



A Generalization of the Sukhatme's Test for Two-Sample Scale Problem

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ABSTRACT

In this paper, we present nonparametric tests for the two-sample scale problem. The proposed tests include as special case the B.V. Sukhatme, Ann. Math. Stat. 28 (1957), 188–194, and J.V. Deshpande, K. Kusum, Aust. J. Stat. 26 (1984), 16–24 tests. The asymptotic distribution of the test statistics is derived and its Pitman efficiency is worked out with respect to some competing tests. For the illustrative purpose, a numerical example for a real life data set is provided. The simulation study is carried out to assess the power of proposed tests.

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1. INTRODUCTION

Let X_1, \dots, X_{n_1} and Y_1, \dots, Y_{n_2} be two independent random samples from populations X and Y having absolutely continuous distribution functions (cdfs) $F(x)$ and $G(x)$, respectively with common known quantile ξ_q of order q , that is, $F(\xi_q) = G(\xi_q) = q$, $0 \leq q \leq 1$. Without loss of generality, we assume that ξ_q is zero for prespecified q . Suppose that the populations X and Y are alike except differing in their scale parameters. Thus, if we take $G(x) = F(x/\theta)$, then we wish to test the null hypothesis:

$$H_0: \theta = 1.$$

against the alternative hypothesis:

$$H_A: \theta > 1. \quad (1)$$

Under the null hypothesis, X 's and Y 's are alike, but under the alternative, Y 's will have more variation than X 's.

For the above problem, with the condition that two distributions have same median, many nonparametric tests are available in literature including Mood [1], Sukhatme [2,3], Ansari and Bradley [4], Siegel and Tukey [5], Capon [6], Klotz [7], Tamura [8–10], Yanagawa [11], Kochar and Gupta [12], Kössler [13,14], ÖzTÜRK [15], Kössler and Kumar [16], and references cited therein.

Nonparametric tests for the two-sample scale problem with common quantile different from median was initially proposed by Deshpande and Kusum [17], which was further modified by Kusum [18], Mehra and Rao [19], Mahajan *et al.* [20], and Kössler and Kumar [21]. This type of problem has several practical applications.

As an example, consider the survey of Hills M. and M345 course team of The Open University, given in Hand *et al.* [22], in which the survey team asked from two groups of students to guess the width of a lecture hall in the metres for group 1 and in feet for group 2. In this survey, 5% of students guess the width less than the common width for both the groups, which shows that both groups do not have the same median but rather have same quantile of order 0.05. Now, in such a case the testing problem is to check whether the variation in guessing in metres is more than guessing in feet or not, when both groups have common quantile of order 0.05.

The tests are proposed in Section 2 and their distributions are established in Section 3. The comparison of tests with respect to (w.r.t.) some existing tests, in terms of Pitman asymptotic relative efficiencies (AREs) is given in Section 4. To see the implementation of proposed tests,

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an illustrative example is provided in Section 5. In Section 6, Monte Carlo simulation study is carried out to assess the performance of proposed tests.

2. THE PROPOSED TESTS

Consider m and j as fixed nonnegative integers such that $1 \leq 2m + 1 \leq n_1$ and $1 \leq j \leq n_2$. Define the following two kernels $h^{(1)}$ and $h^{(2)}$ as

$$h^{(1)}(X_1, \dots, X_{2m+1}; Y_j) = \begin{cases} 1, & \text{if } 0 \leq M_X \leq Y_j \text{ and } X_i, Y_j \geq 0; i = 1, \dots, 2m + 1 \\ & \text{or } Y_j \leq M_X \leq 0 \text{ and } X_i, Y_j \leq 0; i = 1, \dots, 2m + 1 \\ 0, & \text{otherwise} \end{cases}$$

$$h^{(2)}(X_1, \dots, X_{2m+1}; Y_j) = \begin{cases} 1, & \text{if } 0 \leq M_X \leq Y_j \text{ and } X_i, Y_j \geq 0; i = 1, \dots, 2m + 1 \\ & \text{or } Y_j \leq M_X \leq 0 \text{ and } X_i, Y_j \leq 0; i = 1, \dots, 2m + 1 \\ -1, & \text{if } 0 \leq Y_j \leq M_X \text{ and } X_i, Y_j \geq 0; i = 1, \dots, 2m + 1 \\ & \text{or } M_X \leq Y_j \leq 0 \text{ and } X_i, Y_j \leq 0; i = 1, \dots, 2m + 1 \\ 0, & \text{otherwise.} \end{cases}$$

where $M_X = \text{Median of } X_1, \dots, X_{2m+1}$.

The two-sample U -statistics associated with kernel $h^{(c)}$, $c = 1, 2$ is defined as

$$U_m^{(c)}(X_1, \dots, X_{n_1}; Y_1, \dots, Y_{n_2}) = \left[\binom{n_1}{2m+1} \binom{n_2}{1} \right]^{-1} \sum_s \sum_{j=1}^{n_2} h^{(c)}(X_{i_1}, \dots, X_{i_{2m+1}}; Y_j), \quad (2)$$

where s is summation extended over all possible combinations (i_1, \dots, i_{2m+1}) of $(2m + 1)$ integers chosen from $(1, \dots, n_1)$. The test rejects H_0 in favor of H_A for large values of $U_m^{(c)}$, $c = 1, 2$.

In particular

1. For $m = 0$, the test statistics $U_m^{(1)}$ corresponds to test statistics of Sukhatme [2].
2. For $m = 0$, the test statistics $U_m^{(2)}$ corresponds to test statistics of Deshpande and Kusum [17].

Thus the proposed U -statistics $U_m^{(1)}$ and $U_m^{(2)}$ are the extended version of tests of Sukhatme [2] and Deshpande and Kusum [17], respectively.

3. DISTRIBUTION OF THE TEST STATISTICS

The expected value of $U_m^{(c)}$ is

$$E(U_m^{(c)}) = \left[\binom{n_1}{2m+1} \binom{n_2}{1} \right]^{-1} \sum_c \sum_{j=1}^{n_2} E[h^{(c)}(X_{i_1}, \dots, X_{i_{2m+1}}; Y_j)]$$

$$= E[h^{(c)}(X_{i_1}, \dots, X_{i_{2m+1}}; Y_j)].$$

For $c = 1$,

$$E(U_m^{(1)}) = \int_0^\infty P(M_X \leq t) P(Y_j = t) dt + \int_{-\infty}^0 P(M_X \geq t) P(Y_j = t) dt.$$

Under H_0 ,

$$E(U_m^{(1)}) = \frac{1}{2} \left((1-q)^{2m+2} + q^{2m+2} \right). \quad (3)$$

For $c = 2$,

$$E(U_m^{(2)}) = \int_0^\infty P(M_X \leq t) P(Y_j = t) dt + \int_{-\infty}^0 P(M_X \geq t) P(Y_j = t) dt.$$

$$- \int_0^\infty P(M_X \geq t) P(Y_j = t) dt - \int_{-\infty}^0 P(M_X \leq t) P(Y_j = t) dt.$$

Under H_0 ,

$$E\left(U_m^{(c)}\right) = 0 \text{ for all values of } m. \quad (4)$$

The following theorem provides us the asymptotic normality of $U_m^{(c)}$ which follows from the well-known theory of U-Statistics (see Lehmann [23]).

Theorem 3.1. Let $N = n_1 + n_2$. The asymptotic distribution of $N^{1/2} \left[U_m^{(c)} - E\left(U_m^{(c)}\right) \right]$ as $N \rightarrow \infty$ in such a way that that $(n_1/N) \rightarrow \lambda$, $0 < \lambda < 1$, is normal with mean zero and variance, $\sigma^2\left(U_m^{(c)}\right)$, as

$$\sigma^2\left(U_m^{(c)}\right) = \frac{(2m+1)^2 \zeta_{10}^{(c)}}{\lambda} + \frac{\zeta_{01}^{(c)}}{1-\lambda}. \quad (5)$$

Here

$$\zeta_{10}^{(c)} = E\left[\left(h^{(c)}(x, X_2, \dots, X_{2m+1}; Y_1)\right)^2\right] - \left[E\left(U_m^{(c)}\right)\right]^2$$

and

$$\zeta_{01}^{(c)} = E\left[\left(h^{(c)}(X_1, X_2, \dots, X_{2m+1}; y)\right)^2\right] - \left[E\left(U_m^{(c)}\right)\right],$$

where

$$h^{(c)}(x, X_2, \dots, X_{2m+1}; Y_1) = E[h^{(c)}(X_1, X_2, \dots, X_{2m+1}; Y_1) | X_1 = x]$$

and

$$h^{(c)}(X_1, X_2, \dots, X_{2m+1}; y) = E[h^{(c)}(X_1, X_2, \dots, X_{2m+1}; Y_1) | Y_1 = y].$$

