)\\&\geqslant\frac{1}{2}\left(\min_{i_1i_2\cdots i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\}-\max_{i_1i_2\cdots i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\}\right.\\&\quad\left.+\min_{i_1i_2\cdots i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}-\max_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right)\\&=s\left(\alpha^-\right).\end{aligned}$$

Next, we will consider the following three cases:

i. When $s(\alpha)<s\left(\alpha^+\right)$ and $s(\alpha)>s\left(\alpha^-\right)$, the conclusion (2) in **Theorem 3.5** holds.

ii. When $s(\alpha)=s\left(\alpha^+\right)$, we have $\alpha=\alpha^+$, that is, $\mu^l=\max_{i_1,i_2,\cdots,i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\}$, $\mu^u=\max_{i_1,i_2,\cdots,i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}$, $\nu^l=\min_{i_1,i_2,\cdots,i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\}$, and $\nu^u=\min_{i_1,i_2,\cdots,i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}$. Hence, by the **Definition 2.9**, we get

$$\begin{aligned}h(\alpha)&=\frac{1}{2}\left(\mu^l+\mu^u+\nu^l+\nu^u\right)\\&=\frac{1}{2}\left(\max_{i_1i_2\cdots i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\}+\max_{i_1i_2\cdots i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}\right.\\&\quad\left.+\min_{i_1i_2\cdots i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\}+\min_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right)\\&=h\left(\alpha^+\right).\end{aligned}$$

In this case, according to the **Theorem 3.1** and **Definition 2.10**, we obtain $\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)_{i_1}=\alpha^+$. Due to the arbitrariness of i_1, we get

$$\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)=\left(\alpha^+,\cdots,\alpha^+\right)\in\mathbf{IVIF}^{n_1}.$$

iii. When $s(\alpha)=s(\alpha^-)$, we have $\alpha=\alpha^-$, that is, $\mu^l=\min_{i_1,i_2,\cdots,i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\}$, $\mu^u=\min_{i_1,i_2,\cdots,i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}$, $\nu^l=\max_{i_1,i_2,\cdots,i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\}$, and $\nu^u=\max_{i_1,i_2,\cdots,i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}$. Thus

$$\begin{aligned}h(\alpha)&=\frac{1}{2}\left(\mu^l+\mu^u+\nu^l+\nu^u\right)\\&=\frac{1}{2}\left(\min_{i_1i_2\cdots i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\}+\min_{i_1i_2\cdots i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}\right.\\&\quad\left.+\max_{i_1i_2\cdots i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\}+\max_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right)\\&=h(\alpha^-).\end{aligned}$$

In this case, on the basis of the **Theorem 3.1** and **Definition 2.10**, we have $\mathrm{GIIFWA}(\tilde{\mathcal{A}}_{IF}\circ X_2\circ X_3\circ\cdots\circ X_m)_{i_1}=\alpha^-$ for arbitrary $i_1\in[n_1]$. Then

$$\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)=(\alpha^-,\cdots,\alpha^-)\in\mathbf{IVIF}^{n_1}.$$

Therefore, based on cases (i), (ii), and (iii), we can see that the conclusion (2) in **Theorem 3.5** holds.

This completes the proof of **Theorem 3.5**.

Theorem 3.6. *Let* $\tilde{\mathcal{A}}_{IVIF}=\left(a_{i_1i_2\cdots i_m}\right)_{n_1\times n_2\times\cdots\times n_m}\in\mathbf{T_{IVIF}}(m,n_1\times n_2\times\cdots\times n_m)$ *be a mth-order interval-valued intuitionistic fuzzy tensor, where* $a_{i_1i_2\cdots i_m}=\left(\left[\mu^l_{i_1i_2\cdots i_m},\mu^u_{i_1i_2\cdots i_m}\right],\left[\nu^l_{i_1i_2\cdots i_m},\nu^u_{i_1i_2\cdots i_m}\right]\right)$. *And* $X_2=\left(x^2_1,\cdots,x^2_{i_2},\cdots,x^2_{n_2}\right)^T,\cdots,X_m=\left(x^m_1,\cdots,x^m_{i_m},\cdots,x^m_{n_m}\right)^T$ *are the exponential weight vectors of* $a_{:i_2:\cdots:}$ $(i_2=1,2,\cdots,n_2),\cdots,$ $a_{:\cdots:i_m}$ $(i_m=1,2,\cdots,n_m)$, *respectively, and* $\sum_{i_2=1}^{n_2}x^2_{i_2}=1$, $x^2_{i_2}\geqslant 0$; $\cdots$; $\sum_{i_m=1}^{n_m}x^m_{i_m}=1$, $x^m_{i_m}\geqslant 0$. *Then we have the following properties of GIIFWG operator:*

1. (Idempotency). If all the elements of $\tilde{\mathcal{A}}_{IVIF}\in\mathbf{T_{IVIF}}(m,n_1\times n_2\times\cdots\times n_m)$ are equal, that is, $a_{i_1i_2\cdots i_m}=\alpha$, $i_1\in[n_1],i_2\in[n_2],\cdots,i_m\in[n_m]$, then

$$\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)=(\alpha,\alpha,\cdots,\alpha)^T\in\mathbf{IVIF}^{n_1}$$

2. (Boundedness). For any $X_2,X_3,\cdots,X_m$, we have $(\alpha^-,\cdots,\alpha^-)^T\leqslant\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF}\circ X_2\circ X_3\circ\cdots\circ X_m\right)\leqslant\left(\alpha^+,\cdots,\alpha^+\right)^T$, where

$$\begin{aligned}\alpha^-&=\left(\left[\min_{i_1i_2\cdots i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\},\min_{i_1i_2\cdots i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}\right],\right.\\&\quad\left.\left[\max_{i_1i_2\cdots i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\},\max_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right]\right),\\\alpha^+&=\left(\left[\max_{i_1i_2\cdots i_m}\left\{\mu^l_{i_1i_2\cdots i_m}\right\},\max_{i_1i_2\cdots i_m}\left\{\mu^u_{i_1i_2\cdots i_m}\right\}\right],\right.\\&\quad\left.\left[\min_{i_1i_2\cdots i_m}\left\{\nu^l_{i_1i_2\cdots i_m}\right\},\min_{i_1i_2\cdots i_m}\left\{\nu^u_{i_1i_2\cdots i_m}\right\}\right]\right)\end{aligned}$$

and $(\alpha^-,\cdots,\alpha^-)^T,\left(\alpha^+,\cdots,\alpha^+\right)^T\in\mathrm{IVIF}^{n_1}$.

Proof. The proof of the **Theorem 3.6** is similar to the proof of **Theorem 3.5**.

4. ALGORITHM

In this section, we will employ the generalized GIIFWA and GIIFWG operators to devise a new approach for solving the multiple attribute group decision-making problems with high-dimension data. The concrete steps of the algorithm are listed as follows:

Step 1. The interval-valued intuitionistic fuzzy decision matrices are transformed into interval-valued intuitionistic fuzzy tensor $\tilde{\mathcal{A}}_{IVIF}$;

Step 2. According to the **Theorems 3.1** or **3.2**, we utilize the GIIFWA operator:

$$\dot{\tilde{c}}_{i_1} = \text{GIIFWA}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right)_{i_1}$$

or the GIIFWG operator:

$$\ddot{\tilde{c}}_{i_1} = \text{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\right)_{i_1}$$

to aggregate all the elements $a_{i_1 i_2 \cdots i_m}$ $(i_1 \in [n_1], i_2 \in [n_2], \cdots, i_m \in [n_m])$ of the interval-valued intuitionistic fuzzy tensor $\tilde{\mathcal{A}}_{IVIF}$ and get the values $\dot{\tilde{c}}_{i_1}$ (or $\ddot{\tilde{c}}_{i_1}$) corresponding to the alternatives A_{i_1} $(i_1 \in [n_1])$;

Step 3. Calculate the sores $s\left(\dot{\tilde{c}}_{i_1}\right)$ $\left(\text{or } s\left(\ddot{\tilde{c}}_{i_1}\right)\right)$ and the accuracy degrees $h\left(\dot{\tilde{c}}_{i_1}\right)$ $\left(\text{or } h\left(\ddot{\tilde{c}}_{i_1}\right)\right)$ $(i_1 \in [n_1])$ by the **Definitions 2.8** and **2.9**.

Step 4. Rank the alternatives A_{i_1} $(i_1 \in [n_1])$ by the **Definition 2.10**, and then obtain the best desirable alternative.

5. APPLICATION EXAMPLES AND DISCUSSION

5.1. Interval-Valued Intuitionistic Fuzzy Multiple Attribute Group Decision-Making

In this subsection, we apply the GIIFWA and GIIFWG operators to solving the interval-valued intuitionistic fuzzy multiple attribute group decision-making problem with the numerical example used in Qiu [15].

5.1.1. Numerical example

In this example, let us assume that someone intends to buy a car and consults a set of experts. The car supplier x_{i_1} $(i_1 = 1, 2, \cdots, 5)$ are evaluated by four decision-makers e_{i_2} $(i_2 = 1, 2, 3, 4)$, and each decision-maker evaluates the alternatives based on five different characteristics c_{i_3} $(i_3 = 1, 2, \cdots, 5)$. The interval-valued intuitionistic fuzzy decision matrix proposed by e_{i_2} $(i_2 = 1, 2, 3, 4)$ are listed in the Tables 1–4, and the weighted vector of the four experts is $X_2 = (0.3, 0.2, 0.3, 0.2)^T$, and the weighted vector of the five characteristics is $X_3 = (0.2, 0.15, 0.2, 0.3, 0.15)^T$. Due to space limitations, the original interval-valued intuitionistic fuzzy decision matrices are omitted in this paper. For a detailed description, please see Qiu [15].

