Journal of Statistical Theory and Applications
Vol. **18(2)**, June 2019, *pp.* 129–135
DOI: https://doi.org/10.2991/jsta.d.190515.001; ISSN 1538-7887
https://www.atlantis-press.com/journals/jsta

Generalized Order Statistics from Marshall–Olkin Extended Exponential Distribution

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ARTICLE INFO

Article History

Received 11 Sept 2017
Accepted 04 Dec 2019

Keywords

Generalized order statistics
order statistics
record values
marginal moment generating function
joint moment generating function
Marshall–Olkin extended exponential distribution
characterization

ABSTRACT

A.W. Marshall, I. Olkin, Biometrika. 84 (1997), 641–652, introduced an interesting method of adding a new parameter to an existing distribution. The resulting new distribution is called as Marshall–Olkin extended distribution. In this paper some recurrence relations for marginal and joint moment generating function of generalized order statistics from Marshall–Olkin extended exponential distribution are derived, and the characterization results are presented. Further, the results are deduced for order statistics and record values.

1. INTRODUCTION

The Marshall–Olkin extended exponential distribution is considered as a probability model for the lifetime of the product, if the lifetime shows a large variability.

A random variable X is said to have the Marshall–Olkin extended exponential distribution if its *pdf* is of the form (Marshall and Olkin [1])

$$f(x) = \frac{\lambda e^{-x}}{[1-(1-\lambda)e^{-x}]^2}, \qquad x > 0, \lambda > 0, \tag{1}$$

and corresponding survival function

$$\overline{F}(x) = \frac{\lambda e^{-x}}{[1-(1-\lambda)\,e^{-x}]}, \qquad x > 0, \lambda > 0. \tag{2}$$

Now in view of (1) and (2), we have

$$\overline{F}(x) = [1-(1-\lambda)\,e^{-x}]\,f(x)\,. \tag{3}$$

Let $n \geq 2$ be a given integer and $\widetilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{R}^{n-1}, k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \quad \text{for} \quad 1 \leq i \leq n-1.$$

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The random variables $X(1,n,\widetilde{m},k), X(2,n,\widetilde{m},k), \ldots, X(n,n,\widetilde{m},k)$ are said to be generalized order statistics (*gos*) from an absolutely continuous distribution function $F()$ with the probability density funtion (*pdf*) $f()$, if their joint density function is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{4}$$

on the cone $F^{-1}(0) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$.

If $m_i = m = 0; i = 1 \ldots n-1, k = 1$, we obtain the joint *pdf* of the order statistics and for $m = -1, k \in N$, we get joint *pdf* of $k-th$ record values. (Kamps [2]).

In view of (4), the marginal *pdf* of $r-th$ *gos* $X(r,n,m,k)$ is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[\overline{F}(x)\right]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x)) \tag{5}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is

$$\begin{aligned} f_{X(r,n,m,k).X(s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!\;(s-r-1)!}\left[\overline{F}(x)\right]^m g_m^{r-1}(F(x)) \\ &\times\left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1} f(x) f(y),\ x<y, \end{aligned} \tag{6}$$

where $\overline{F}(x) = 1 - F(x)$

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \quad m \ne -1 \\ \log\left(\dfrac{1}{1-x}\right) & , \quad m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m\, dt, \quad x \in [0,1].$$

Relations for marginal and joint moment generating functions of record values and *gos* for some specific distributions are investigated by several authors in literature. For more detailed survey one may refer to Ahsanullah and Raqab [3], Raqab and Ahsanullah [4,5], Saran and Pandey [6], Al-Hussaini *et al.* [7,8], and references therein. Here in this paper some recurrence relations for marginal and joint moment generating function of generalized order statistics from Marshall–Olkin extended exponential distribution are derived. Further the results are deduced for order statistics and $k-th$ record values.

Let us denote the moment generating function of $X(r,n,m,k)$ by $M_{X(r,n,m,k)}(t)$ and it $j-th$ derivative by $M^{(j)}_{X(r,n,m,k)}(t)$

$$M_{X_{(r,n,m,k)}}(t) = \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{tx}\left[\overline{F}(x)\right]^{\gamma_i-1} g_m^{r-1}(F(x)) f(x)\, dx \tag{7}$$

and the joint moment generating function of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$ is denoted by $M_{X(r,s,n,m,k)}(t_1,t_2)$ and its $i,j-th$ partial derivative by $M^{(i,j)}_{X(r,s,n,m,k)}(t_1,t_2)$

$$\begin{aligned} M_{X(r,s,n,m,k)}(t_1,t_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x}^{\infty} e^{t_1x+t_2y}\left[\overline{F}(x)\right]^m f(x) \\ &\times\ g_m^{r-1}(F(x))\left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1} f(y)\, dydx. \end{aligned} \tag{8}$$

Using the relation (3), we shall derived the recurrence relation for moment generating function(*mgf*) of generalized order statistics from Marshall–Olkin extended exponential distribution.