Under H_0 , after some involved computations, we establish the asymptotic null variance, $\sigma_0^2\left(U_m^{(c)}\right)$ as

$$\sigma_0^2\left(U_m^{(c)}\right) = \frac{(2m+1)^2 \rho_m^{(c)}}{\lambda(1-\lambda)}, \quad (6)$$

where for $c = 1$,

$$\begin{aligned} \rho_m^{(1)} = & \left[\left\{ \sum_{i=m+1}^{2m} \sum_{j=m+1}^{2m} \binom{2m}{i} \binom{2m}{j} \left\{ \sum_{r=0}^i \sum_{s=0}^j \binom{i}{r} \binom{j}{s} \frac{(-1)^{i+j+r+s}}{(2m-r+1)(2m-s+1)} \right\} \right\} \right. \\ & + 2 \left(\binom{2m}{m} \right) \left\{ \sum_{i=m+1}^{2m} \binom{2m}{i} \sum_{r=0}^i \binom{i}{r} \frac{(-1)^{i+r}}{(2m-r+1)} \right\} \times \left\{ \sum_{s=0}^m \binom{m}{s} \frac{(-1)^{m+s}}{(2m-s+1)(2m-s+2)} \right\} \\ & \left. + \left(\binom{2m}{m} \right)^2 \left\{ \sum_{r=0}^m \sum_{s=0}^m \binom{m}{r} \binom{m}{s} \frac{(-1)^{r+s}}{(2m-r+1)(2m-s+1)(4m-r-s+3)} \right\} \right] \\ & \times \left[(1-q)^{4m+3} + q^{4m+3} \right] - \left[E\left(U_m^{(1)}\right) \right]^2 \end{aligned}$$

and for $c = 2$,

$$\begin{aligned} \rho_m^{(2)} = & \left(\binom{2m}{m} \right)^2 \left[4 \left\{ \sum_{k=0}^m \sum_{l=0}^m \binom{m}{k} \binom{m}{l} \frac{(-1)^{k+l}}{(2m-k+1)(2m-l+1)(4m-k-l+3)} \right\} \right. \\ & - 4 \left\{ \sum_{l=0}^m \binom{m}{l} \frac{(-1)^{m+l}}{(2m-l+1)(2m-l+2)} \right\} \times \left\{ \sum_{k=0}^m \binom{m}{k} \frac{(-1)^{m+k}}{(2m-k+1)} \right\} \\ & \left. + \left\{ \sum_{k=0}^m \sum_{l=0}^m \binom{m}{k} \binom{m}{l} \frac{(-1)^{k+l}}{(2m-k+1)(2m-l+1)} \right\} \right] \times \left[(1-q)^{4m+3} + q^{4m+3} \right]. \end{aligned}$$

4. ASYMPTOTIC RELATIVE EFFICIENCY

In this Section, we compare the tests based on $U_m^{(c)}$ relative to some existing tests, in the sense of Pitman AREs. The efficacy of the test statistics $U_m^{(c)}$ under the sequence of local alternatives, $\theta_N = N^{-1/2}\theta$, is

$$\begin{aligned} e^2(U_m^{(c)}) &= \lim_{N \rightarrow \infty} \frac{\left[\frac{d}{d\theta} E(U_m^{(c)}) \Big|_{\theta=1} \right]^2}{N \sigma_0^2(U_m^{(c)})} \\ &= \frac{c^2 \left[(2m+1) \binom{2m}{m} \right]^2}{\sigma_0^2(U_m^{(c)})} \left[\int_0^\infty (F(y) - q)^m (1 - F(y))^m f^2(y) dy \right. \\ &\quad \left. - \int_{-\infty}^0 (F(y))^m (q - F(y))^m f^2(y) dy \right]^2. \end{aligned} \quad (7)$$

Now, we compare the proposed tests based on $U_m^{(c)}$ w.r.t. some existing tests for two-sample scale problem, namely, Sukhatme [2] test (S), Deshpande and Kusum [17] test (DK), Kusum [18] test (K), Mahajan *et al.* [20] test (MGA), and some members of Kössler and Kumar [21] test (T_k). We also compare proposed tests $U_m^{(1)}$ and $U_m^{(2)}$ with each other as well.

The efficacies of S , DK , K , MGA , and T_k tests are

$$\begin{aligned} e^2(S) &= \frac{\lambda(1-\lambda)}{\frac{1}{12} - q^2(1-q)^2} \left\{ \int_{-\infty}^\infty |y| f^2(y) dy \right\}^2 \\ e^2(DK) &= \frac{4\lambda(1-\lambda)}{\frac{1}{3} - q(1-q)} \left\{ \int_{-\infty}^\infty |y| f^2(y) dy \right\}^2 \\ e^2(K) &= \frac{112\lambda(1-\lambda)}{q^7 + (1-q)^7} \left\{ \int_{-\infty}^\infty |y| [F(y) - q]^2 f^2(y) dy \right\}^2 \\ e^2(MGA) &= \frac{831600\lambda(1-\lambda)}{131(q^{11} + (1-q)^{11})} \left\{ \int_{-\infty}^0 y F^2(y) [q - F(y)]^2 f^2(y) dy \right. \\ &\quad \left. - \int_0^\infty y [F(y) - q]^2 [1 - F(y)]^2 f^2(y) dy \right\}^2 \\ e^2(T_k) &= \frac{4k^2(4k-1)\lambda(1-\lambda)}{\left(q^{4k-1} + (1-q)^{4k-1} \right)} \left\{ \int_{-\infty}^\infty |y| [F(y) - q]^{2k-2} f^2(y) dy \right\}^2. \end{aligned} \quad (8)$$

In the following Tables 1–10, we have computed the AREs of $U_m^{(1)}$ and $U_m^{(2)}$ tests w.r.t. competing tests for some underlying distributions.

Table 1 | AREs of $U_m^{(1)}$ w.r.t. competing tests for uniform distribution.

q or $(1-q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.1	0	1.0000	0.8086	0.3546	1.4079	0.2257	0.1655	0.8086
	1	0.2913	0.2355	0.1033	0.4101	0.0657	0.0482	0.3511
	2	0.1314	0.1062	0.0466	0.1849	0.0296	0.0217	0.1849
	3	0.0730	0.0590	0.0259	0.1028	0.0165	0.0121	0.1122
	4	0.0460	0.0372	0.0163	0.0648	0.0104	0.0076	0.0750
0.2	0	1.0000	0.7506	0.3548	1.4187	0.2274	0.1669	0.7506
	1	0.1806	0.1355	0.0641	0.2562	0.0411	0.0301	0.2178
	2	0.0719	0.0540	0.0255	0.1020	0.0164	0.0120	0.1020
	3	0.0384	0.0288	0.0136	0.0544	0.0087	0.0064	0.0594
	4	0.0237	0.0178	0.0084	0.0337	0.0054	0.0040	0.0390
0.3	0	1.0000	0.7859	0.4105	1.7155	0.2750	0.2037	0.7859
	1	0.1256	0.0987	0.0515	0.2154	0.0345	0.0256	0.1753
	2	0.0421	0.0331	0.0173	0.0723	0.0116	0.0086	0.0722
	3	0.0215	0.0169	0.0088	0.0368	0.0059	0.0044	0.0406
	4	0.0131	0.0103	0.0054	0.0225	0.0036	0.0027	0.0263

Table 1 | AREs of $U_m^{(1)}$ w.r.t. competing tests for uniform distribution.

q or $(1 - q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.4	0	1.0000	0.9067	0.4617	2.1225	0.3403	0.2710	0.9067
	1	0.1426	0.1293	0.0658	0.3027	0.0485	0.0386	0.2238
	2	0.0333	0.0302	0.0154	0.0707	0.0113	0.0090	0.0707
	3	0.0137	0.0124	0.0063	0.0291	0.0047	0.0037	0.0345
	4	0.0075	0.0068	0.0035	0.0160	0.0026	0.0020	0.0210
0.5	0	1.0000	1.0000	0.4286	1.7013	0.2727	0.2000	1.0000
	1	0.6863	0.6863	0.2941	1.1675	0.1872	0.1373	1.0000
	2	0.5878	0.5878	0.2519	1.0000	0.1603	0.1176	1.0000
	3	0.5383	0.5383	0.2307	0.9158	0.1468	0.1077	1.0000
	4	0.5079	0.5079	0.2177	0.8641	0.1385	0.1016	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan *et al.* test; w.r.t.: With respect to.

Table 2 | AREs of $U_m^{(1)}$ w.r.t. competing tests for normal distribution.

q or $(1 - q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.1	0	1.0000	0.8086	0.5867	0.9251	0.5485	0.5452	0.8086
	1	0.3977	0.3216	0.2333	0.3679	0.2181	0.2168	0.3511
	2	0.1999	0.1616	0.1173	0.1849	0.1096	0.1090	0.1849
	3	0.1174	0.0950	0.0689	0.1086	0.0644	0.0640	0.1122
	4	0.0766	0.0619	0.0449	0.0708	0.0420	0.0417	0.0750
0.2	0	1.0000	0.7506	0.5693	0.9229	0.5292	0.5221	0.7506
	1	0.2496	0.1873	0.1421	0.2303	0.1321	0.1303	0.2178
	2	0.1106	0.0830	0.0629	0.1020	0.0585	0.0577	0.1020
	3	0.0622	0.0467	0.0354	0.0574	0.0329	0.0325	0.0594
	4	0.0397	0.0298	0.0226	0.0367	0.0210	0.0207	0.0390
0.3	0	1.0000	0.7859	0.6525	1.0998	0.6272	0.6217	0.7859
	1	0.1751	0.1376	0.1143	0.1926	0.1099	0.1089	0.1753
	2	0.0656	0.0516	0.0428	0.0722	0.0412	0.0408	0.0722
	3	0.0353	0.0278	0.0231	0.0389	0.0222	0.0220	0.0406
	4	0.0222	0.0175	0.0145	0.0245	0.0139	0.0138	0.0263
0.4	0	1.0000	0.9067	0.7546	1.3814	0.7938	0.8398	0.9067
	1	0.1951	0.1769	0.1472	0.2695	0.1549	0.1638	0.2238
	2	0.0512	0.0464	0.0386	0.0707	0.0406	0.0430	0.0707
	3	0.0224	0.0203	0.0169	0.0309	0.0178	0.0188	0.0345
	4	0.0128	0.0116	0.0096	0.0176	0.0101	0.0107	0.0210
0.5	0	1.0000	1.0000	0.7232	1.1531	0.6690	0.6577	1.0000
	1	0.9151	0.9151	0.6618	1.0552	0.6122	0.6018	1.0000
	2	0.8672	0.8672	0.6272	1.0000	0.5802	0.5704	1.0000
	3	0.8360	0.8360	0.6046	0.9640	0.5593	0.5499	1.0000
	4	0.8137	0.8137	0.5885	0.9382	0.5443	0.5351	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan *et al.* test; w.r.t.: With respect to.