We now implement our algorithm to solve this problem.

Step 1. If the interval-valued intuitionistic fuzzy tensor and the GIIFWA operator are employed for expressing data in Tables 1–4, then $\tilde{\mathcal{A}}_{IVIF} = \left(a_{i_1 i_2 i_3}\right)_{5\times4\times5} \in \mathbf{T}_{\mathbf{IVIF}}(3, 5\times4\times5)$, where its elements $a_{i_1 i_2 i_3} = \left(\left[\mu^l_{i_1 i_2 i_3}, \mu^u_{i_1 i_2 i_3}\right], \left[\nu^l_{i_1 i_2 i_3}, \nu^u_{i_1 i_2 i_3}\right]\right)$, and $a_{i_1::}$ $(i_1 = [5])$ represent five suppliers, $a_{:i_2:}$ $(i_2 = [4])$ represent four experts and $a_{::i_3}$ $(i_3 = [5])$ represent five different characteristics. The details are as follows:

$$a_{111} = ([0.3, 0.4], [0.4, 0.6]), a_{112} = ([0.5, 0.6], [0.1, 0.2]),$$
$$a_{113} = ([0.6, 0.7], [0.2, 0.3]), a_{114} = ([0.7, 0.8], [0.0, 0.1]),$$
$$a_{115} = ([0.6, 0.7], [0.2, 0.3]), a_{121} = ([0.4, 0.5], [0.3, 0.4]),$$
$$a_{122} = ([0.5, 0.6], [0.1, 0.2]), a_{123} = ([0.6, 0.7], [0.2, 0.3]),$$
$$a_{124} = ([0.7, 0.8], [0.1, 0.2]), a_{125} = ([0.7, 0.8], [0.0, 0.2]),$$
$$a_{131} = ([0.4, 0.6], [0.3, 0.4]), a_{132} = ([0.5, 0.7], [0.0, 0.2]),$$
$$a_{133} = ([0.5, 0.6], [0.2, 0.4]), a_{134} = ([0.6, 0.8], [0.1, 0.2]),$$
$$a_{135} = ([0.4, 0.7], [0.2, 0.3]), a_{141} = ([0.3, 0.4], [0.4, 0.5]),$$
$$a_{142} = ([0.8, 0.9], [0.1, 0.1]), a_{143} = ([0.7, 0.8], [0.1, 0.2]),$$
$$a_{144} = ([0.4, 0.5], [0.3, 0.5]), a_{145} = ([0.2, 0.4], [0.3, 0.6]),$$
$$a_{211} = ([0.6, 0.8], [0.1, 0.2]), a_{212} = ([0.6, 0.7], [0.2, 0.3]),$$
$$a_{213} = ([0.2, 0.3], [0.4, 0.6]), a_{214} = ([0.5, 0.6], [0.1, 0.3]),$$
$$a_{215} = ([0.7, 0.8], [0.0, 0.2]), a_{221} = ([0.6, 0.8], [0.1, 0.2]),$$
$$a_{222} = ([0.5, 0.6], [0.3, 0.4]), a_{223} = ([0.4, 0.5], [0.3, 0.4]),$$
$$a_{224} = ([0.4, 0.6], [0.3, 0.4]), a_{225} = ([0.4, 0.7], [0.1, 0.3]),$$
$$a_{231} = ([0.5, 0.8], [0.1, 0.2]), a_{232} = ([0.3, 0.5], [0.2, 0.3]),$$
$$a_{233} = ([0.3, 0.6], [0.2, 0.4]), a_{234} = ([0.4, 0.5], [0.2, 0.4]),$$
$$a_{235} = ([0.3, 0.6], [0.2, 0.3]), a_{241} = ([0.5, 0.7], [0.1, 0.3]),$$
$$a_{242} = ([0.4, 0.7], [0.2, 0.3]), a_{243} = ([0.4, 0.5], [0.2, 0.2]),$$
$$a_{244} = ([0.6, 0.8], [0.1, 0.2]), a_{245} = ([0.2, 0.3], [0.0, 0.1]),$$
$$a_{311} = ([0.5, 0.8], [0.1, 0.2]), a_{312} = ([0.7, 0.8], [0.0, 0.1]),$$
$$a_{313} = ([0.5, 0.5], [0.4, 0.5]), a_{314} = ([0.2, 0.3], [0.2, 0.4]),$$
$$a_{315} = ([0.4, 0.6], [0.2, 0.3]), a_{321} = ([0.5, 0.6], [0.3, 0.4]),$$
$$a_{322} = ([0.5, 0.7], [0.1, 0.2]), a_{323} = ([0.5, 0.6], [0.3, 0.4]),$$
$$a_{324} = ([0.3, 0.4], [0.2, 0.5]), a_{325} = ([0.6, 0.7], [0.2, 0.3]),$$
$$a_{331} = ([0.5, 0.6], [0.0, 0.1]), a_{332} = ([0.5, 0.8], [0.1, 0.2]),$$
$$a_{333} = ([0.4, 0.7], [0.2, 0.3]), a_{334} = ([0.2, 0.4], [0.2, 0.3]),$$
$$a_{335} = ([0.5, 0.8], [0.0, 0.2]), a_{341} = ([0.2, 0.4], [0.1, 0.2]),$$
$$a_{342} = ([0.4, 0.5], [0.2, 0.4]), a_{343} = ([0.5, 0.8], [0.0, 0.1]),$$
$$a_{344} = ([0.4, 0.6], [0.2, 0.3]), a_{345} = ([0.5, 0.6], [0.2, 0.3]),$$
$$a_{411} = ([0.2, 0.3], [0.4, 0.5]), a_{412} = ([0.5, 0.7], [0.1, 0.3]),$$
$$a_{413} = ([0.6, 0.7], [0.1, 0.2]), a_{414} = ([0.4, 0.5], [0.1, 0.3]),$$
$$a_{415} = ([0.6, 0.9], [0.0, 0.1]), a_{421} = ([0.5, 0.6], [0.3, 0.4]),$$
$$a_{422} = ([0.7, 0.8], [0.0, 0.1]), a_{423} = ([0.4, 0.5], [0.2, 0.4]),$$
$$a_{424} = ([0.5, 0.7], [0.1, 0.2]), a_{425} = ([0.5, 0.7], [0.2, 0.3]),$$
$$a_{431} = ([0.5, 0.7], [0.1, 0.3]), a_{432} = ([0.4, 0.6], [0.0, 0.1]),$$
$$a_{433} = ([0.3, 0.5], [0.2, 0.4]), a_{434} = ([0.7, 0.9], [0.0, 0.1]),$$
$$a_{435} = ([0.3, 0.5], [0.2, 0.2]), a_{441} = ([0.7, 0.8], [0.0, 0.2]),$$

Table 1 Interval-valued intuitionistic fuzzy decision matrix proposed by e_1.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.3,0.4], [0.4,0.6])	([0.5,0.6], [0.1,0.2])	([0.6,0.7], [0.2,0.3])	([0.7,0.8], [0.0,0.1])	([0.6,0.7], [0.2,0.3])
A_2	([0.6,0.8], [0.1,0.2])	([0.6,0.7], [0.2,0.3])	([0.2,0.3], [0.4,0.6])	([0.5,0.6], [0.1,0.3])	([0.7,0.8], [0.0,0.2])
A_3	([0.5,0.8], [0.1,0.2])	([0.7,0.8], [0.0,0.1])	([0.5,0.5], [0.4,0.5])	([0.2,0.3], [0.2,0.4])	([0.4,0.6], [0.2,0.3])
A_4	([0.2,0.3], [0.4,0.5])	([0.5,0.7], [0.1,0.3])	([0.6,0.7], [0.1,0.2])	([0.4,0.5], [0.1,0.3])	([0.6,0.9], [0.0,0.1])
A_5	([0.6,0.8], [0.1,0.2])	([0.3,0.5], [0.4,0.5])	([0.4,0.6], [0.3,0.4])	([0.6,0.8], [0.1,0.2])	([0.5,0.6], [0.2,0.3])

Table 2 Interval-valued intuitionistic fuzzy decision matrix proposed by e_2.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.4,0.5], [0.3,0.4])	([0.5,0.6], [0.1,0.2])	([0.6,0.7], [0.2,0.3])	([0.7,0.8], [0.1,0.2])	([0.7,0.8], [0.0,0.2])
A_2	([0.6,0.8], [0.1,0.2])	([0.5,0.6], [0.3,0.4])	([0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.3,0.4])	([0.4,0.7], [0.1,0.3])
A_3	([0.5,0.6], [0.3,0.4])	([0.5,0.7], [0.1,0.2])	([0.5,0.6], [0.3,0.4])	([0.3,0.4], [0.2,0.5])	([0.6,0.7], [0.2,0.3])
A_4	([0.5,0.6], [0.3,0.4])	([0.7,0.8], [0.0,0.1])	([0.4,0.5], [0.2,0.4])	([0.5,0.7], [0.1,0.2])	([0.5,0.7], [0.2,0.3])
A_5	([0.4,0.7], [0.2,0.3])	([0.5,0.6], [0.2,0.4])	([0.3,0.6], [0.3,0.4])	([0.6,0.8], [0.1,0.2])	([0.4,0.5], [0.2,0.3])