2. MARGINAL MOMENT GENERATING FUNCTION

Lemma 2.1. For the distribution given in (1) and $1 \leq r \leq n$, $M_{X(r,n,m,k)}(t)$ exists.

Proof: Since,

$$g_m^{r-1}(F(x)) = \frac{1}{(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \left(\bar{F}(x)\right)^{(m+1)i}.$$

Therefore, (7) may also be written as

$$M_{X(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_0^{\infty} e^{tx} \left(\bar{F}(x)\right)^{\gamma_{r-i}-1} f(x)\,dx.$$

Now in view of (2) and (3), we have

$$\begin{aligned} M_{X(r,n,m,k)}(t) &= \frac{C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \lambda^{\gamma_{r-i}} \\ &\quad \times \int_0^{\infty} \frac{e^{-(\gamma_{r-i}-t)x}}{\left[1-(1-\lambda)e^{-x}\right]^{\gamma_{r-i}+1}} dx. \end{aligned}$$

Since,

$$\int_0^{\infty} \frac{e^{-\mu x}}{(1-\beta e^{-x})^{\rho}} dx = B(\mu, 1)\,{}_2F_1(\rho, \mu; \mu+1; \beta),$$

where $B(m, n)$ is complete beta function

and

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n\ z^n}{(c)_n\ n!}$$

$$(x)_n = x(x+1)\ldots(x+n-1), \quad n > 0.$$

(Gradshteyn and Ryzhik [9])

Therefore,

$$\begin{aligned} M_{X(r,n,m,k)}(t) &= \frac{C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \lambda^{\gamma_{r-i}} \\ &\quad \times B(\gamma_{r-i}-t, 1)\,{}_2F_1(\gamma_{r-i}+1, \gamma_{r-i}-t; \gamma_{r-i}-t+1; (1-\lambda)). \end{aligned}$$

Hence the lemma.

Lemma 2.2. For $2 \leq r \leq n, n \geq 2, k = 1, 2, \ldots$

i.
$$M_{X(r,n,m,k)}(t) - M_{X(r-1,n,m,k)}(t) = \frac{C_{r-1}}{\gamma_r\ (r-1)!}\ t \int_{-\infty}^{\infty} e^{tx} \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}(F(x))\,dx \tag{9}$$

ii.
$$\begin{aligned} &M_{X(r-1,n,m,k)}(t) - M_{X(r-1,n-1,m,k)}(t) \\ &= -\frac{(m+1)\,C_{r-2}^{(n)}}{\gamma_1\ (r-2)!}\ t \int_{-\infty}^{\infty} e^{tx} \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}(F(x))\,dx \end{aligned} \tag{10}$$

iii.
$$M_{X(r,n,m,k)}(t) - M_{X(r-1,n-1,m,k)}(t) = \frac{C_{r-1}}{\gamma_1\ (r-1)!}\ t \int_{-\infty}^{\infty} e^{tx} \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}(F(x))\,dx. \tag{11}$$

Proof: Relations (9–11) can be seen in view of Athar and Islam [10].

Theorem 2.1. Fix a positive integer k. For $n \in \mathbb{N}$, $m \in \mathbb{R}$, $2 \le r \le n$, $n \ge 2$ and $j = 1, 2, \ldots$,

$$\begin{aligned} M^{(j+1)}_{X(r,n,m,k)}(t) &= \frac{\gamma_r}{\gamma_r - t} M^{(j+1)}_{X(r-1,n,m,k)}(t) - \frac{t(1-\lambda)}{\gamma_r - t} M^{(j+1)}_{X(r,n,m,k)}(t-1) \\ &\quad + \frac{j+1}{\gamma_r - t}\left\{ M^{(j)}_{X(r,n,m,k)}(t) - (1-\lambda) M^{(j)}_{X(r,n,m,k)}(t-1) \right\}, \end{aligned} \tag{12}$$

where $M^{(j)}_{X(r,n,m,k)}(t)$ is the $j-th$ derivative of $M_{X(r,n,m,k)}(t)$.