Table 3 | AREs of $U_m^{(1)}$ w.r.t. competing tests for logistic distribution.

q or $(1 - q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.1	0	1.0000	0.8086	0.6438	0.8756	0.6475	0.6812	0.8086
	1	0.4129	0.3339	0.2658	0.3615	0.2674	0.2813	0.3511
	2	0.2112	0.1707	0.1360	0.1849	0.1367	0.1439	0.1849
	3	0.1253	0.1013	0.0807	0.1097	0.0812	0.0854	0.1122
	4	0.0823	0.0665	0.0530	0.0720	0.0533	0.0561	0.0750
0.2	0	1.0000	0.7506	0.6201	0.8753	0.6161	0.6396	0.7506
	1	0.2590	0.1944	0.1606	0.2267	0.1596	0.1656	0.2178
	2	0.1166	0.0875	0.0723	0.1021	0.0719	0.0746	0.1020
	3	0.0662	0.0497	0.0411	0.0580	0.0408	0.0424	0.0594
	4	0.0425	0.0319	0.0264	0.0372	0.0262	0.0272	0.0390

(continued)

Table 3 | AREs of $U_m^{(1)}$ w.r.t. competing tests for logistic distribution. (Continued)

q or $(1 - q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.3	0	1.0000	0.7859	0.7073	1.0442	0.7229	0.7511	0.7859
	1	0.1816	0.1427	0.1285	0.1896	0.1313	0.1364	0.1753
	2	0.0692	0.0543	0.0489	0.0722	0.0500	0.0519	0.0722
	3	0.0376	0.0295	0.0266	0.0392	0.0272	0.0282	0.0406
	4	0.0238	0.0187	0.0168	0.0248	0.0172	0.0179	0.0263
0.4	0	1.0000	0.9067	0.8179	1.3189	0.9106	1.0060	0.9067
	1	0.2012	0.1824	0.1645	0.2653	0.1832	0.2024	0.2238
	2	0.0536	0.0486	0.0438	0.0707	0.0488	0.0539	0.0707
	3	0.0236	0.0214	0.0193	0.0312	0.0215	0.0238	0.0345
	4	0.0135	0.0123	0.0111	0.0178	0.0123	0.0136	0.0210
0.5	0	1.0000	1.0000	0.7859	1.1109	0.7688	0.7877	1.0000
	1	0.9385	0.9385	0.7375	1.0425	0.7214	0.7392	1.0000
	2	0.9002	0.9002	0.7074	1.0000	0.6920	0.7091	1.0000
	3	0.8738	0.8738	0.6867	0.9707	0.6717	0.6883	1.0000
	4	0.8542	0.8542	0.6713	0.9490	0.6567	0.6729	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t.: With respect to.

Table 4 | AREs of $U_m^{(1)}$ w.r.t. competing tests for Laplace distribution.

q or $(1 - q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.1	0	1.0000	0.8086	0.7937	0.7794	0.9290	1.0704	0.8086
	1	0.4436	0.3587	0.3521	0.3458	0.4121	0.4748	0.3511
	2	0.2373	0.1918	0.1883	0.1849	0.2204	0.2539	0.1849
	3	0.1453	0.1175	0.1153	0.1133	0.1350	0.1555	0.1122
	4	0.0977	0.0790	0.0775	0.0761	0.0907	0.1045	0.0750
0.2	0	1.0000	0.7506	0.7651	0.8107	0.8528	0.9374	0.7506
	1	0.2719	0.2041	0.2080	0.2204	0.2319	0.2549	0.2178
	2	0.1259	0.0945	0.0963	0.1021	0.1074	0.1180	0.1020
	3	0.0728	0.0546	0.0557	0.0590	0.0621	0.0682	0.0594
	4	0.0474	0.0356	0.0362	0.0384	0.0404	0.0444	0.0390
0.3	0	1.0000	0.7859	0.8434	1.0247	0.9259	0.9890	0.7859
	1	0.1839	0.1445	0.1551	0.1884	0.1703	0.1818	0.1753
	2	0.0705	0.0554	0.0595	0.0722	0.0653	0.0697	0.0722
	3	0.0384	0.0302	0.0324	0.0394	0.0356	0.0380	0.0406
	4	0.0243	0.0191	0.0205	0.0249	0.0225	0.0241	0.0263
0.4	0	1.0000	0.9067	0.9080	1.3571	1.0485	1.1730	0.9067
	1	0.1978	0.1793	0.1796	0.2684	0.2074	0.2320	0.2238
	2	0.0521	0.0472	0.0473	0.0706	0.0546	0.0611	0.0707
	3	0.0227	0.0206	0.0207	0.0309	0.0239	0.0267	0.0345
	4	0.0130	0.0118	0.0118	0.0176	0.0136	0.0152	0.0210
0.5	0	1.0000	1.0000	0.8216	1.1185	0.8107	0.8302	1.0000
	1	0.9341	0.9341	0.7675	1.0448	0.7573	0.7755	1.0000
	2	0.8941	0.8941	0.7347	1.0000	0.7249	0.7423	1.0000
	3	0.8669	0.8669	0.7123	0.9696	0.7028	0.7197	1.0000
	4	0.8469	0.8469	0.6959	0.9473	0.6866	0.7031	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t.: With respect to.

Table 5 | AREs of $U_m^{(1)}$ w.r.t. competing tests for Cauchy distribution.

q or $(1 - q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.1	0	1.0000	0.8086	1.0503	0.6981	1.6322	2.4497	0.8086
	1	0.4798	0.3879	0.5039	0.3349	0.7831	1.1753	0.3511
	2	0.2649	0.2142	0.2782	0.1849	0.4324	0.6489	0.1849
	3	0.1649	0.1334	0.1732	0.1151	0.2692	0.4040	0.1122
	4	0.1118	0.0904	0.1175	0.0781	0.1826	0.2740	0.0750

Table 5 | AREs of $U_m^{(1)}$ w.r.t. competing tests for Cauchy distribution.

q or $(1 - q)$	m	Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(2)}$
0.2	0		1.0000	0.7506	0.9851	0.7099	1.4326	2.0008	0.7506
	1		0.2991	0.2245	0.2947	0.2123	0.4285	0.5985	0.2178
	2		0.1438	0.1079	0.1416	0.1021	0.2059	0.2876	0.1020
	3		0.0849	0.0637	0.0837	0.0603	0.1217	0.1699	0.0594
	4		0.0559	0.0420	0.0551	0.0397	0.0801	0.1119	0.0390
0.3	0		1.0000	0.7859	1.0879	0.8547	1.5550	2.0871	0.7859
	1		0.2086	0.1639	0.2270	0.1783	0.3244	0.4354	0.1753
	2		0.0845	0.0664	0.0919	0.0722	0.1314	0.1763	0.0722
	3		0.0476	0.0374	0.0517	0.0406	0.0740	0.0993	0.0406
	4		0.0308	0.0242	0.0335	0.0263	0.0479	0.0643	0.0263
0.4	0		1.0000	0.9067	1.2479	1.1145	1.8715	2.5871	0.9067
	1		0.2243	0.2034	0.2800	0.2500	0.4199	0.5804	0.2238
	2		0.0634	0.0575	0.0791	0.0706	0.1186	0.1639	0.0707
	3		0.0289	0.0262	0.0361	0.0322	0.0541	0.0748	0.0345
	4		0.0169	0.0153	0.0211	0.0189	0.0317	0.0438	0.0210
0.5	0		1.0000	1.0000	1.2118	0.9881	1.5808	1.9958	1.0000
	1		1.0145	1.0145	1.2293	1.0023	1.6036	2.0246	1.0000
	2		1.0122	1.0122	1.2266	1.0000	1.6001	2.0201	1.0000
	3		1.0062	1.0062	1.2192	0.9941	1.5905	2.0079	1.0000
	4		0.9995	0.9995	1.2111	0.9875	1.5799	1.9946	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t: With respect to.