Table 3 Interval-valued intuitionistic fuzzy decision matrix proposed by e_3.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.4,0.6], [0.3,0.4])	([0.5,0.7], [0.0,0.2])	([0.5,0.6], [0.2,0.4])	([0.6,0.8], [0.1,0.2])	([0.4,0.7], [0.2,0.3])
A_2	([0.5,0.8], [0.1,0.2])	([0.3,0.5], [0.2,0.3])	([0.3,0.6], [0.2,0.4])	([0.4,0.5], [0.2,0.4])	([0.3,0.6], [0.2,0.3])
A_3	([0.5,0.6], [0.0,0.1])	([0.5,0.8], [0.1,0.2])	([0.4,0.7], [0.2,0.3])	([0.2,0.4], [0.2,0.3])	([0.5,0.8], [0.0,0.2])
A_4	([0.5,0.7], [0.1,0.3])	([0.4,0.6], [0.0,0.1])	([0.3,0.5], [0.2,0.4])	([0.7,0.9], [0.0,0.1])	([0.3,0.5], [0.2,0.2])
A_5	([0.7,0.8], [0.0,0.1])	([0.4,0.6], [0.0,0.2])	([0.4,0.7], [0.2,0.3])	([0.3,0.5], [0.1,0.3])	([0.6,0.7], [0.1,0.2])

Table 4 Interval-valued intuitionistic fuzzy decision matrix proposed by e_4.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.3,0.4], [0.4,0.5])	([0.8,0.9], [0.1,0.1])	([0.7,0.8], [0.1,0.2])	([0.4,0.5], [0.3,0.5])	([0.2,0.4], [0.3,0.6])
A_2	([0.5,0.7], [0.1,0.3])	([0.4,0.7], [0.2,0.3])	([0.4,0.5], [0.2,0.2])	([0.6,0.8], [0.1,0.2])	([0.2,0.3], [0.0,0.1])
A_3	([0.2,0.4], [0.1,0.2])	([0.4,0.5], [0.2,0.4])	([0.5,0.8], [0.0,0.1])	([0.4,0.6], [0.2,0.3])	([0.5,0.6], [0.2,0.3])
A_4	([0.7,0.8], [0.0,0.2])	([0.5,0.7], [0.1,0.2])	([0.6,0.7], [0.1,0.3])	([0.4,0.5], [0.1,0.2])	([0.7,0.8], [0.1,0.2])
A_5	([0.5,0.6], [0.2,0.4])	([0.5,0.8], [0.0,0.2])	([0.4,0.7], [0.2,0.3])	([0.3,0.6], [0.2,0.3])	([0.7,0.8], [0.0,0.1])

$$
\begin{aligned}
&a_{442} = ([0.5,0.7],[0.1,0.2]), a_{443} = ([0.6,0.7],[0.1,0.3]),\\
&a_{444} = ([0.4,0.5],[0.1,0.2]), a_{445} = ([0.7,0.8],[0.1,0.2]),\\
&a_{511} = ([0.6,0.8],[0.1,0.2]), a_{512} = ([0.3,0.5],[0.4,0.5]),\\
&a_{513} = ([0.4,0.6],[0.3,0.4]), a_{514} = ([0.6,0.8],[0.1,0.2]),\\
&a_{515} = ([0.5,0.6],[0.2,0.3]), a_{521} = ([0.4,0.7],[0.2,0.3]),\\
&a_{522} = ([0.5,0.6],[0.2,0.4]), a_{523} = ([0.3,0.6],[0.3,0.4]),\\
&a_{524} = ([0.6,0.8],[0.1,0.2]), a_{525} = ([0.4,0.5],[0.2,0.3]),\\
&a_{531} = ([0.7,0.8],[0.0,0.1]), a_{532} = ([0.4,0.6],[0.0,0.2]),\\
&a_{533} = ([0.4,0.7],[0.2,0.3]), a_{534} = ([0.3,0.5],[0.1,0.3]),\\
&a_{535} = ([0.6,0.7],[0.1,0.2]), a_{541} = ([0.5,0.6],[0.2,0.4]),\\
&a_{542} = ([0.5,0.8],[0.0,0.2]), a_{543} = ([0.4,0.7],[0.2,0.3]),\\
&a_{544} = ([0.3,0.6],[0.2,0.3]), a_{545} = ([0.7,0.8],[0.0,0.1]).
\end{aligned}
$$

Step 2. By the **Theorem 3.1**, and $\tilde{\mathcal{A}}_{IVIF} \in \mathbf{T_{IVIF}}(3, 5\times 4\times 5)$, according to the experts weight X_2 and the characteristics weight X_3 in Qiu [15], we have

$$
\begin{aligned}
&\text{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3\right)\\
&= \left(\left[1-\prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(1-\mu^l_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}},\right.\right.\\
&\quad \left.1-\prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(1-\mu^u_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}}\right],\\
&\quad \left.\left[\prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(\nu^l_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}}, \prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(\nu^u_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}}\right]\right)\\
&= (([0.551,0.651],[0.000,0.269]),([0.460,0.657],\\
&\quad [0.000,0.290]),([0.431,0.570],[0.000,0.264]),\\
&\quad ([0.511,0.661],[0.000,0.224]),([0.487,0.645],\\
&\quad [0.000,0.255]))^T.
\end{aligned}
$$

that is,

$$
\begin{aligned}
x_1 &= ([0.551,0.651],[0.000,0.267]),\\
x_2 &= ([0.460,0.657],[0.000,0.290]),\\
x_3 &= ([0.431,0.570],[0.000,0.264]),\\
x_4 &= ([0.511,0.661],[0.000,0.224]),\\
x_5 &= ([0.487,0.645],[0.000,0.255]).
\end{aligned}
$$

Step 3. To rank the IVIFNs $x_{i_1}(i_1 = [5])$, we calculate the scores $s(x_{i_1})(i_1 = [5])$ by the **Definition 2.8**. $s(x_1) = 0.466$, $s(x_2) = 0.414$, $s(x_3) = 0.369$, $s(x_4) = 0.474$, $s(x_5) = 0.438$.

Step 4. By the scores $s(x_{i_1})$ result, the ranking order of all the alternatives is generated as $x_4 > x_1 > x_5 > x_2 > x_3$. Therefore, the best car supplier is x_4.

We can also replace the GIIFWA with the GIIFWG to resolve this problem. The difference starts from step 2.

Step 2'. By the **Theorem 3.2**, we have

$$
\begin{aligned}
&\text{GIIFWG}\left(\tilde{\mathcal{A}}_{IVIF} \circ X_2 \circ X_3\right)\\
&= \left(\left[\prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(\mu^l_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}}, \prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(\mu^u_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}}\right],\right.\\
&\quad \left[1-\prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(1-\nu^l_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}},\right.\\
&\quad \left.\left.1-\prod_{i_2=1}^{4}\prod_{i_3=1}^{5}\left(1-\nu^u_{i_1i_2i_3}\right)^{x^2_{i_2}x^3_{i_3}}\right]\right)
\end{aligned}
$$

$$= (([0.503, 0.641], [0.186, 0.323]), ([0.421, 0.598], [0.178, 0.349]), ([0.387, 0.560], [0.171, 0.307]), ([0.466, 0.623], [0.125, 0.280]), ([0.450, 0.659], [0.158, 0.282])).$$

Then, we get

$$x_1 = ([0.503, 0.641], [0.186, 0.323]),$$
$$x_2 = ([0.421, 0.598], [0.178, 0.349]),$$
$$x_3 = ([0.387, 0.560], [0.171, 0.307]),$$
$$x_4 = ([0.466, 0.623], [0.125, 0.280]),$$
$$x_5 = ([0.450, 0.659], [0.158, 0.282]).$$

Step 3'. In order to rank the IVIFNs $x_{i_1}(i_1 = [5])$, we calculate the scores $s(x_{i_1})(i_1 = 1, 2, \cdots, 5)$ by the **Definition 2.8**, then we get $s(x_1) = 0.318$, $s(x_2) = 0.246$, $s(x_3) = 0.234$, $s(x_4) = 0.342$, $s(x_5) = 0.334$.

Step 4'. Then, by the scores $s(x_i)$ result, the ranking order of all the alternatives is generated as $x_4 > x_5 > x_1 > x_2 > x_3$. Therefore, the optimal car supplier is x_4.

5.1.2. Discussion

In this subsection, we try to explain the difference between our results with GIIFWA and GIIFWG operators and those in Qiu [15].

1. The comparison of the results is shown in Table 5. By using the same data and weight information, the results calculated by GIIFWG operator are the same as the results in Qiu [15]. However, the results calculated by GIIFWA operator are slightly different from that in Qiu [15].
2. The reason for the slightly different results calculated by the GIIFWA and GIIFWG operators is that the ranking result of GIIFWG operator is more accurate because zero-valued elements in expert preferences do not affect the calculation process; the GIIFWG operator ensure the reasonable of the alternative ranking in this numerical example.
3. Compared with the method in Qiu [15], we get the same calculation result with the GIIFWG operator which is more simple in establishing and computing model.