Proof: On application of (3) in (9), we get

$$\begin{aligned} & M_{X(r,n,m,k)}(t) - M_{X(r-1,n,m,k)}(t) \\ &= \frac{t}{\gamma_r} \frac{C_{r-1}}{(r-1)!} \int_0^\infty e^{tx} \left[1 - (1-\lambda) e^{-x}\right] \left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x)\, dx \end{aligned}$$

$$M_{X(r,n,m,k)}(t) - M_{X(r-1,n,m,k)}(t) = \frac{t}{\gamma_r}\left\{ M_{X(r,n,m,k)}(t) - (1-\lambda) M_{X(r,n,m,k)}(t-1) \right\}. \tag{13}$$

Differentiating both the sides of (13) $(j+1)$ times $w.r.t.\ t$ and rearranging the terms, we get the required result.

Remark 2.1. For $m = 0, k = 1$, the recurrence relation for marginal moment generating function of order statistics from Marshall–Olkin extended exponential distribution is

$$\begin{aligned} M^{(j+1)}_{X_{r:n}}(t) &= \frac{n-r+1}{n-r+1-t} M^{(j+1)}_{X_{r-1:n}}(t) - \frac{t(1-\lambda)}{n-r+1-t} M^{(j+1)}_{X_{r:n}}(t-1) \\ &\quad + \frac{j+1}{n-r+1-t}\left\{ M^{(j)}_{X_{r:n}}(t) - (1-\lambda) M^{(j)}_{X_{r:n}}(t-1) \right\}. \end{aligned} \tag{14}$$

At $\lambda = 1$ in (14), we get the result for standard exponential distribution.

Remark 2.2. Letting $m \to -1$ in (12), we get the recurrence relation for marginal moment generating function of $k-th$ upper record values as

$$\begin{aligned} M^{(j+1)}_{X^{(k)}_{U(r)}}(t) &= \frac{k}{k-t} M^{(j+1)}_{X^{(k)}_{U(r-1)}}(t) - \frac{t(1-\lambda)}{k-t} M^{(j+1)}_{X^{(k)}_{U(r)}}(t-1) \\ &\quad + \frac{j+1}{k-t}\left\{ M^{(j)}_{X^{(k)}_{U(r)}}(t) - (1-\lambda) M^{(j)}_{X^{(k)}_{U(r)}}(t-1) \right\}. \end{aligned}$$

Theorem 2.2. For distribution as given in (1) and $2 \le r \le n$, $n \ge 2$, $j = 1, 2, \ldots$

i. $$\begin{aligned} M^{(j+1)}_{X(r,n,m,k)}(t) &= (1-\lambda) M^{(j+1)}_{X(r,n,m,k)}(t-1) \\ &\quad - \left(\tfrac{j+1}{t}\right)\left\{ M^{(j)}_{X(r,n,m,k)}(t) - (1-\lambda) M^{(j)}_{X(r,n,m,k)}(t-1) \right\} \\ &\quad - \frac{\gamma_1 \gamma_r}{(m+1)(r-1)t}\left\{ M^{(j+1)}_{X(r-1,n,m,k)}(t) - M^{(j+1)}_{X(r-1,n-1,m,k)}(t) \right\}. \end{aligned}$$

ii. $$\begin{aligned} M^{(j+1)}_{X(r,n,m,k)}(t) &= \left(\tfrac{j+1}{\gamma_1 - t}\right)\left\{ M^{(j)}_{X(r,n,m,k)}(t) - (1-\lambda) M^{(j)}_{X(r,n,m,k)}(t-1) \right\} \\ &\quad + \left(\tfrac{\gamma_1}{\gamma_1 - t}\right) M^{(j+1)}_{X(r-1,n-1,m,k)}(t) - (1-\lambda) M^{(j+1)}_{X(r,n,m,k)}(t-1). \end{aligned}$$

Proof: Results can be established on the lines of Theorem 2.1 in view of (10) and (11).

3. JOINT MOMENT GENERATING FUNCTION

Lemma 3.1. For $1 \leq r < s < n-1$, $n \geq 2$ and $k = 1, 2, ...$

$$\begin{aligned} &M_{X(r,s,n,m,k)}(t_1, t_2) - M_{X(r,s-1,n,m,k)}(t_1, t_2) \\ &= \frac{t_2}{\gamma_s} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{x}^{\infty} e^{t_1 x + t_2 y} \left[\overline{F}(x)\right]^m f(x) g_m^{r-1}((F(x)) \\ &\times \left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1} \left[\overline{F}(y)\right]^{\gamma_s} dydx. \end{aligned} \tag{15}$$

Proof: Relation (15) can be established in view of Athar and Islam [10].