Table 6 | AREs of $U_m^{(2)}$ w.r.t. competing tests for uniform distribution.

q or $(1 - q)$	m	Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(1)}$
0.1	0		1.2367	1.0000	0.4385	1.7412	0.2791	0.2047	1.2367
	1		0.8295	0.6708	0.2941	1.1679	0.1872	0.1373	2.8480
	2		0.7103	0.5743	0.2518	1.0000	0.1603	0.1176	5.4072
	3		0.6504	0.5259	0.2306	0.9157	0.1468	0.1077	8.9105
	4		0.6138	0.4963	0.2176	0.8641	0.1385	0.1016	13.3405
0.2	0		1.3323	1.0000	0.4727	1.8901	0.3030	0.2223	1.3323
	1		0.8290	0.6222	0.2941	1.1761	0.1885	0.1383	4.5904
	2		0.7049	0.5291	0.2501	1.0000	0.1603	0.1176	9.8014
	3		0.6452	0.4843	0.2289	0.9153	0.1467	0.1077	16.8216
	4		0.6088	0.4569	0.2160	0.8637	0.1385	0.1016	25.6632
0.3	0		1.2724	1.0000	0.5223	2.1828	0.3499	0.2592	1.2724
	1		0.7166	0.5631	0.2941	1.2292	0.1970	0.1460	5.7058
	2		0.5829	0.4581	0.2393	1.0000	0.1603	0.1187	13.8383
	3		0.5285	0.4154	0.2169	0.9066	0.1453	0.1077	24.6283
	4		0.4978	0.3912	0.2043	0.8540	0.1369	0.1014	37.9751
0.4	0		1.1029	1.0000	0.5092	2.3409	0.3753	0.2989	1.1029
	1		0.6371	0.5777	0.2941	1.3523	0.2168	0.1726	4.4677
	2		0.4711	0.4272	0.2175	1.0000	0.1603	0.1277	14.1440
	3		0.3973	0.3602	0.1834	0.8432	0.1352	0.1077	28.9889
	4		0.3600	0.3265	0.1662	0.7642	0.1225	0.0976	47.7677
0.5	0		1.0000	1.0000	0.4286	1.7013	0.2727	0.2000	1.0000
	1		0.6863	0.6863	0.2941	1.1675	0.1872	0.1373	1.0000
	2		0.5878	0.5878	0.2519	1.0000	0.1603	0.1176	1.0000
	3		0.5383	0.5383	0.2307	0.9158	0.1468	0.1077	1.0000
	4		0.5079	0.5079	0.2177	0.8641	0.1385	0.1016	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t: With respect to.

Table 7 | AREs of $U_m^{(2)}$ w.r.t. competing tests for normal distribution.

q or $(1-q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(1)}$
0.1	0	1.2367	1.0000	0.7255	1.1441	0.6783	0.6742	1.2367
	1	1.1325	0.9158	0.6644	1.0477	0.6212	0.6175	2.8480
	2	1.0809	0.8740	0.6341	1.0000	0.5929	0.5893	5.4072
	3	1.0466	0.8463	0.6140	0.9682	0.5740	0.5706	8.9105
	4	1.0215	0.8260	0.5993	0.9450	0.5603	0.5569	13.3405
0.2	0	1.3323	1.0000	0.7585	1.2296	0.7051	0.6955	1.3323
	1	1.1456	0.8599	0.6522	1.0573	0.6063	0.5981	4.5904
	2	1.0836	0.8133	0.6169	1.0000	0.5734	0.5657	9.8014
	3	1.0462	0.7853	0.5956	0.9656	0.5537	0.5462	16.8216
	4	1.0196	0.7653	0.5805	0.9410	0.5396	0.5323	25.6632
0.3	0	1.2724	1.0000	0.8303	1.3995	0.7982	0.7911	1.2724
	1	0.9995	0.7855	0.6522	1.0993	0.6270	0.6214	5.7058
	2	0.9092	0.7146	0.5933	1.0000	0.5704	0.5653	13.8383
	3	0.8699	0.6837	0.5677	0.9568	0.5457	0.5409	24.6283
	4	0.8458	0.6647	0.5519	0.9303	0.5306	0.5259	37.9751
0.4	0	1.1029	1.0000	0.8323	1.5236	0.8755	0.9262	1.1029
	1	0.8714	0.7902	0.6576	1.2039	0.6917	0.7318	4.4677
	2	0.7239	0.6564	0.5462	1.0000	0.5746	0.6079	14.1440
	3	0.6475	0.5871	0.4886	0.8945	0.5140	0.5438	28.9889
	4	0.6075	0.5508	0.4584	0.8392	0.4822	0.5101	47.7677
0.5	0	1.0000	1.0000	0.7232	1.1531	0.6690	0.6577	1.0000
	1	0.9151	0.9151	0.6618	1.0552	0.6122	0.6018	1.0000
	2	0.8672	0.8672	0.6272	1.0000	0.5802	0.5704	1.0000
	3	0.8360	0.8360	0.6046	0.9640	0.5593	0.5499	1.0000
	4	0.8137	0.8137	0.5885	0.9382	0.5443	0.5351	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t.: With respect to.

Table 8 | AREs of $U_m^{(2)}$ w.r.t. competing tests for logistic distribution.

q or $(1-q)$	$m \setminus$ Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(1)}$
0.1	0	1.2367	1.0000	0.7963	1.0829	0.8008	0.8425	1.2367
	1	1.1759	0.9508	0.7571	1.0296	0.7614	0.8011	2.8480
	2	1.1421	0.9235	0.7354	1.0000	0.7396	0.7781	5.4072
	3	1.1171	0.9033	0.7192	0.9781	0.7234	0.7610	8.9105
	4	1.0977	0.8876	0.7068	0.9611	0.7108	0.7478	13.3405
0.2	0	1.3323	1.0000	0.8262	1.1661	0.8208	0.8521	1.3323
	1	1.1888	0.8923	0.7372	1.0405	0.7324	0.7603	4.5904
	2	1.1425	0.8576	0.7085	1.0000	0.7039	0.7307	9.8014
	3	1.1134	0.8357	0.6905	0.9745	0.6859	0.7121	16.8216
	4	1.0915	0.8192	0.6769	0.9553	0.6724	0.6981	25.6632
0.3	0	1.2724	1.0000	0.9001	1.3287	0.9199	0.9558	1.2724
	1	1.0364	0.8145	0.7331	1.0821	0.7492	0.7784	5.7058
	2	0.9577	0.7526	0.6774	1.0000	0.6923	0.7193	13.8383
	3	0.9242	0.7264	0.6538	0.9651	0.6681	0.6942	24.6283
	4	0.9035	0.7101	0.6391	0.9434	0.6531	0.6786	37.9751
0.4	0	1.1029	1.0000	0.9021	1.4546	1.0042	1.1095	1.1029
	1	0.8988	0.8150	0.7351	1.1855	0.8184	0.9042	4.4677
	2	0.7582	0.6875	0.6202	1.0000	0.6904	0.7628	14.1440
	3	0.6843	0.6205	0.5597	0.9025	0.6231	0.6884	28.9889
	4	0.6456	0.5854	0.5280	0.8515	0.5878	0.6495	47.7677
0.5	0	1.0000	1.0000	0.7859	1.1109	0.7688	0.7877	1.0000
	1	0.9385	0.9385	0.7375	1.0425	0.7214	0.7392	1.0000
	2	0.9002	0.9002	0.7074	1.0000	0.6920	0.7091	1.0000
	3	0.8738	0.8738	0.6867	0.9707	0.6717	0.6883	1.0000
	4	0.8542	0.8542	0.6713	0.9490	0.6567	0.6729	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t.: With respect to.

Table 9 | AREs of $U_m^{(2)}$ w.r.t. competing tests for Laplace distribution.

q or $(1 - q)$	m	Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(1)}$
0.1	0		1.2367	1.0000	0.9816	0.9639	1.1489	1.3237	1.2367
	1		1.2634	1.0216	1.0028	0.9847	1.1736	1.3523	2.8480
	2		1.2830	1.0375	1.0184	1.0000	1.1919	1.3733	5.4072
	3		1.2951	1.0472	1.0279	1.0094	1.2031	1.3862	8.9105
	4		1.3031	1.0536	1.0342	1.0156	1.2105	1.3947	13.3405
0.2	0		1.3323	1.0000	1.0193	1.0800	1.1361	1.2489	1.3323
	1		1.2484	0.9370	0.9551	1.0120	1.0646	1.1702	4.5904
	2		1.2336	0.9259	0.9438	1.0000	1.0520	1.1564	9.8014
	3		1.2241	0.9188	0.9366	0.9923	1.0439	1.1475	16.8216
	4		1.2154	0.9123	0.9299	0.9853	1.0365	1.1394	25.6632
0.3	0		1.2724	1.0000	1.0731	1.3039	1.1782	1.2583	1.2724
	1		1.0490	0.8244	0.8846	1.0749	0.9712	1.0373	5.7058
	2		0.9759	0.7669	0.8230	1.0000	0.9035	0.9650	13.8383
	3		0.9447	0.7424	0.7967	0.9680	0.8747	0.9342	24.6283
	4		0.9245	0.7266	0.7797	0.9474	0.8560	0.9143	37.9751
0.4	0		1.1029	1.0000	1.0014	1.4967	1.1564	1.2937	1.1029
	1		0.8835	0.8011	0.8022	1.1989	0.9264	1.0363	4.4677
	2		0.7369	0.6682	0.6691	1.0000	0.7727	0.8644	14.1440
	3		0.6604	0.5988	0.5997	0.8963	0.6925	0.7747	28.9889
	4		0.6204	0.5625	0.5633	0.8419	0.6505	0.7277	47.7677
0.5	0		1.0000	1.0000	0.8216	1.1185	0.8107	0.8302	1.0000
	1		0.9341	0.9341	0.7675	1.0448	0.7573	0.7755	1.0000
	2		0.8941	0.8941	0.7347	1.0000	0.7249	0.7423	1.0000
	3		0.8669	0.8669	0.7123	0.9696	0.7028	0.7197	1.0000
	4		0.8469	0.8469	0.6959	0.9473	0.6866	0.7031	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t: With respect to.