5.2. Dynamic Interval-Valued Intuitionistic Fuzzy Multiple Attribute Group Decision-Making

In this subsection, we will use a practical example which is a slightly revised version of *Case illustration* in Xu and Yager [35] to illustrate the efficiency and universal applicability of the presented algorithm.

5.2.1. Practical example

Located in Central China and the middle reaches of the Changjiang (Yangtze) River, Hubei Province is distributed in a transitional belt where physical conditions and landscapes are on the transition from north to south and from east to west. Thus, Hubei Province is well known as a land of rice and fish since the region enjoys some of the favorable physical conditions, with a diversity of natural resources and the suitability for growing various crops. At the same time, however, there are also some restrictive factors for developing agriculture, such as a tight man-land relation between a constant degradation of natural resources and a growing population pressure on land resource reserve. Despite cherishing a burning desire to promote their standard of living, people living in the area are frustrated because they have no ability to enhance their power to accelerate economic development because of a dramatic decline in quantity and quality of natural resources and a deteriorating environment. Based on the distinctness and differences in environment and natural resources, Hubei Province can be roughly divided into seven agroecological regions: Y_1-Wuhan-Ezhou-Huanggang; Y_2-Northeast of Hubei; Y_3-Southeast of Hubei; Y_4-Jianghan region; Y_5-North of Hubei; Y_6-Northwest of Hubei; Y_7-Southwest of Hubei. In order to prioritize these agroecological regions $Y_i\,(i = 1, 2, \cdots, 7)$ with respect to their comprehensive functions, a committee comprised of three experts $E_l\,(l = 1, 2, 3)$ has been set up to provide assessment information on $Y_i\,(i = 1, 2, \cdots, 7)$. The attributes which are considered here in assessment of $Y_i\,(i = 1, 2, \cdots, 7)$ are (1) G_1 is ecological benefit, (2) G_2 is economic benefit, and (3) G_3 is social benefit. The committee evaluates the performance of agroecological regions $Y_i\,(i = 1, 2, \cdots, 7)$ in the years 2004 – 2006 according to the attributes $G_j\,(j = 1, 2, 3)$, and constructs, respectively, the interval-valued intuitionistic fuzzy decision matrices $R\left(t_k^l\right)(l, k = 1, 2, 3)$ (here, t_1^l denotes the year "2004," t_2^l denotes the year "2005," and t_3^l denotes the year "2006") as listed in Tables 6–14. Let $\omega = (1/6, 2/6, 3/6)^T$ be the weight vector of the years $t_k^l\,(k = 1, 2, 3)$, $\lambda = (0.5, 0.2, 0.3)^T$ be the weight vector of the experts $E_l(l = 1, 2, 3)$, and $\xi = (0.3, 0.4, 0.3)^T$ be the weight vector of the attributes $G_j\,(j = 1, 2, 3)$.

Step 1. If the interval-valued intuitionistic fuzzy tensor and the GIIFWA operator are employed for expressing data in Tables 6–14, then $\tilde{\mathcal{A}}_{IVIF} = \left(a_{i_1 i_2 i_3 i_4}\right)_{7\times3\times3\times3} \in \mathbf{T}_{\mathbf{IVIF}}(4, 7\times3\times3\times3)$, where its elements $a_{i_1 i_2 i_3 i_4} = \left(\left[\mu^l_{i_1 i_2 i_3 i_4}, \mu^u_{i_1 i_2 i_3 i_4}\right], \left[\nu^l_{i_1 i_2 i_3 i_4}, \nu^u_{i_1 i_2 i_3 i_4}\right]\right)$, and $a_{i_1:::}\,(i_1 \in [7])$ represent seven agroecological regions, $a_{:i_2::}\,(i_2 \in [3])$ represent three years, $a_{::i_3:}\,(i_3 \in [3])$ represent three experts, and $a_{:::i_4}\,(i_4 \in [3])$ represent three attributes. The details are as follows:

$$a_{1111} = ([0.8, 0.9], [0.0, 0.1]), a_{1112} = ([0.7, 0.8], [0.1, 0.2]),$$
$$a_{1113} = ([0.6, 0.8], [0.0, 0.2]), a_{1121} = ([0.5, 0.6], [0.2, 0.3]),$$
$$a_{1122} = ([0.2, 0.6], [0.1, 0.2]), a_{1123} = ([0.3, 0.6], [0.2, 0.3]),$$
$$a_{1131} = ([0.3, 0.6], [0.1, 0.3]), a_{1132} = ([0.2, 0.5], [0.2, 0.5]),$$
$$a_{1133} = ([0.2, 0.5], [0.3, 0.4]), a_{1211} = ([0.7, 0.8], [0.1, 0.2]),$$
$$a_{1212} = ([0.8, 0.9], [0.0, 0.1]), a_{1213} = ([0.7, 0.9], [0.0, 0.1]),$$
$$a_{1221} = ([0.2, 0.6], [0.3, 0.4]), a_{1222} = ([0.2, 0.5], [0.3, 0.4]),$$
$$a_{1223} = ([0.4, 0.5], [0.2, 0.5]), a_{1231} = ([0.4, 0.6], [0.1, 0.3]),$$
$$a_{1232} = ([0.2, 0.6], [0.1, 0.2]), a_{1233} = ([0.2, 0.5], [0.2, 0.4]),$$
$$a_{1311} = ([0.6, 0.7], [0.1, 0.3]), a_{1312} = ([0.7, 0.9], [0.0, 0.1]),$$
$$a_{1313} = ([0.8, 0.9], [0.0, 0.1]), a_{1321} = ([0.4, 0.6], [0.2, 0.3]),$$

Table 5 The comparison among the results of the GIIFWA and GIIFWG operators in this paper and the results of Qiu.

		Sort Function		The Optimal
Method	Results	Values	Preference order	Car supplier
Qiu's [15] method	$x_1 = ([0.350, 0.774], [0.226, 0.349])$	$s(x_1) = 0.203$	$x_4 \succ x_5 \succ x_1 \succ x_2 \succ x_3$	x_4
	$x_2 = ([0.423, 0.692], [0.171, 0.227])$	$s(x_2) = 0.181$		
	$x_3 = ([0.318, 0.698], [0.272, 0.302])$	$s(x_3) = 0.169$		
	$x_4 = ([0.259, 0.740], [0.191, 0.200])$	$s(x_4) = 0.235$		
	$x_5 = ([0.392, 0.646], [0.185, 0.193])$	$s(x_5) = 0.222$		
GIIFWA operator	$x_1 = ([0.551, 0.650], [0.000, 0.269])$	$s(x_1) = 0.466$	$x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$	x_4
	$x_2 = ([0.460, 0.657], [0.000, 0.290])$	$s(x_2) = 0.414$		
	$x_3 = ([0.431, 0.570], [0.000, 0.264])$	$s(x_3) = 0.369$		
	$x_4 = ([0.511, 0.661], [0.000, 0.224])$	$s(x_4) = 0.474$		
	$x_5 = ([0.487, 0.645], [0.000, 0.255])$	$s(x_5) = 0.438$		
GIIFWG operator	$x_1 = ([0.503, 0.641], [0.186, 0.323])$	$s(x_1) = 0.318$	$x_4 \succ x_5 \succ x_1 \succ x_2 \succ x_3$	x_4
	$x_2 = ([0.421, 0.598], [0.178, 0.349])$	$s(x_2) = 0.246$		
	$x_3 = ([0.387, 0.560], [0.171, 0.307])$	$s(x_3) = 0.234$		
	$x_4 = ([0.466, 0.623], [0.125, 0.280])$	$s(x_4) = 0.342$		
	$x_5 = ([0.450, 0.659], [0.158, 0.282])$	$s(x_5) = 0.334$		

GIIFWA, generalized interval-valued intuitionistic fuzzy weighted averaging; GIIFWG, generalized interval-valued intuitionistic fuzzy weighted geometric .

Table 6 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_1^1\right)$.

	G_1	G_2	G_3
Y_1	([0.8,0.9], [0.0,0.1])	([0.7,0.8], [0.1,0.2])	([0.6,0.8], [0.0,0.2])
Y_2	([0.6,0.7], [0.2,0.3])	([0.5,0.7], [0.2,0.3])	([0.5,0.6], [0.2,0.3])
Y_3	([0.4,0.5], [0.2,0.4])	([0.5,0.6], [0.2,0.3])	([0.4,0.6], [0.1,0.2])
Y_4	([0.7,0.8], [0.1,0.2])	([0.6,0.8], [0.0,0.1])	([0.6,0.7], [0.1,0.2])
Y_5	([0.5,0.7], [0.1,0.3])	([0.7,0.8], [0.1,0.2])	([0.4,0.5], [0.2,0.4])
Y_6	([0.2,0.3], [0.5,0.6])	([0.3,0.5], [0.4,0.5])	([0.4,0.6], [0.3,0.4])
Y_7	([0.4,0.5], [0.3,0.4])	([0.2,0.5], [0.3,0.5])	([0.4,0.7], [0.2,0.3])

Table 7 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_1^2\right)$.