Theorem 3.1. For the distribution as given in (1) and $n \in \mathbb{N}$, $m \in \mathbb{R}$, $1 \leq r < s-1 \leq n$, $k \geq 1$,

$$\begin{aligned} M^{(i,j+1)}_{X(r,s,n,m,k)}(t_1, t_2) = &\left(\frac{\gamma_s}{\gamma_s - t_2}\right) M^{(i,j+1)}_{X(r,s-1,n,m,k)}(t_1, t_2) \\ &- \frac{(1-\lambda)t_2}{\gamma_s - t_2} M^{(i,j+1)}_{X(r,s-1,n,m,k)}(t_1, t_2 - 1) \\ &+ \frac{j+1}{\gamma_s - t_2} \left\{M^{(i,j)}_{X(r,s,n,m,k)}(t_1, t_2) - (1-\lambda) M^{(i,j)}_{X(r,s-1,n,m,k)}(t_1, t_2 - 1)\right\}. \end{aligned} \tag{16}$$

Proof: In view of (3) and (15), we have

$$\begin{aligned} &M_{X(r,s,n,m,k)}(t_1, t_2) - M_{X(r,s-1,n,m,k)}(t_1, t_2) \\ &= \frac{t_2}{\gamma_s} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{0}^{\infty} \int_{x}^{\infty} e^{t_1 x + t_2 y} \left[\overline{F}(x)\right]^m f(x) g_m^{r-1}(F(x)) \\ &\times \left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1} \left[\overline{F}(y)\right]^{\gamma_s - 1} [1 - (1-\lambda)e^{-y}] f(y)\, dydx. \end{aligned}$$

After simplification, we get

$$\begin{aligned} &M_{X(r,s,n,m,k)}(t_1, t_2) - M_{X(r,s-1,n,m,k)}(t_1, t_2) \\ &= \frac{t_2}{\gamma_s} \left\{M_{X(r,s,n,m,k)}(t_1, t_2) - (1-\lambda) M_{X(r,s-1,n,m,k)}(t_1, t_2 - 1)\right\}. \end{aligned} \tag{17}$$

Differentiating (17) i times $w.r.t.$ t_1 and $j+1$ times $w.r.t.$ t_2, we get the required result.

Remark 3.1. Putting $m = 0$ and $k = 1$ in Theorem 3.1, we get the recurrence relations for joint moment generating function of order statistics and at $m \to -1$, we get the result for $k-th$ upper record values.

Remark 3.2. Theorem 2.1 can be deduced from Theorem 3.1 by letting $t_1 \to 0$.

4. CHARACTERIZATIONS

This section contains characterization results for the given distribution using the recurrence relations for the marginal as well as joint moment generating functions using *Müntz-Szász* theorem.

Theorem 4.1. The necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1) is that

$$M_{X(r,n,m,k)}(t) - M_{X(r-1,n,m,k)}(t) = \frac{t}{\gamma_r} \left\{M_{X(r,n,m,k)}(t) - (1-\lambda) M_{X(r,n,m,k)}(t-1)\right\}. \tag{18}$$

Proof: The necessary part follows immediately from (13). On the other hand if the recurrence relation (18) is satisfied, then

$$\frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} \left[\overline{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x)\, dx$$

$$-\frac{C_{r-2}}{(r-2)!} \int_{-\infty}^{\infty} e^{tx} \left[\overline{F}(x)\right]^{\gamma_{r-1} - 1} g_m^{r-2}(F(x)) f(x)\, dx$$

$$= \frac{t}{\gamma_r}\left\{\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{tx}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}(F(x)) f(x)\,dx - (1-\lambda)\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{(t-1)x}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}(F(x)) f(x)\,dx\right\}.$$

Integrating the first part on the left-hand side of the equation by parts, treating $-\frac{d}{dx}\left[\overline{F}(x)\right]^{\gamma_r}$ for integration and the rest of the terms for differentiation, we get after simplification that

$$\frac{t}{\gamma_r}\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{tx}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}(F(x))\left[-\left[\overline{F}(x)\right]+f(x)-(1-\lambda)e^{-x}f(x)\right]dx = 0.$$

Using *Müntz – Sza's̀z* theorem (See, Hwang and Lin [11]), we get

$$f(x) = \frac{\lambda e^{-x}}{\left[1-(1-\lambda)e^{-x}\right]^2},$$

which proves that $f(x)$ has the form (1).