Table 10 | AREs of $U_m^{(2)}$ w.r.t. competing tests for Cauchy distribution.

q or $(1 - q)$	m	Test	S	DK or T_1	K or T_2	MGA	T_3	T_4	$U_m^{(1)}$
0.1	0		1.2367	1.0000	1.2990	0.8633	2.0186	3.0296	1.2367
	1		1.3663	1.1048	1.4351	0.9537	2.2301	3.3470	2.8480
	2		1.4326	1.1584	1.5047	1.0000	2.3382	3.5094	5.4072
	3		1.4698	1.1885	1.5438	1.0260	2.3990	3.6006	8.9105
	4		1.4924	1.2067	1.5675	1.0418	2.4359	3.6559	13.3405
0.2	0		1.3323	1.0000	1.3124	0.9458	1.9086	2.6656	1.3323
	1		1.3732	1.0307	1.3527	0.9748	1.9672	2.7475	4.5904
	2		1.4087	1.0574	1.3877	1.0000	2.0180	2.8185	9.8014
	3		1.4281	1.0719	1.4068	1.0138	2.0458	2.8573	16.8216
	4		1.4383	1.0796	1.4169	1.0210	2.0605	2.8778	25.6632
0.3	0		1.2724	1.0000	1.3843	1.0875	1.9786	2.6557	1.2724
	1		1.1904	0.9356	1.2951	1.0175	1.8511	2.4845	5.7058
	2		1.1701	0.9196	1.2729	1.0000	1.8194	2.4420	13.8383
	3		1.1688	0.9185	1.2715	0.9989	1.8174	2.4393	24.6283
	4		1.1687	0.9185	1.2715	0.9989	1.8173	2.4392	37.9751
0.4	0		1.1029	1.0000	1.3765	1.2293	2.0642	2.8535	1.1029
	1		1.0022	0.9087	1.2508	1.1170	1.8757	2.5930	4.4677
	2		0.8971	0.8135	1.1197	1.0000	1.6791	2.3212	14.1440
	3		0.8387	0.7605	1.0468	0.9349	1.5698	2.1700	28.9889
	4		0.8092	0.7338	1.0100	0.9020	1.5146	2.0937	47.7677
0.5	0		1.0000	1.0000	1.2118	0.9881	1.5808	1.9958	1.0000
	1		1.0145	1.0145	1.2293	1.0023	1.6036	2.0246	1.0000
	2		1.0122	1.0122	1.2266	1.0000	1.6001	2.0201	1.0000
	3		1.0062	1.0062	1.2192	0.9941	1.5905	2.0079	1.0000
	4		0.9995	0.9995	1.2111	0.9875	1.5799	1.9946	1.0000

ARE: Asymptotic relative efficiencies; DK: Deshpande and Kusum test; MGA: Mahajan et al. test; w.r.t: With respect to.

From the ARE tables, we observe the following:

1. For light-tailed distributions, like uniform distribution, $U_m^{(1)}$ and $U_m^{(2)}$ tests perform as good as or better than S and MGA tests for some specific choices of m and q . However the optimal choice of m is
 - (a) For $q = 0.5$, $U_m^{(1)}$ and $U_m^{(2)}$ tests are more efficient than MGA test for $m \leq 2$ with maximum efficiency achieved at $m = 0$.
 - (b) For $q \neq 0.5$, $U_m^{(2)}$ test is more efficient than S test for $m = 0$ and is more efficient than MGA test for $m \leq 2$ with maximum efficiency achieved at $m = 0$.
2. For medium-tailed distributions, like normal distribution, $U_m^{(1)}$ and $U_m^{(2)}$ tests perform as good as or better than S and MGA tests for some specific choices of m and q . However the optimal choice of m is
 - (a) For $q \in (0.3, 0.7)$, then $U_m^{(1)}$ and $U_m^{(2)}$ tests are more efficient than S and MGA test for $m = 0$.
 - (b) For $q \notin (0.3, 0.7)$, $U_m^{(2)}$ test is more efficient than S and MGA tests for $m \leq 2$, with maximum efficiency achieved at $m = 0$.
3. For large tail distributions, like Cauchy distribution, $U_m^{(1)}$ and $U_m^{(2)}$ tests performs better than its competing tests. The optimal choice of m is
 - (a) For $q = 0.5$, $U_m^{(1)}$ and $U_m^{(2)}$ tests are more efficient than the competing tests for $m \leq 2$ with maximum efficiency achieved at $m = 1$.
 - (b) For $q \neq 0.5$, $U_m^{(2)}$ tests are more efficient than its competing tests with maximum efficiency is for m as large as possible for $q \notin (0.3, 0.7)$, otherwise the maximum efficiency for $U_m^{(2)}$ tests is achieved at $m = 0$.
4. ARE of $U_m^{(2)}$ test w.r.t. $U_m^{(1)}$ test doesn't depend upon the underlying distribution. Moreover, the $U_m^{(2)}$ test is asymptotically equivalent to $U_m^{(1)}$ test for $q = 0.5$ and $U_m^{(2)}$ test is always more efficient than $U_m^{(1)}$ test for $q \neq 0.5$. Thus, in general, one should use $U_m^{(2)}$ test in comparison to $U_m^{(1)}$ test to gain more efficiency.

5. AN ILLUSTRATIVE EXAMPLE

To see the execution of the tests based on $U_m^{(c)}$, we consider the data of the survey of Hills M. and M345 course team of The Open University, given in Hand *et al.* [22]. In this experiment, two groups of 44 and 69 students were asked to guess the width of a lecture hall in metres and feet, respectively. It is of relevance to check whether there is greater variation in guessing the width in metres in comparison to guessing the width in feet.

By using Kolmogorov–Smirnov test, we have seen that the data set follows Cauchy distribution at 5% level of significance and has common quantile of order 0.05, that is, $q = 0.05$. Therefore for $U_m^{(2)}$ test, by using the observation 3, made in Section 4, one should consider m as large as possible to have maximum gain in efficiency in comparison to competing tests.

The values of computed $U_m^{(1)}$ and $U_m^{(2)}$ tests statistics, and the competing test statistics along with their p -values are given in Table 11.

We note that at 5% level of significance, the null hypothesis of same variability in guessing the width in metres in comparison to guessing the width in feet is rejected by tests K, T_2, T_3, T_4 , and $U_m^{(2)}(m = 1, 2)$. However, for tests $S, DK, T_1, MGA, U_m^{(1)}(m = 0, 1, 2)$, and $U_m^{(2)}(m = 0)$ tests the null hypothesis is not rejected.

6. SIMULATION STUDY

In this section, using Monte Carlo simulation technique, we have computed the estimated power of $U_m^{(1)}$ and $U_m^{(2)}$ tests for sample size n_1 and n_2 with $n_1, n_2 = 10, 15, 20, 25, 30, 40$. The computation of power is based on 10,000 repetitions, by generating the data from three

Table 11 | Computed values of test statistics and p -values for different tests.

Test	S	DK or T_1	K or T_2	MGA	T_3	T_4
Test statistics	0.536399	0.189655	0.348273	0.248532	0.425221	0.464311
p -value	0.163061	0.118561	0.033079	0.112129	0.018887	0.013862
Test	$U_m^{(1)}$			$U_m^{(2)}$		
m	0	1	2	0	1	2
Test statistics	0.536399	0.467569	0.400347	0.189655	0.129979	0.066549
p -value	0.163061	0.309472	0.414866	0.118561	0.012831	0.011439

common distributions, namely, (i) uniform, (ii) normal, and (iii) Cauchy. The scale parameters considered are $\theta = 1.5$ (0.5) 3 and level of significance is fixed at 5%.

The idea behind the selecting these three distributions, is that the uniform, normal, and Cauchy have short, medium, and heavy tail, respectively. So it is of relevance to see the test performance for these distributions in terms of power.

The estimated powers are given in Tables 12–17.

Table 12 | Estimated power of $U_m^{(1)}$ for uniform distribution.