	G_1	G_2	G_3
Y_1	([0.5,0.6], [0.2,0.3])	([0.2,0.6], [0.1,0.2])	([0.3,0.6], [0.2,0.3])
Y_2	([0.4,0.5], [0.1,0.3])	([0.2,0.6], [0.1,0.4])	([0.4,0.5], [0.3,0.5])
Y_3	([0.4,0.5], [0.2,0.3])	([0.7,0.8], [0.1,0.2])	([0.5,0.7], [0.2,0.3])
Y_4	([0.4,0.5], [0.2,0.3])	([0.2,0.6], [0.1,0.3])	([0.2,0.8], [0.1,0.2])
Y_5	([0.3,0.5], [0.2,0.3])	([0.3,0.6], [0.1,0.3])	([0.3,0.6], [0.1,0.2])
Y_6	([0.3,0.6], [0.2,0.3])	([0.2,0.7], [0.1,0.2])	([0.2,0.6], [0.1,0.4])
Y_7	([0.4,0.6], [0.2,0.3])	([0.4,0.5], [0.1,0.2])	([0.4,0.5], [0.2,0.3])

Table 8 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_1^3\right)$.

	G_1	G_2	G_3
Y_1	([0.3,0.6], [0.1,0.3])	([0.2,0.5], [0.2,0.5])	([0.2,0.5], [0.3,0.4])
Y_2	([0.3,0.5], [0.2,0.5])	([0.3,0.5], [0.3,0.4])	([0.2,0.6], [0.2,0.3])
Y_3	([0.4,0.6], [0.1,0.3])	([0.3,0.4], [0.2,0.3])	([0.3,0.6], [0.1,0.2])
Y_4	([0.3,0.5], [0.1,0.3])	([0.3,0.5], [0.2,0.3])	([0.2,0.7], [0.1,0.2])
Y_5	([0.2,0.6], [0.1,0.2])	([0.2,0.5], [0.1,0.4])	([0.4,0.5], [0.2,0.3])
Y_6	([0.3,0.5], [0.2,0.3])	([0.3,0.5], [0.1,0.2])	([0.3,0.5], [0.2,0.4])
Y_7	([0.4,0.7], [0.1,0.3])	([0.2,0.7], [0.2,0.3])	([0.4,0.8], [0.1,0.2])

Table 9 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_2^1\right)$.

	G_1	G_2	G_3
Y_1	([0.7,0.8], [0.1,0.2])	([0.8,0.9], [0.0,0.1])	([0.7,0.9], [0.0,0.1])
Y_2	([0.5,0.7], [0.1,0.2])	([0.6,0.7], [0.1,0.3])	([0.4,0.5], [0.2,0.4])
Y_3	([0.3,0.5], [0.1,0.3])	([0.4,0.5], [0.1,0.3])	([0.3,0.6], [0.3,0.4])
Y_4	([0.6,0.7], [0.1,0.2])	([0.7,0.8], [0.1,0.2])	([0.5,0.7], [0.1,0.3])
Y_5	([0.5,0.7], [0.2,0.3])	([0.5,0.7], [0.1,0.3])	([0.4,0.6], [0.2,0.3])
Y_6	([0.3,0.4], [0.4,0.6])	([0.2,0.4], [0.5,0.6])	([0.4,0.5], [0.4,0.5])
Y_7	([0.3,0.5], [0.3,0.5])	([0.4,0.6], [0.3,0.4])	([0.4,0.5], [0.2,0.4])

Table 10 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_2^2\right)$.

	G_1	G_2	G_3
Y_1	([0.2,0.6], [0.3,0.4])	([0.2,0.5], [0.3,0.4])	([0.4,0.5], [0.2,0.5])
Y_2	([0.3,0.5], [0.2,0.3])	([0.3,0.5], [0.1,0.2])	([0.3,0.5], [0.2,0.4])
Y_3	([0.4,0.5], [0.1,0.2])	([0.2,0.4], [0.2,0.3])	([0.1,0.5], [0.2,0.3])
Y_4	([0.2,0.7], [0.1,0.3])	([0.2,0.7], [0.1,0.3])	([0.3,0.6], [0.2,0.4])
Y_5	([0.4,0.5], [0.2,0.3])	([0.3,0.5], [0.1,0.2])	([0.4,0.5], [0.1,0.3])
Y_6	([0.3,0.6], [0.2,0.3])	([0.3,0.7], [0.1,0.2])	([0.3,0.6], [0.1,0.4])
Y_7	([0.3,0.5], [0.1,0.2])	([0.3,0.5], [0.2,0.4])	([0.1,0.8], [0.1,0.2])

Table 11 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_2^3\right)$.

	G_1	G_2	G_3
Y_1	([0.4,0.6], [0.1,0.3])	([0.2,0.6], [0.1,0.2])	([0.2,0.5], [0.2,0.4])
Y_2	([0.4,0.5], [0.3,0.5])	([0.4,0.5], [0.1,0.2])	([0.2,0.8], [0.1,0.2])
Y_3	([0.2,0.6], [0.2,0.3])	([0.2,0.7], [0.1,0.2])	([0.3,0.7], [0.2,0.3])
Y_4	([0.1,0.6], [0.2,0.3])	([0.1,0.7], [0.2,0.3])	([0.3,0.6], [0.1,0.4])
Y_5	([0.4,0.7], [0.1,0.2])	([0.2,0.6], [0.2,0.3])	([0.3,0.7], [0.1,0.2])
Y_6	([0.4,0.5], [0.2,0.3])	([0.2,0.5], [0.1,0.2])	([0.3,0.5], [0.2,0.3])
Y_7	([0.4,0.5], [0.2,0.1])	([0.2,0.7], [0.1,0.3])	([0.3,0.6], [0.1,0.2])

$$a_{1322} = ([0.3, 0.6], [0.1, 0.3]), a_{1323} = ([0.3, 0.6], [0.2, 0.4]),$$
$$a_{1331} = ([0.3, 0.5], [0.2, 0.4]), a_{1332} = ([0.3, 0.5], [0.1, 0.2]),$$
$$a_{1333} = ([0.3, 0.6], [0.2, 0.3]), a_{2111} = ([0.6, 0.7], [0.2, 0.3]),$$
$$a_{2112} = ([0.5, 0.7], [0.2, 0.3]), a_{2113} = ([0.5, 0.6], [0.2, 0.3]),$$
$$a_{2121} = ([0.4, 0.5], [0.1, 0.3]), a_{2122} = ([0.2, 0.6], [0.1, 0.4]),$$
$$a_{2123} = ([0.4, 0.5], [0.3, 0.5]), a_{2131} = ([0.3, 0.5], [0.2, 0.5]),$$
$$a_{2132} = ([0.3, 0.5], [0.3, 0.4]), a_{2133} = ([0.2, 0.6], [0.2, 0.3]),$$

$$a_{2211} = ([0.5, 0.7], [0.1, 0.2]), a_{2212} = ([0.6, 0.7], [0.1, 0.3]),$$
$$a_{2213} = ([0.4, 0.5], [0.2, 0.4]), a_{2221} = ([0.3, 0.5], [0.2, 0.3]),$$
$$a_{2222} = ([0.3, 0.5], [0.1, 0.2]), a_{2223} = ([0.3, 0.5], [0.2, 0.4]),$$
$$a_{2231} = ([0.4, 0.5], [0.3, 0.5]), a_{2232} = ([0.4, 0.5], [0.1, 0.2]),$$
$$a_{2233} = ([0.2, 0.8], [0.1, 0.2]), a_{2311} = ([0.4, 0.6], [0.1, 0.2]),$$
$$a_{2312} = ([0.5, 0.7], [0.1, 0.2]), a_{2313} = ([0.6, 0.7], [0.1, 0.3]),$$
$$a_{2321} = ([0.1, 0.7], [0.2, 0.3]), a_{2322} = ([0.2, 0.7], [0.1, 0.2]),$$

Table 12 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_3^1\right)$.

	G_1	G_2	G_3
Y_1	([0.6,0.7], [0.1,0.3])	([0.7,0.9], [0.0,0.1])	([0.8,0.9], [0.0,0.1])
Y_2	([0.4,0.6], [0.1,0.2])	([0.5,0.7], [0.1,0.2])	([0.6,0.7], [0.1,0.3])
Y_3	([0.2,0.4], [0.2,0.3])	([0.3,0.6], [0.2,0.3])	([0.4,0.6], [0.2,0.4])
Y_4	([0.7,0.8], [0.0,0.1])	([0.8,0.9], [0.0,0.1])	([0.4,0.7], [0.2,0.3])
Y_5	([0.5,0.6], [0.2,0.3])	([0.4,0.5], [0.1,0.2])	([0.6,0.7], [0.2,0.3])
Y_6	([0.2,0.3], [0.5,0.6])	([0.3,0.5], [0.3,0.4])	([0.3,0.6], [0.2,0.4])
Y_7	([0.5,0.6], [0.3,0.4])	([0.2,0.3], [0.4,0.5])	([0.7,0.8], [0.1,0.2])

Table 13 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_3^2\right)$.