Theorem 4.2. The necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1) is that

$$M_{X(r,s,n,m,k)}(t_1,t_2) - M_{X(r,s-1,n,m,k)}(t_1,t_2) = \frac{t_2}{\gamma_s}\left\{M_{X(r,s,n,m,k)}(t_1,t_2) - (1-\lambda)M_{X(r,s-1,n,m,k)}(t_1,t_2-1)\right\}. \tag{19}$$

Proof: The necessary part follows immediately from (17). On the other hand if the recurrence relation (19) is satisfied, then

$$\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^{\infty}\int_x^{\infty} e^{t_1x+t_2y}\left[\overline{F}(x)\right]^m f(x) g_m^{r-1}(F(x))\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s-1} f(y)\,dydx - \frac{C_{s-2}}{(r-1)!(s-r-2)!}\int_0^{\infty}\int_x^{\infty} e^{t_1x+t_2y}\left[\overline{F}(x)\right]^m f(x) g_m^{r-1}(F(x))$$
$$\times\left[\overline{F}(y)\right]^{\gamma_{s-1}-1}\left[h_m(F(y))-h_m(F(x))\right]^{s-r-2} f(y)\,dydx$$

$$= \frac{t_2}{\gamma_s}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^{\infty}\int_x^{\infty} e^{t_1x+t_2y}\left[\overline{F}(x)\right]^m f(x) g_m^{r-1}(F(x))$$
$$\times\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1} f(y)\,dydx - (1-\lambda)\frac{t_2}{\gamma_s}\frac{C_{s-1}}{(r-1)!(s-r-1)!}$$
$$\times\int_0^{\infty}\int_x^{\infty} e^{t_1x+t_2y}\left[\overline{F}(x)\right]^m f(x) g_m^{r-1}(F(x))\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1} f(y)\,dydx.$$

Integrating the first part on the left-hand side of the equation by parts, treating $-\frac{d}{dy}\left[\overline{F}(y)\right]^{\gamma_s}$ for integration and the rest of the terms for differentiation, we get after simplification that

$$\frac{t_2}{\gamma_s}\frac{C_{r-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_x^{\infty} e^{t_1x+t_2y}\left[\overline{F}(x)\right]^m g_m^{r-1}(F(x))$$
$$\times\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1}\left[-\left[\overline{F}(y)\right]+f(y)-(1-\lambda)e^{-y}f(y)\right]dydx = 0.$$

Now on application of *Müntz – Sza's̀z* theorem (See, Hwang and Lin [11]), we get

$$f(x) = \frac{\lambda e^{-x}}{\left[1-(1-\lambda)e^{-x}\right]^2}$$

which proves that $f(x)$ has the form (1).

ACKNOWLEDGMENT

Authors are thankful to Professor M. Ahsanullah, Editor in Chief, *JSTA* and learned referee who spent their valuable time to review this manuscript.

REFERENCES

1. A.W. Marshall, I. Olkin, Biometrika. 84 (1997), 641–652.
2. U. Kamps, A Concept of Generalized Order Statistics, B.G. Teubner Stuttgart, Germany, 1995.
3. M. Ahsanullah, M.Z. Raqab, Stoch. Model. Appl. 2 (2) (1999), 35–48.
4. M.Z. Raqab, M. Ahsanullah, J. Appl. Stat. Sci. 10 (2000), 27–36.
5. M.Z. Raqab, M. Ahsanullah, Pak. J. Stat. 19 (1) (2003), 1–13.
6. J. Saran, A. Pandey, Metron. 61 (2003), 27–33. https://ideas.repec.org/a/mtn/ancoec/030104.html
7. E.K. Al-Hussaini, A.A. Ahmad, M.A. Al-Kashif, Metrika. 61 (2005), 199–220.
8. E.K. Al-Hussaini, A.A. Ahmad, M.A. Al-Kashif, J. Stat. Theory Appl. 6 (2007), 134–155.
9. I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Edited by A. Jeffrey and D. Zwillinger (7th Ed.), pp. 335, Academic Press, New York, 2007.
10. H. Athar, H.M. Islam, Metron. (2004), 327–337. https://econpapers.repec.org/RePEc:mtn:ancoec:040303
11. J.S. Hwang, G.D. Lin, Analysis. 4 (1984), 143–160.