q or $(1 - q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \backslash m$	0	1	2	0	1	2	0	1	2
10, 10	1.5	0.2377	0.1446	0.1060	0.2465	0.1029	0.0516	0.2643	0.2199	0.1735
	2	0.3466	0.2615	0.2299	0.3512	0.2267	0.1413	0.3718	0.3267	0.3140
	2.5	0.4754	0.3726	0.3157	0.4876	0.3145	0.1967	0.5079	0.4614	0.4497
	3	0.4992	0.4112	0.3760	0.5017	0.3710	0.2718	0.6220	0.4825	0.4750
15, 10	1.5	0.2556	0.1653	0.1298	0.2678	0.1250	0.0829	0.2798	0.2317	0.2252
	2	0.3679	0.2877	0.2520	0.3910	0.2478	0.1665	0.4029	0.3675	0.3538
	2.5	0.4821	0.4013	0.3517	0.5231	0.3476	0.2115	0.5367	0.5021	0.4694
	3	0.5377	0.4524	0.4064	0.5423	0.4013	0.2995	0.6718	0.5319	0.5168
15, 15	1.5	0.2713	0.1861	0.1492	0.2814	0.1448	0.1033	0.2940	0.2528	0.2466
	2	0.3912	0.3093	0.2816	0.4355	0.2789	0.1834	0.4412	0.4120	0.3975
	2.5	0.5113	0.4415	0.3902	0.5499	0.3850	0.2365	0.5671	0.5317	0.5120
	3	0.5701	0.4836	0.4380	0.5797	0.4320	0.3218	0.7374	0.5690	0.5498
20, 15	1.5	0.3055	0.2076	0.1709	0.3244	0.1653	0.1345	0.3383	0.2936	0.2855
	2	0.4323	0.3517	0.3084	0.4676	0.3034	0.2069	0.4855	0.4534	0.4412
	2.5	0.5435	0.4690	0.4125	0.5892	0.4097	0.2578	0.6054	0.5630	0.5427
	3	0.6211	0.5019	0.4584	0.6283	0.4525	0.3629	0.7990	0.5987	0.5811
20, 20	1.5	0.3397	0.2375	0.1967	0.3421	0.1899	0.1656	0.3622	0.3210	0.3118
	2	0.4650	0.3819	0.3419	0.5116	0.3364	0.2246	0.5249	0.4921	0.4720
	2.5	0.5783	0.4926	0.4520	0.6311	0.4465	0.2790	0.6420	0.6112	0.6009
	3	0.6608	0.5420	0.4879	0.6691	0.4819	0.3935	0.8323	0.6317	0.6202
25, 20	1.5	0.3694	0.2542	0.2122	0.3980	0.2067	0.1872	0.4047	0.3618	0.3447
	2	0.4976	0.4315	0.3601	0.5567	0.3512	0.2457	0.5685	0.5431	0.5230
	2.5	0.6062	0.5436	0.5011	0.6638	0.4951	0.3010	0.6828	0.6642	0.6435
	3	0.6811	0.5994	0.5114	0.7298	0.5078	0.4225	0.8693	0.6913	0.6513
25, 25	1.5	0.4009	0.3019	0.2610	0.4330	0.2518	0.2091	0.4428	0.4029	0.3892
	2	0.5272	0.4628	0.4025	0.5906	0.3970	0.2679	0.6019	0.5772	0.5699
	2.5	0.6351	0.6013	0.5510	0.7110	0.5412	0.3315	0.7292	0.7018	0.6910
	3	0.7215	0.6519	0.5721	0.7820	0.5699	0.4576	0.9086	0.7114	0.7005
30, 30	1.5	0.4464	0.3517	0.3167	0.4765	0.3035	0.2335	0.4835	0.4667	0.4487
	2	0.5511	0.5029	0.4398	0.6514	0.4305	0.2984	0.6626	0.6220	0.6165
	2.5	0.6535	0.6520	0.5999	0.7688	0.5921	0.3629	0.7709	0.7517	0.7374
	3	0.7914	0.7013	0.6310	0.8610	0.6223	0.4898	0.9389	0.7690	0.7589
40, 40	1.5	0.4938	0.4044	0.3759	0.5149	0.3649	0.2639	0.5250	0.4990	0.4814
	2	0.5906	0.5514	0.4895	0.7170	0.4819	0.3247	0.7278	0.6814	0.6678
	2.5	0.7036	0.7130	0.6606	0.8125	0.6566	0.3945	0.8311	0.8155	0.8006
	3	0.8688	0.7412	0.6975	0.9456	0.6836	0.5476	0.9652	0.8432	0.8399

Table 13 | Estimated power of $U_m^{(1)}$ for normal distribution.

q or $(1 - q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \backslash m$	0	1	2	0	1	2	0	1	2
10, 10	1.5	0.3923	0.2295	0.1917	0.1868	0.1135	0.0758	0.2571	0.2103	0.2176
	2	0.5539	0.2892	0.2503	0.3992	0.1527	0.1079	0.4983	0.3006	0.2824
	2.5	0.6802	0.3207	0.2913	0.5606	0.1816	0.1322	0.6520	0.3802	0.3036
	3	0.7717	0.3961	0.3263	0.7412	0.2084	0.1624	0.7506	0.4712	0.3238

(continued)

Table 13 | Estimated power of $U_m^{(1)}$ for normal distribution. (Continued)

q or $(1-q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \backslash m$	0	1	2	0	1	2	0	1	2
15, 10	1.5	0.4407	0.2531	0.2802	0.1973	0.1442	0.1108	0.2861	0.2421	0.2284
	2	0.6203	0.3265	0.2779	0.4561	0.1778	0.1445	0.5228	0.3445	0.2903
	2.5	0.7209	0.3876	0.3383	0.6284	0.1966	0.1621	0.6849	0.4169	0.3264
	3	0.8319	0.4161	0.3691	0.7783	0.2111	0.1909	0.7829	0.4701	0.4232
15, 15	1.5	0.4712	0.2822	0.2561	0.2376	0.1655	0.1427	0.3213	0.2690	0.2316
	2	0.5998	0.3670	0.3492	0.5147	0.1908	0.1799	0.6285	0.3739	0.3167
	2.5	0.7716	0.4314	0.3614	0.6813	0.2123	0.2013	0.7906	0.4545	0.3433
	3	0.9064	0.5029	0.4553	0.8718	0.2304	0.2177	0.8878	0.5254	0.7545
20, 15	1.5	0.5001	0.3168	0.2877	0.2924	0.1885	0.1613	0.3626	0.2803	0.2520
	2	0.6312	0.4028	0.3789	0.5598	0.2103	0.1981	0.6689	0.4452	0.3469
	2.5	0.8128	0.4733	0.4622	0.7502	0.2292	0.2104	0.8281	0.5674	0.3902
	3	0.9155	0.5614	0.5033	0.9040	0.2586	0.2336	0.9197	0.6591	0.5508
20, 20	1.5	0.5233	0.3527	0.3219	0.3362	0.2067	0.1906	0.3881	0.2992	0.2751
	2	0.6778	0.4556	0.4177	0.6190	0.2214	0.2115	0.7088	0.4718	0.3972
	2.5	0.8602	0.5141	0.4834	0.8123	0.2409	0.2256	0.8853	0.6190	0.4713
	3	0.9330	0.6233	0.5646	0.9536	0.2760	0.2569	0.9549	0.7105	0.6243
25, 20	1.5	0.5713	0.3967	0.3536	0.3818	0.2412	0.2278	0.4286	0.3235	0.3038
	2	0.6966	0.4920	0.4562	0.6645	0.2723	0.2416	0.7611	0.5474	0.4589
	2.5	0.8870	0.5672	0.5251	0.8605	0.2916	0.2550	0.9109	0.7105	0.5472
	3	0.9424	0.6713	0.6109	0.9661	0.3278	0.2714	0.9688	0.7979	0.7094
25, 25	1.5	0.6051	0.4535	0.4078	0.4202	0.2813	0.2589	0.4638	0.3377	0.3165
	2	0.7203	0.5687	0.5087	0.7127	0.3225	0.2726	0.8083	0.5742	0.4991
	2.5	0.9013	0.6079	0.5566	0.9008	0.3609	0.3018	0.9404	0.7448	0.6393
	3	0.9516	0.7123	0.6643	0.9743	0.4035	0.3537	0.9775	0.8564	0.7739
30, 30	1.5	0.6768	0.5191	0.4519	0.4612	0.3562	0.3109	0.5116	0.3714	0.3506
	2	0.7425	0.6286	0.5628	0.7794	0.3843	0.3217	0.8663	0.6413	0.5624
	2.5	0.9309	0.7008	0.6234	0.9389	0.4354	0.3663	0.9703	0.8005	0.6955
	3	0.9731	0.8013	0.7256	0.9822	0.4912	0.4108	0.9873	0.8976	0.8211
40, 40	1.5	0.6908	0.6124	0.5327	0.5277	0.4729	0.4019	0.5804	0.5083	0.4139
	2	0.7612	0.7075	0.6560	0.8513	0.5296	0.4386	0.9024	0.8159	0.6528
	2.5	0.9489	0.7716	0.7384	0.9544	0.5763	0.4896	0.9818	0.9593	0.8244
	3	0.9875	0.8624	0.8290	0.9956	0.6204	0.5516	0.9909	0.9912	0.9377

Table 14 | Estimated power of $U_m^{(1)}$ for Cauchy distribution.

q or $(1-q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \backslash m$	0	1	2	0	1	2	0	1	2
10, 10	1.5	0.2213	0.2109	0.2016	0.1710	0.1409	0.1280	0.1838	0.2523	0.2137
	2	0.3830	0.3613	0.3329	0.3034	0.2825	0.2513	0.3186	0.4001	0.3662
	2.5	0.4910	0.4805	0.4645	0.3878	0.3619	0.3127	0.4216	0.5127	0.4623
	3	0.5405	0.5318	0.5298	0.4481	0.4132	0.3809	0.4980	0.5778	0.5319
15, 10	1.5	0.3166	0.3095	0.2949	0.1850	0.1644	0.1425	0.1965	0.2712	0.2351
	2	0.3902	0.3720	0.3590	0.3114	0.3003	0.2808	0.3206	0.4224	0.3683
	2.5	0.5178	0.5103	0.5017	0.4067	0.3825	0.3413	0.4308	0.5334	0.4877
	3	0.5613	0.5527	0.5493	0.4660	0.4335	0.4115	0.5336	0.5966	0.5622
15, 15	1.5	0.3280	0.3305	0.3160	0.1943	0.1808	0.1640	0.2095	0.2879	0.2523
	2	0.4019	0.3909	0.3825	0.3642	0.3224	0.3025	0.3914	0.4598	0.4312
	2.5	0.5421	0.5358	0.5272	0.4865	0.4236	0.3738	0.5287	0.5971	0.5534
	3	0.5744	0.5819	0.5935	0.5112	0.4748	0.4526	0.6509	0.6603	0.6578
20, 15	1.5	0.3401	0.3366	0.3271	0.2098	0.2019	0.1839	0.2225	0.2997	0.2660
	2	0.4429	0.4221	0.4066	0.4013	0.3604	0.3246	0.4124	0.4772	0.4455
	2.5	0.5765	0.5639	0.5433	0.5222	0.4757	0.4215	0.5524	0.6214	0.5815
	3	0.6779	0.6525	0.6240	0.5620	0.5354	0.4918	0.6812	0.7009	0.6924