	G_1	G_2	G_3
Y_1	([0.4,0.6], [0.2,0.3])	([0.3,0.6], [0.1,0.3])	([0.3,0.6], [0.2,0.4])
Y_2	([0.1,0.7], [0.2,0.3])	([0.2,0.7], [0.1,0.2])	([0.5,0.6], [0.1,0.3])
Y_3	([0.5,0.7], [0.2,0.3])	([0.5,0.6], [0.1,0.3])	([0.4,0.5], [0.1,0.2])
Y_4	([0.1,0.7], [0.2,0.3])	([0.2,0.7], [0.1,0.3])	([0.3,0.6], [0.1,0.2])
Y_5	([0.4,0.5], [0.1,0.3])	([0.2,0.6], [0.1,0.4])	([0.1,0.7], [0.2,0.3])
Y_6	([0.5,0.6], [0.1,0.3])	([0.4,0.6], [0.2,0.4])	([0.2,0.6], [0.1,0.3])
Y_7	([0.2,0.7], [0.1,0.2])	([0.2,0.8], [0.1,0.2])	([0.1,0.8], [0.1,0.2])

Table 14 Interval-valued intuitionistic fuzzy decision matrix $R\left(t_3^3\right)$.

	G_1	G_2	G_3
Y_1	([0.3,0.5], [0.2,0.4])	([0.3,0.5], [0.1,0.2])	([0.3,0.6], [0.2,0.3])
Y_2	([0.3,0.7], [0.2,0.3])	([0.3,0.5], [0.1,0.4])	([0.2,0.5], [0.2,0.4])
Y_3	([0.4,0.7], [0.2,0.3])	([0.4,0.5], [0.1,0.3])	([0.5,0.7], [0.1,0.2])
Y_4	([0.2,0.8], [0.1,0.2])	([0.2,0.8], [0.1,0.2])	([0.2,0.7], [0.1,0.2])
Y_5	([0.2,0.8], [0.1,0.2])	([0.2,0.5], [0.1,0.3])	([0.1,0.7], [0.2,0.3])
Y_6	([0.2,0.7], [0.1,0.3])	([0.1,0.7], [0.2,0.3])	([0.2,0.6], [0.3,0.4])
Y_7	([0.2,0.8], [0.1,0.2])	([0.4,0.5], [0.2,0.3])	([0.1,0.6], [0.2,0.4])

$a_{2323} = ([0.5, 0.6], [0.1, 0.3]), a_{2331} = ([0.3, 0.7], [0.2, 0.3]),$

$a_{2332} = ([0.3, 0.5], [0.1, 0.4]), a_{2333} = ([0.2, 0.5], [0.2, 0.4]),$

$a_{3111} = ([0.4, 0.5], [0.2, 0.4]), a_{3112} = ([0.5, 0.6], [0.2, 0.3]),$

$a_{3113} = ([0.4, 0.6], [0.1, 0.2]), a_{3121} = ([0.4, 0.5], [0.2, 0.3]),$

$a_{3122} = ([0.7, 0.8], [0.1, 0.2]), a_{3123} = ([0.5, 0.7], [0.2, 0.3]),$

$a_{3131} = ([0.4, 0.6], [0.1, 0.3]), a_{3132} = ([0.3, 0.4], [0.2, 0.3]),$

$a_{3133} = ([0.3, 0.6], [0.1, 0.2]), a_{3211} = ([0.3, 0.5], [0.1, 0.3]),$

$a_{3212} = ([0.4, 0.5], [0.1, 0.3]), a_{3213} = ([0.3, 0.6], [0.3, 0.4]),$

$a_{3221} = ([0.4, 0.5], [0.1, 0.2]), a_{3222} = ([0.2, 0.4], [0.2, 0.3]),$

$a_{3223} = ([0.1, 0.5], [0.2, 0.3]), a_{3231} = ([0.2, 0.6], [0.2, 0.3]),$

$a_{3232} = ([0.2, 0.7], [0.1, 0.2]), a_{3233} = ([0.3, 0.7], [0.2, 0.3]),$

$a_{3311} = ([0.2, 0.4], [0.2, 0.3]), a_{3312} = ([0.3, 0.6], [0.2, 0.3]),$

$a_{3313} = ([0.4, 0.6], [0.2, 0.4]), a_{3321} = ([0.5, 0.7], [0.2, 0.3]),$

$a_{3322} = ([0.5, 0.6], [0.1, 0.3]), a_{3323} = ([0.4, 0.5], [0.1, 0.2]),$

$a_{3331} = ([0.4, 0.7], [0.2, 0.3]), a_{3332} = ([0.4, 0.5], [0.1, 0.3]),$

$a_{3333} = ([0.5, 0.7], [0.1, 0.2]), a_{4111} = ([0.7, 0.8], [0.1, 0.2]),$

$a_{4112} = ([0.6, 0.8], [0.0, 0.1]), a_{4113} = ([0.6, 0.7], [0.1, 0.2]),$

$a_{4121} = ([0.4, 0.5], [0.2, 0.3]), a_{4122} = ([0.2, 0.6], [0.1, 0.3]),$

$a_{4123} = ([0.2, 0.8], [0.1, 0.2]), a_{4131} = ([0.3, 0.5], [0.1, 0.3]),$

$a_{4132} = ([0.3, 0.5], [0.2, 0.3]), a_{4133} = ([0.2, 0.7], [0.1, 0.2]),$

$a_{4211} = ([0.6, 0.7], [0.1, 0.2]), a_{4212} = ([0.7, 0.8], [0.1, 0.2]),$

$a_{4213} = ([0.5, 0.7], [0.1, 0.3]), a_{4221} = ([0.2, 0.7], [0.1, 0.3]),$

$a_{4222} = ([0.2, 0.7], [0.1, 0.3]), a_{4223} = ([0.3, 0.6], [0.2, 0.4]),$

$a_{4231} = ([0.1, 0.6], [0.2, 0.3]), a_{4232} = ([0.1, 0.7], [0.2, 0.3]),$

$a_{4233} = ([0.3, 0.6], [0.1, 0.4]), a_{4311} = ([0.7, 0.8], [0.0, 0.1]),$

$a_{4312} = ([0.8, 0.9], [0.0, 0.1]), a_{4313} = ([0.4, 0.7], [0.2, 0.3]),$

$a_{4321} = ([0.1, 0.7], [0.2, 0.3]), a_{4322} = ([0.2, 0.7], [0.1, 0.3]),$

$a_{4323} = ([0.3, 0.6], [0.1, 0.2]), a_{4331} = ([0.2, 0.8], [0.1, 0.2]),$

$a_{4332} = ([0.2, 0.8], [0.1, 0.2]), a_{4333} = ([0.2, 0.7], [0.1, 0.2]),$

$a_{5111} = ([0.5, 0.7], [0.1, 0.3]), a_{5112} = ([0.7, 0.8], [0.1, 0.2]),$

$a_{5113} = ([0.4, 0.5], [0.2, 0.4]), a_{5121} = ([0.3, 0.5], [0.2, 0.3]),$

$a_{5122} = ([0.3, 0.6], [0.1, 0.3]), a_{5123} = ([0.3, 0.6], [0.1, 0.2]),$

$a_{5131} = ([0.2, 0.6], [0.1, 0.2]), a_{5132} = ([0.2, 0.5], [0.1, 0.4]),$

$a_{5133} = ([0.4, 0.5], [0.2, 0.3]), a_{5211} = ([0.5, 0.7], [0.2, 0.3]),$

$a_{5212} = ([0.5, 0.7], [0.1, 0.3]), a_{5213} = ([0.4, 0.6], [0.2, 0.3]),$

$a_{5221} = ([0.4, 0.5], [0.2, 0.3]), a_{5222} = ([0.3, 0.5], [0.1, 0.2]),$

$a_{5223} = ([0.4, 0.5], [0.1, 0.3]), a_{5231} = ([0.4, 0.7], [0.1, 0.2]),$

$a_{5232} = ([0.2, 0.6], [0.2, 0.3]), a_{5233} = ([0.3, 0.7], [0.1, 0.2]),$

$a_{5311} = ([0.5, 0.6], [0.2, 0.3]), a_{5312} = ([0.4, 0.5], [0.1, 0.2]),$

$a_{5313} = ([0.6, 0.7], [0.2, 0.3]), a_{5321} = ([0.4, 0.5], [0.1, 0.3]),$

$a_{5322} = ([0.2, 0.6], [0.1, 0.4]), a_{5323} = ([0.1, 0.7], [0.2, 0.3]),$

$a_{5331} = ([0.2, 0.8], [0.1, 0.2]), a_{5332} = ([0.2, 0.5], [0.1, 0.3]),$

$a_{5333} = ([0.1, 0.7], [0.2, 0.3]), a_{6111} = ([0.2, 0.3], [0.5, 0.6]),$

$a_{6112} = ([0.3, 0.5], [0.4, 0.5]), a_{6113} = ([0.4, 0.6], [0.3, 0.4]),$

$a_{6121} = ([0.3, 0.6], [0.2, 0.3]), a_{6122} = ([0.2, 0.7], [0.1, 0.2]),$

$a_{6123} = ([0.2, 0.6], [0.1, 0.4]), a_{6131} = ([0.3, 0.5], [0.2, 0.3]),$

$a_{6132} = ([0.3, 0.5], [0.1, 0.2]), a_{6133} = ([0.3, 0.5], [0.2, 0.4]),$

$a_{6211} = ([0.3, 0.4], [0.4, 0.6]), a_{6212} = ([0.2, 0.4], [0.5, 0.6]),$

$a_{6213} = ([0.4, 0.5], [0.4, 0.5]), a_{6221} = ([0.3, 0.6], [0.2, 0.3]),$

$a_{6222} = ([0.3, 0.7], [0.1, 0.2]), a_{6223} = ([0.3, 0.6], [0.1, 0.4]),$

$a_{6231} = ([0.4, 0.5], [0.2, 0.3]), a_{6232} = ([0.2, 0.5], [0.1, 0.2]),$

$a_{6233} = ([0.3, 0.5], [0.2, 0.3]), a_{6311} = ([0.2, 0.3], [0.5, 0.6]),$

$a_{6312} = ([0.3, 0.5], [0.3, 0.4]), a_{6313} = ([0.3, 0.6], [0.2, 0.4]),$

$a_{6321} = ([0.5, 0.6], [0.1, 0.3]), a_{6322} = ([0.4, 0.6], [0.2, 0.4]),$

$a_{6323} = ([0.2, 0.6], [0.1, 0.3]), a_{6331} = ([0.2, 0.7], [0.1, 0.3]),$

$a_{6332} = ([0.1, 0.7], [0.2, 0.3]), a_{6333} = ([0.2, 0.6]], [0.3, 0.4],$

$a_{7111} = ([0.4, 0.5], [0.3, 0.4]), a_{7112} = ([0.2, 0.5], [0.3, 0.5]),$

$a_{7113} = ([0.4, 0.7], [0.2, 0.3]), a_{7121} = ([0.4, 0.6], [0.2, 0.3]),$

$a_{7122} = ([0.4, 0.5], [0.1, 0.2]), a_{7123} = ([0.4, 0.5], [0.2, 0.3]),$

$a_{7131} = ([0.4, 0.7], [0.1, 0.3]), a_{7132} = ([0.2, 0.7], [0.2, 0.3]),$

$a_{7133} = ([0.4, 0.8], [0.1, 0.2]), a_{7211} = ([0.3, 0.5], [0.3, 0.5]),$

$a_{7212} = ([0.4, 0.6], [0.3, 0.4]), a_{7213} = ([0.4, 0.5], [0.2, 0.4]),$

$a_{7221} = ([0.3, 0.5], [0.1, 0.2]), a_{7222} = ([0.3, 0.5], [0.2, 0.4]),$

$a_{7223} = ([0.1, 0.8], [0.1, 0.2]), a_{7231} = ([0.4, 0.5], [0.2, 0.1]),$

$a_{7232} = ([0.2, 0.7], [0.1, 0.3]), a_{7233} = ([0.3, 0.6], [0.1, 0.2]),$

$a_{7311} = ([0.5, 0.6], [0.3, 0.4]), a_{7312} = ([0.2, 0.3], [0.4, 0.5]),$

$a_{7313} = ([0.7, 0.8], [0.1, 0.2]), a_{7321} = ([0.2, 0.7], [0.1, 0.2]),$

$a_{7322} = ([0.2, 0.8], [0.1, 0.2]), a_{7323} = ([0.1, 0.8], [0.1, 0.2]),$

$$a_{7331} = ([0.2, 0.8], [0.1, 0.2]), a_{7332} = ([0.4, 0.5], [0.2, 0.3]),$$
$$a_{7333} = ([0.1, 0.6], [0.2, 0.4]).$$

Step 2. By **Theorem 3.1**, $\tilde{\mathcal{A}}_{IVIF} \in \mathbf{T_{IVIF}}(4, 7 \times 3 \times 3 \times 3)$. Let $X_2 = \omega$ (the years weight), $X_3 = \lambda$ (the decision-makers weight), and $X_4 = \xi$ (the attributes weight), we have

$$\begin{aligned}
&\mathrm{GIIFWA}\left(\tilde{\mathcal{A}}_{IF} \circ X_2 \circ X_3 \circ X_4\right)\\
&= \left(\left[1 - \prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(1 - \mu^{l}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}},\right.\right.\\
&\quad \left. 1 - \prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(1 - \mu^{u}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}}\right],\\
&\quad \left[\prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(\nu^{l}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}},\right.\\
&\quad \left.\left.\prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(\nu^{u}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}}\right]\right)\\
&= \left(([0.556, 0.754], [0.000, 0.207]), ([0.415, 0.630],\right.\\
&\quad [0.134, 0.283]), ([0.363, 0.581], [0.153, 0.290]),\\
&\quad ([0.479, 0.749], [0.000, 0.208]), ([0.391, 0.632],\\
&\quad [0.135, 0.273]), ([0.279, 0.542], [0.233, 0.384]),\\
&\quad \left.([0.349, 0.626], [0.183, 0.304])\right)^{T}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
Y_1 &= ([0.556, 0.754], [0.000, 0.207]),\\
Y_2 &= ([0.415, 0.630], [0.134, 0.283]),\\
Y_3 &= ([0.363, 0.581], [0.153, 0.290]),\\
Y_4 &= ([0.479, 0.749], [0.000, 0.208]),\\
Y_5 &= ([0.391, 0.632], [0.135, 0.273]),\\
Y_6 &= ([0.279, 0.542], [0.233, 0.384]),\\
Y_7 &= ([0.349, 0.626], [0.183, 0.304]).
\end{aligned}$$

Step 3. To rank the IVIFNs Y_{i_1} ($i_1 \in [7]$), we calculate the scores $s\left(Y_{i_1}\right)(i_1 \in [7])$ by the **Definition 2.8**. Then, we have $s(Y_1) = 0.552$, $s(Y_2) = 0.314$, $s(Y_3) = 0.251$, $s(Y_4) = 0.510$, $s(Y_5) = 0.307$, $s(Y_6) = 0.102$, $s(Y_7) = 0.244$.

Step 4. By the scores $s\left(Y_{i_1}\right)$ result, the ranking order of all the alternatives is generated as $Y_1 > Y_4 > Y_2 > Y_5 > Y_3 > Y_7 > Y_6$. Therefore, the agroecological region with the most comprehensive functions is Y_1-Wuhan-Ezhou-Huanggang.

We can also replace the GIIFWA with the GIIFWG to resolve this problem. The difference starts from step 2.

Step 2'. By the **Theorem 3.2**, we have

$$\begin{aligned}
&\mathrm{GIIFWG}\left(\tilde{\mathcal{A}}_{IF} \circ X_2 \circ X_3 \circ X_4\right)\\
&= \left(\left[\prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(\mu^{l}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}},\right.\right.\\
&\quad \left.\prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(\mu^{u}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}}\right],\\
&\quad \left[1 - \prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(1 - \nu^{l}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}},\right.\\
&\quad \left.\left.1 - \prod_{i_2=1}^{3}\prod_{i_3=1}^{3}\prod_{i_4=1}^{3}\left(1 - \nu^{u}_{i_1 i_2 i_3 i_4}\right)^{x^2_{i_2} x^3_{i_3} x^4_{i_4}}\right]\right)\\
&= \left(([0.444, 0.684], [0.107, 0.249]), ([0.369, 0.610],\right.\\
&\quad [0.147, 0.302]), ([0.334, 0.562], [0.165, 0.299]),\\
&\quad ([0.346, 0.724], [0.104, 0.231]), ([0.332, 0.610],\\
&\quad [0.145, 0.282]), ([0.258, 0.514], [0.285, 0.419]),\\
&\quad \left.[0.290, 0.575], [0.214, 0.338])\right)^{T}
\end{aligned}$$

Then, we get

$$\begin{aligned}
Y_1 &= ([0.444, 0.684], [0.107, 0.249]),\\
Y_2 &= ([0.369, 0.610], [0.147, 0.302]),\\
Y_3 &= ([0.334, 0.562], [0.165, 0.299]),\\
Y_4 &= ([0.346, 0.724], [0.104, 0.231]),\\
Y_5 &= ([0.332, 0.610], [0.145, 0.282]),\\
Y_6 &= ([0.258, 0.514], [0.285, 0.419]),\\
Y_7 &= ([0.290, 0.575], [0.214, 0.338]).
\end{aligned}$$

Step 3'. To rank the IVIFNs Y_{i_1} ($i_1 \in [7]$), we calculate the scores $s\left(Y_{i_1}\right)(i_1 \in [7])$ by the **Definition 2.8**, then we get $s(Y_1) = 0.386$, $s(Y_2) = 0.266$, $s(Y_3) = 0.216$, $s(Y_4) = 0.367$, $s(Y_5) = 0.258$, $s(Y_6) = 0.034$, $s(Y_7) = 0.156$.

Step 4'. By the scores $s\left(Y_{i_1}\right)$ result, the ranking order of all the alternatives is generated as $Y_1 \succ Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_7 \succ Y_6$. Therefore, the agroecological region with the most comprehensive functions is also Y_1-Wuhan-Ezhou-Huanggang.

5.2.2. Discussion

1. The comparison of the results is shown in Table 15. By using the same data and weight information, we get the same results calculated by the GIIFWA and GIIFWG operators. That is, the agroecological region with the most comprehensive functions is Y_1-Wuhan-Ezhou-Huanggang.
2. The GIIFWA and GIIFWG operators proposed in this paper can effectively solve the dynamic multiple attribute group decision-making problem (four-dimensional data) through analyzing the above practical decision-making problem. Therefore, in order to solve the actual decision problem of high-dimensional data, the proposed methods have better adaptability. For example, it can effectively deal with multiple attribute group decision-making problem (three-dimensional data), dynamic multiple attribute group decision-making problem (four-dimensional data), and practical decision problems with higher dimension data.