Table 14 | Estimated power of $U_m^{(1)}$ for Cauchy distribution.

q or $(1 - q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \backslash m$	0	1	2	0	1	2	0	1	2
20, 20	1.5	0.3728	0.3650	0.3352	0.2201	0.2160	0.2007	0.2344	0.3116	0.2778
	2	0.4813	0.4635	0.4423	0.4247	0.3993	0.3743	0.4345	0.5972	0.4819
	2.5	0.6116	0.6053	0.5819	0.6057	0.5388	0.4710	0.6078	0.6813	0.6427
	3	0.7105	0.6996	0.6626	0.6108	0.5813	0.5520	0.7345	0.7724	0.7535
25, 20	1.5	0.3869	0.3744	0.3625	0.2377	0.2279	0.2215	0.2526	0.3319	0.2893
	2	0.5292	0.5037	0.4896	0.4589	0.4334	0.4017	0.4775	0.5588	0.5110
	2.5	0.6402	0.6206	0.6035	0.6260	0.5893	0.5280	0.6593	0.7179	0.6823
	3	0.7512	0.7300	0.6947	0.6613	0.6224	0.5903	0.7842	0.8308	0.8009
25, 25	1.5	0.4683	0.4635	0.4580	0.2565	0.2412	0.2328	0.2743	0.3524	0.3120
	2	0.5810	0.5222	0.5099	0.4991	0.4624	0.4456	0.5148	0.5977	0.5546
	2.5	0.6811	0.6690	0.6413	0.6778	0.6383	0.5779	0.7011	0.7685	0.7407
	3	0.8203	0.7991	0.7340	0.7124	0.6735	0.6329	0.8222	0.8533	0.8335
30, 30	1.5	0.4891	0.4828	0.4799	0.2756	0.2589	0.2501	0.2875	0.3713	0.3490
	2	0.6224	0.6015	0.5876	0.5463	0.5013	0.4897	0.5691	0.6364	0.6012
	2.5	0.7320	0.7056	0.6737	0.7318	0.6879	0.6388	0.7693	0.7922	0.7806
	3	0.8913	0.8843	0.8715	0.7639	0.7320	0.6879	0.8773	0.8997	0.8812
40, 40	1.5	0.5120	0.5007	0.4914	0.3144	0.2892	0.2785	0.3670	0.4009	0.3850
	2	0.6736	0.6507	0.6219	0.6017	0.5625	0.5382	0.6852	0.7355	0.7180
	2.5	0.7762	0.7519	0.7395	0.7923	0.7634	0.7179	0.8048	0.8125	0.8005
	3	0.9427	0.9266	0.9146	0.8514	0.8017	0.7685	0.9530	0.9770	0.9640

Table 15 | Estimated power of $U_m^{(2)}$ for uniform distribution.

q or $(1 - q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \backslash m$	0	1	2	0	1	2	0	1	2
10, 10	1.5	0.2819	0.2357	0.2015	0.3054	0.2142	0.1517	0.2597	0.2117	0.1621
	2	0.3912	0.3419	0.3220	0.4206	0.3230	0.2967	0.3723	0.3229	0.3016
	2.5	0.5519	0.4910	0.4517	0.5736	0.4725	0.4114	0.5068	0.4622	0.4399
	3	0.6306	0.6070	0.5830	0.6519	0.5812	0.5520	0.6210	0.4892	0.4624
15, 10	1.5	0.3010	0.2614	0.2487	0.3500	0.2443	0.1943	0.2793	0.2225	0.2219
	2	0.4349	0.3767	0.3543	0.4517	0.3611	0.3272	0.3992	0.3616	0.3590
	2.5	0.5720	0.5120	0.4871	0.5998	0.5019	0.4612	0.5364	0.5010	0.4711
	3	0.6910	0.6385	0.6220	0.6909	0.6057	0.5711	0.6619	0.2399	0.5168
15, 15	1.5	0.3450	0.2875	0.2676	0.3809	0.2698	0.2319	0.2941	0.2540	0.2410
	2	0.4629	0.4250	0.4006	0.4928	0.4095	0.3630	0.4431	0.4210	0.3803
	2.5	0.5810	0.5577	0.5230	0.6347	0.5367	0.4915	0.5679	0.5334	0.5116
	3	0.7565	0.6611	0.6498	0.7325	0.6348	0.6028	0.7320	0.5688	0.5499
20, 15	1.5	0.3629	0.3117	0.2989	0.4110	0.3001	0.2610	0.3389	0.2913	0.2813
	2	0.4930	0.4478	0.4223	0.5333	0.4373	0.3979	0.4818	0.4529	0.4426
	2.5	0.6044	0.5883	0.5622	0.6759	0.5771	0.5351	0.5990	0.5637	0.5437
	3	0.7977	0.6850	0.6733	0.7710	0.6692	0.6377	0.7997	0.5986	0.5818
20, 20	1.5	0.3920	0.3408	0.3258	0.4419	0.3298	0.2912	0.3614	0.3277	0.3180
	2	0.5450	0.4826	0.4647	0.5740	0.4636	0.4320	0.5227	0.4915	0.4712
	2.5	0.6570	0.6219	0.6008	0.7123	0.6029	0.5678	0.6390	0.6123	0.6013
	3	0.8419	0.7184	0.7029	0.8229	0.6980	0.6636	0.8334	0.6378	0.6207
25, 20	1.5	0.4347	0.3775	0.3660	0.4722	0.3412	0.3161	0.4010	0.3610	0.3420
	2	0.5661	0.5210	0.5110	0.5934	0.5117	0.4707	0.5666	0.5409	0.5218
	2.5	0.6955	0.6506	0.6336	0.7436	0.6320	0.5830	0.6813	0.6618	0.6407
	3	0.8910	0.7519	0.7375	0.8651	0.7321	0.6923	0.8740	0.7024	0.6515
25, 25	1.5	0.4891	0.4256	0.4095	0.4931	0.4002	0.3781	0.4398	0.4018	0.3810
	2	0.5997	0.5412	0.5278	0.6247	0.5290	0.5079	0.6050	0.5799	0.5658
	2.5	0.7338	0.6878	0.6550	0.7725	0.6627	0.6185	0.7208	0.7037	0.6997
	3	0.9316	0.7818	0.7655	0.9021	0.7617	0.7259	0.9089	0.7128	0.7011

(continued)

Table 15 | Estimated power of $U_m^{(2)}$ for uniform distribution. (Continued)

q or $(1 - q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \setminus m$	0	1	2	0	1	2	0	1	2
30, 30	1.5	0.5378	0.4610	0.4503	0.5240	0.4495	0.4179	0.4872	0.4634	0.4410
	2	0.6412	0.5899	0.5619	0.6755	0.5776	0.5446	0.6613	0.6260	0.6160
	2.5	0.7745	0.7398	0.7147	0.7991	0.7194	0.6756	0.7719	0.7585	0.7315
	3	0.9512	0.8591	0.8329	0.9503	0.8360	0.7607	0.9385	0.7687	0.7585
40, 40	1.5	0.5660	0.5090	0.4887	0.5538	0.4979	0.4638	0.5214	0.4966	0.4891
	2	0.6823	0.6413	0.6257	0.7135	0.6320	0.5824	0.7229	0.6897	0.6690
	2.5	0.8421	0.7829	0.7592	0.8356	0.7728	0.7188	0.8320	0.8119	0.8094
	3	0.9789	0.9127	0.8856	0.9727	0.9005	0.8222	0.9649	0.8430	0.8390

Table 16 | Estimated power of $U_m^{(2)}$ for normal distribution.