6. CONCLUSION

As a generalization of fuzzy decision matrix, this paper has presented the concept of mth-order interval-valued intuitionistic fuzzy tensor and related properties. The GIIFWA and GIIFWG operators by the product of tensor with vector have been obtained and found effective to deal with the multiple attribute group decision-making and dynamic multiple attribute group decision-making problems in an interval-valued intuitionistic condition. Two typical examples have also been provided to demonstrate the efficiency and universal applicability of the proposed method.

Table 15 The comparison between the results of the GIIFWA and GIIFWG operators in this paper.

Method	Results	Sort Function Values	Preference Order	The Agroecological Region with the Most Comprehensive Functions
GIIFWA operator	$Y_1 = ([0.556, 0.754], [0.000, 0.207])$	$s(Y_1) = 0.552$	$Y_1 \succ Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_7 \succ Y_6$	Y_1
	$Y_2 = ([0.415, 0.630], [0.134, 0.283])$	$s(Y_2) = 0.314$		
	$Y_3 = ([0.363, 0.581], [0.153, 0.290])$	$s(Y_3) = 0.251$		
	$Y_4 = ([0.479, 0.749], [0.000, 0.208])$	$s(Y_4) = 0.510$		
	$Y_5 = ([0.391, 0.632], [0.135, 0.273])$	$s(Y_5) = 0.307$		
	$Y_6 = ([0.279, 0.542], [0.233, 0.384])$	$s(Y_6) = 0.102$		
	$Y_7 = ([0.349, 0.626], [0.183, 0.304])$	$s(Y_7) = 0.244$		
GIIFWG operator	$Y_1 = ([0.444, 0.684], [0.107, 0.249])$	$s(Y_1) = 0.386$	$Y_1 \succ Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_7 \succ Y_6$	Y_1
	$Y_2 = ([0.369, 0.610], [0.147, 0.302])$	$s(Y_2) = 0.266$		
	$Y_3 = ([0.334, 0.562], [0.165, 0.299])$	$s(Y_3) = 0.216$		
	$Y_4 = ([0.346, 0.724], [0.104, 0.231])$	$s(Y_4) = 0.367$		
	$Y_5 = ([0.332, 0.610], [0.145, 0.282])$	$s(Y_5) = 0.258$		
	$Y_6 = ([0.258, 0.514], [0.285, 0.419])$	$s(Y_6) = 0.034$		
	$Y_7 = ([0.290, 0.575], [0.214, 0.338])$	$s(Y_7) = 0.156$		

GIIFWA, generalized interval-valued intuitionistic fuzzy weighted averaging; GIIFWG, generalized interval-valued intuitionistic fuzzy weighted geometric.

ACKNOWLEDGMENT

The work was supported in part by the National Natural Science Foundation of China (Grant nos. 11571292 and 11747104), the General Project of Hunan Provincial Natural Science Foundation of China (2016JJ2043 and 2019JJ50125), the Postgraduate Innovation Project of Hunan Province of China (CX2017B263), Since the Guangxi Municipality Project for the Basic Ability Enhancement of Young and Middle-Aged Teachers (KY2016YB532) has been completed, the project is deleted in the final version.

REFERENCES

[1] Z. Xu, X. Cai, Intuitionistic Fuzzy Information Aggregation - Theory and Applications, Science Press and Springer-Verlag, Beijing and Berlin, 2012.

[2] Y. Liu, J. Liu, Z. Hong, A multiple attribute decision making approach based on new similarity measures of interval-valued hesitant fuzzy sets. Int. J. Comput. Intell. Syst. 11 (2018), 15–32.

[3] L. Wang, R.M. Rodrandíguez, Y.-M. Wang, A dynamic multi-attribute group emergency decision making method considering experts' hesitation. Int. J. Comput. Intell. Syst. 11 (2018), 163–182.

[4] J. Qin, X. Liu, W. Pedrycz, Multi-attribute group decision making based on Choquet integral under interval-valued intuitionistic fuzzy environment. Int. J. Comput. Intell. Syst. 9 (2016), 133–152.

[5] Y. He, Z. He, Extensions of Atanassovs intuitionistic fuzzy interaction bonferroni means and their application to multiple-attribute decision making. IEEE Trans. Fuzzy Syst. 24 (2016), 558–573.

[6] S.S. Hashemi, S.H.R. Hajiagha, E.K. Zavadskas, H.A. Mahdiraji, Multicriteria group decision making with ELECTRE III method based on interval-valued intuitionistic fuzzy information, Appl. Math. Model. 40 (2016), 1554–1564.

[7] K. Atanassov, T. Krassimir, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986), 87–96.

[8] K. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets Syst. 31 (1989), 343–349.

[9] Z. Xu, A method based on distance measure for interval-valued intuitionistic fuzzy group decision making, Info. Sci. 180 (2010), 181–190.

[10] O. Kabak, B. Ervural, Multiple attribute group decision making: a generic conceptual framework and a classification scheme, Knowledge Based Syst. 123 (2017), 13–30.

[11] Z. Yang, J. Li, L. Huang, Y. Shi, Developing dynamic intuitionistic normal fuzzy aggregation operators for multi-attribute decision-making with time sequence preference, Expert Syst. Appl. 82 (2017), 344–356.

[12] P. Liu, Multiple attribute group decision making method based on interval-valued intuitionistic fuzzy power Heronian aggregation operators, Comput. Ind. Eng. 108 (2017), 199–212.

[13] S.-M. Chen, Z.-C. Huang, Multiattribute decision making based on interval-valued intuitionistic fuzzy values and particle swarm optimization techniques, Info. Sci. 397–398 (2017), 206–218.

[14] C.Y. Wang, S.-M. Chen, An improved multiattribute decision making method based on new score function of interval-valued intuitionistic fuzzy values and linear programming methodology, Info. Sci. 411 (2017), 176–184.

[15] J. Qiu, L. Li, A new approach for multiple attribute group decision making with interval-valued intuitionistic fuzzy information, Appl Soft Comput. 61 (2017), 111–121.

[16] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symb. Comput. 40 (2005), 1302–1324.

[17] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, in Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP '05), Puerto Vallarta, Mexico, 2005, vol. 1, 129–132.

[18] W. Ding, L. Qi, Y. Wei, M-tensors and nonsingular M-tensors, Linear Algebra Appl. 439 (2013), 3264–3278.

[19] W. Ding, Y. Wei, Solving multi-linear systems with M-tensors, J. Sci. Comput. 68 (2016), 689–715.

[20] L. Qi, Symmetric nonnegative tensors and copositive tensors, Linear Algebra Appl. 439 (2013), 228–238.

[21] G. Ni, L. Qi, M. Bai, Geometric measure of entanglement and U-eigenvalues of tensors, SIAM J. Matrix Anal. Appl. 35 (2014), 73–87.
[22] M. Ng, L. Qi, G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, SIAM J. Matrix Anal. Appl. 31 (2010), 1090–1099.
[23] M. Rajesh Kannan, N. Shaked-Monderer, A. Berman, Some properties of strong H-tensors and general H-tensors, Linear Algebra Appl. 476 (2015), 42–55.
[24] Y. Wang, G. Zhou, L. Caccetta, Nonsingular H-tensor and its criteria, J. Ind. Manage. Optim. 12 (2016), 1173–1186.
[25] J.-Y. Shao, A general product of tensors with applications, Linear Algebra Appl. 439 (2013), 2350–2366.
[26] J.-Y. Shao, H.-Y. Shan, L. Zhang, On some properties of the determinants of tensors, Linear Algebra Appl. 439 (2013), 3057–3069.
[27] C. Bu, X. Zhang, J. Zhou, W. Wang, Y. Wei, The inverse, rank and product of tensors, Linear Algebra Appl. 446 (2014), 269–280.
[28] S. Pumplün, Tensor products of nonassociative cyclic algebras, J. Algebra. 451 (2016), 145–165.
[29] O. Giladi, J. Prochno, C. Schütt, N. Tomczak-Jaegermann, E. Werner, On the geometry of projective tensor products, J. Funct. Anal. 273 (2017), 471–495.
[30] J. Gutiérrez García, U. Höhle, T. Kubiak, Tensor products of complete lattices and their application in constructing quantales, Fuzzy Sets Syst. 313 (2017), 43–60.
[31] T. Hilberdink, Matrices with multiplicative entries are tensor products, Linear Algebra Appl. 532 (2017), 179–197.
[32] S. Deng, J. Liu, X. Wang, The Properties of Fuzzy Tensor and Its Application in Multiple Attribute Group Decision Making, IEEE Trans. Fuzzy Syst. 27 (2019), 589–597.
[33] Z. Xu, Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making, Control Decis. 22 (2007), 215–219.
[34] K. Atanassov, Remarks on the intuitionistic fuzzy sets, Fuzzy Sets Syst. 51 (1992), 117–118.
[35] Z. Xu, R.R. Yager, Dynamic intuitionistic fuzzy multi-attribute decision making, Int. J. Approx. Reason. 48 (2008), 246–262.