q or $(1 - q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \setminus m$	0	1	2	0	1	2	0	1	2
10, 10	1.5	0.4218	0.2824	0.2767	0.2993	0.2615	0.1827	0.2776	0.2118	0.2113
	2	0.6494	0.5105	0.5054	0.5189	0.4922	0.3213	0.5052	0.3051	0.2808
	2.5	0.6986	0.6779	0.6625	0.6802	0.6431	0.4385	0.6615	0.3867	0.3016
	3	0.8115	0.7865	0.7750	0.7976	0.7665	0.5211	0.7764	0.4777	0.3265
15, 10	1.5	0.4523	0.2878	0.2845	0.3224	0.2817	0.2135	0.2908	0.2420	0.2217
	2	0.6611	0.5451	0.5312	0.5537	0.5209	0.3623	0.5381	0.3446	0.2907
	2.5	0.7452	0.7277	0.7115	0.7392	0.7013	0.4612	0.7182	0.4175	0.3285
	3	0.8562	0.8268	0.8121	0.8302	0.8024	0.5448	0.8236	0.4760	0.4218
15, 15	1.5	0.4817	0.3248	0.3156	0.3536	0.3101	0.2409	0.3332	0.2634	0.2368
	2	0.6903	0.6413	0.6370	0.6495	0.6205	0.3925	0.6386	0.3752	0.3119
	2.5	0.8254	0.8104	0.7923	0.8122	0.7724	0.5102	0.8095	0.4523	0.3420
	3	0.9193	0.9054	0.8828	0.9097	0.8619	0.5664	0.9005	0.5277	0.7525
20, 15	1.5	0.5229	0.3660	0.3604	0.3813	0.3551	0.2652	0.3772	0.2850	0.2599
	2	0.7217	0.6899	0.6771	0.6982	0.6610	0.4178	0.6828	0.4468	0.3425
	2.5	0.8728	0.8587	0.8392	0.8617	0.8235	0.5389	0.8538	0.5635	0.3934
	3	0.9457	0.9305	0.9197	0.9365	0.9029	0.6149	0.9274	0.6564	0.5537
20, 20	1.5	0.5516	0.3852	0.3798	0.4106	0.3744	0.2911	0.3945	0.2945	0.2798
	2	0.7543	0.7403	0.7292	0.7499	0.7120	0.4325	0.7353	0.4779	0.3937
	2.5	0.9102	0.8992	0.8714	0.9010	0.8536	0.5703	0.8957	0.6140	0.4741
	3	0.9713	0.9648	0.9579	0.9699	0.9411	0.6577	0.9604	0.7175	0.6210
25, 20	1.5	0.5920	0.4476	0.4261	0.4495	0.4058	0.3266	0.4449	0.3257	0.3044
	2	0.7997	0.7837	0.7699	0.7981	0.7539	0.4698	0.7762	0.5450	0.4528
	2.5	0.9356	0.9297	0.8915	0.9344	0.8748	0.6027	0.9253	0.7119	0.5433
	3	0.9855	0.9799	0.9757	0.9801	0.9625	0.6903	0.9736	0.7984	0.7068
25, 25	1.5	0.6317	0.4892	0.4644	0.4927	0.4436	0.3702	0.4837	0.3346	0.3156
	2	0.8322	0.8275	0.8113	0.8306	0.7976	0.4984	0.8129	0.5703	0.4917
	2.5	0.9614	0.9501	0.9280	0.9578	0.9017	0.6378	0.9496	0.7427	0.6334
	3	0.9927	0.9884	0.9808	0.9906	0.9701	0.7286	0.9836	0.8569	0.7705
30, 30	1.5	0.7166	0.5880	0.5522	0.5995	0.5216	0.3916	0.5613	0.3760	0.3567
	2	0.8899	0.8799	0.8684	0.8817	0.8513	0.5399	0.8778	0.6412	0.5672
	2.5	0.9894	0.9796	0.9640	0.9836	0.9422	0.6778	0.9747	0.8039	0.6945
	3	0.9959	0.9943	0.9895	0.9950	0.9811	0.7790	0.9932	0.8920	0.8217
40, 40	1.5	0.8002	0.6203	0.5978	0.6324	0.5727	0.4328	0.6029	0.5078	0.4184
	2	0.9478	0.9224	0.9109	0.9335	0.8997	0.5700	0.9134	0.8133	0.6585
	2.5	0.9965	0.9902	0.9809	0.9918	0.9678	0.7102	0.9890	0.9547	0.8279
	3	0.9987	0.9974	0.9920	0.9979	0.9899	0.8588	0.9967	0.9922	0.9391

Table 17 | Estimated power of $U_m^{(2)}$ for Cauchy distribution.

q or $(1 - q)$		0.1			0.3			0.5		
n_1, n_2	$\theta \setminus m$	0	1	2	0	1	2	0	1	2
10, 10	1.5	0.2299	0.2528	0.2814	0.2317	0.2270	0.2178	0.1790	0.2517	0.2111
	2	0.3907	0.3810	0.4077	0.3644	0.3583	0.3497	0.3180	0.4025	0.3609
	2.5	0.5150	0.5212	0.5386	0.4780	0.4709	0.4510	0.4207	0.5113	0.4665
	3	0.5590	0.5824	0.6243	0.5613	0.5566	0.5414	0.5337	0.5709	0.5370
15, 10	1.5	0.3192	0.3214	0.3441	0.2677	0.2578	0.2440	0.1999	0.2765	0.2396
	2	0.3983	0.4131	0.4319	0.3912	0.3822	0.3787	0.3122	0.4202	0.3693
	2.5	0.5252	0.5336	0.5613	0.4944	0.4913	0.4877	0.4340	0.5398	0.4895
	3	0.6080	0.6329	0.6837	0.6119	0.6002	0.5944	0.5320	0.5910	0.5615
15, 15	1.5	0.3356	0.3417	0.3619	0.2998	0.2907	0.2668	0.2060	0.2893	0.2517
	2	0.4443	0.4734	0.5013	0.4502	0.4376	0.4219	0.3918	0.4528	0.4399
	2.5	0.5492	0.5637	0.6027	0.5413	0.5317	0.5299	0.5214	0.5987	0.5596
	3	0.6891	0.7252	0.7614	0.6927	0.6829	0.6752	0.6574	0.6616	0.6582
20, 15	1.5	0.3494	0.3523	0.3952	0.3008	0.2997	0.2883	0.2241	0.2952	0.2626
	2	0.4609	0.4912	0.5387	0.4767	0.4599	0.4228	0.4117	0.4713	0.4495
	2.5	0.6360	0.6514	0.6926	0.6372	0.6302	0.6212	0.5504	0.6291	0.5895
	3	0.7396	0.7790	0.8052	0.7416	0.7310	0.7243	0.6890	0.7025	0.6993
20, 20	1.5	0.3602	0.3915	0.4337	0.3225	0.3185	0.3004	0.2311	0.3775	0.2701
	2	0.5389	0.5667	0.6012	0.5465	0.5311	0.5110	0.4364	0.5919	0.4809
	2.5	0.6592	0.6921	0.7331	0.6610	0.6550	0.6494	0.6081	0.6808	0.6489
	3	0.7884	0.8395	0.8544	0.7917	0.7805	0.7710	0.7367	0.7779	0.7513
25, 20	1.5	0.3898	0.4309	0.4702	0.3652	0.3541	0.3319	0.2533	0.3384	0.2880
	2	0.5686	0.5914	0.6508	0.5713	0.5660	0.5330	0.4847	0.5556	0.5120
	2.5	0.6991	0.7316	0.7802	0.7053	0.6909	0.6895	0.6625	0.7134	0.6895
	3	0.8497	0.8825	0.9009	0.8541	0.8444	0.8319	0.7877	0.8338	0.8094
25, 25	1.5	0.4705	0.4777	0.4993	0.3962	0.3819	0.3540	0.2789	0.3579	0.3116
	2	0.5912	0.6235	0.6626	0.6021	0.5891	0.5680	0.5162	0.5908	0.5525
	2.5	0.7399	0.7629	0.7941	0.7413	0.7310	0.7284	0.7088	0.7610	0.7430
	3	0.8785	0.9014	0.9325	0.8820	0.8711	0.8614	0.8240	0.8519	0.8398
30, 30	1.5	0.4908	0.4991	0.5395	0.4347	0.4210	0.3917	0.2821	0.3766	0.3411
	2	0.6289	0.6314	0.6997	0.6118	0.6009	0.5883	0.5640	0.6365	0.6014
	2.5	0.7695	0.7920	0.8448	0.7729	0.7618	0.7591	0.7699	0.7927	0.7890
	3	0.9056	0.9310	0.9499	0.9117	0.9190	0.9020	0.8792	0.8997	0.8899
40, 40	1.5	0.5258	0.5440	0.5865	0.4698	0.4608	0.4319	0.3617	0.4075	0.3868
	2	0.6783	0.6939	0.7290	0.6329	0.6197	0.6108	0.6878	0.7393	0.7186
	2.5	0.8220	0.8580	0.8985	0.8360	0.8115	0.8090	0.8065	0.8157	0.8016
	3	0.9569	0.9735	0.9798	0.9688	0.9526	0.9508	0.9538	0.9771	0.9645

Based on the power computations, we have the following observations:

- For uniform distribution, change in scale of the order of 3 is detected:
 - For $q = 0.5$, for random samples of size ≥ 40 for $U_m^{(1)}$ and $U_m^{(2)}$ tests at $m = 0$.
 - For $q \neq 0.5$, change in scale of same order is detected for random samples of size ≥ 30 for $U_m^{(2)}$ tests at $m = 0$. This authenticates the observation 1, of Section 4.
- For normal distribution, change in scale of the order of 3 is detected:
 - For $q \in (0.3, 0.7)$, for random samples of size ≥ 20 for $U_m^{(1)}$ and $U_m^{(2)}$ tests at $m = 0$
 - For $q \notin (0.3, 0.7)$, the change in scale of same order is detected for random samples of size ≥ 20 for $U_m^{(2)}$ tests at $m = 0, 1, 2$ with maximum power is achieved at $m = 0$. This authenticates the observation 2, of Section 4.
- For Cauchy distribution, change in scale of the order of 3 is detected:
 - For $q = 0.5$, for random samples of size ≥ 40 for $U_m^{(1)}$ and $U_m^{(2)}$ tests at $m = 0, 1, 2$ with maximum power is achieved at $m = 1$.
 - For $q \neq 0.5$, the change in scale of same order is detected at random samples of size ≥ 40 , for $U_m^{(2)}$ tests at $m = 0, 1, 2$ with maximum power is achieved at $m = 2$ for $q \notin (0.3, 0.7)$, otherwise maximum power is achieved at $m = 0$. This authenticates the observation 3, of Section 4.

4. Also the power of $U_m^{(2)}$ test is equivalent to $U_m^{(1)}$ test for $q = 0.5$, and the power of $U_m^{(2)}$ test is greater than $U_m^{(1)}$ test for $q \neq 0.5$, for all choices of m . This authenticates the observation 4, of Section 4.
5. For all other $U_m^{(1)}$ and $U_m^{(2)}$ tests, one needs to take larger sample size, to detect the change of scale of the same order. This once again authenticates the computations of AREs as well.

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