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Some Characterization Results Based on the Mean Vitality Function of the First-Order Statistics

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ABSTRACT

In this paper, we investigate some new properties of the mean vitality function (MVF) of a random variable, proposed by A. Toomaj, M. Doostparast, J. Stat. Theory Appl. 13 (2013), 189–195. Specifically, we explore properties of MVF and study under what conditions the MVF of the first-order statistics can uniquely determines the parent distribution. We show that in all distributions the Weibull family, which is commonly used in several fields of applied probability, is characterized through the ratio of the MVF of the first-order statistics to its expectation.

1. INTRODUCTION

The notion of *entropy* as a measure of uncertainty, introduced by Shannon [1], has a fundamental importance in different areas such as probability and statistics, financial analysis, engineering, and information theory; see, for example, Cover and Thomas [2]. For a continuous random variable X with probability density function (pdf) f and cumulative distribution function (cdf) F, the differential entropy of X is defined as $H(X) = -E(\log f(X))$. Throughout this paper "log" stands for the natural logarithm. Recently Rao *et al.* [3] introduced an alternate measure of uncertainty, called it *cumulative residual entropy* (CRE) and is used for measuring the residual uncertainty of a random variable. For a nonnegative random variable X, the *CRE* is defined by

$$\mathcal{E}(X) = -\int_0^\infty \overline{F}(x) \log \overline{F}(x)\, dx, \tag{1}$$

where $\overline{F}(x) = 1 - F(x)$ stands for the reliability function. This measure, in general, is more stable since the distribution function is more regular than the density function and has a lot of mathematical properties. The applications and other properties and some new version of the mentioned measure can be found in Asadi and Zohrevand [4], Baratpour [5], Baratpour and Habibi rad [6], Navarro *et al.* [7], Psarrakos and Navarro [8], Rao [9], and the references therein.

Let X be a random lifetime with cdf F. Moreover, let us suppose that $F(t) = 0$ for $t < 0$, with a finite moment. The mean residual life (MRL) is defined as

$$m(t) = E\left(X - t | X > t\right) = \frac{\int_t^\infty \overline{F}(x)}{\overline{F}(t)} dx, \tag{2}$$

where $\overline{F}(t) > 0$ for $t > 0$. If F is absolutely continuous with the pdf f, then MRL can be rewritten as

$$m(t) = \frac{\int_t^\infty xf(x)}{\overline{F}(t)} dx - t = v(t) - t, \quad t > 0. \tag{3}$$

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The function $v(t) = E\left(X|X\rangle t\right) = \frac{\int_t^\infty xf(x)}{\overline{F}(t)} dx$ is known as the *vitality function* (VF) or life expectancy which introduced by Kupka and Loo [10]. The VF and MRL play important roles in engineering reliability, biomedical science, and survival analyzes; see Kotz and Shanbhag [11], Murari and Sujit [12], Ruiz and Navarro [13], and Bairamov *et al.* [14] and the references therein. It is worth mentioning that the rapid ageing on average of a component needs to low vitality relatively, whereas high vitality implies relatively slow (or even possibly "negative") ageing during the time period. By using (3), Toomaj and Doostparat [15] defined a new measure which called it mean vitality function (MVF) and defined as

$$E(v(X)) = \mathcal{E}(X) + \mu, \tag{4}$$

where $\mathcal{E}(X)$ is given in (1) and $\mu = E(X)$ denotes the expectation of a random variable. It is worth pointing out that the MVF expresses as the sum of two terms, the CRE and the mean value of X. Hence to compute the MVF, one can compute the mentioned measures. Another useful representation for the MVF is given as

$$E(v(X)) = \int_0^\infty xf(x)\psi\left(\overline{F}(x)\right)dx,$$

where $\psi(v) = -\log v,\ 0 < v < 1$. Also, the probability integral transformation $U = \overline{F}(X)$ provides a useful expression as follows:

$$E(v(X)) = \int_0^1 \overline{F}^{-1}(u)\psi(u)\,du = \mu E\left[\psi(Z_F)\right], \tag{5}$$

where $\overline{F}^{-1}(u) = \sup\left\{x : \overline{F}(x) \geq u\right\}$ and Z_F has a pdf

$$h_F(u) = \frac{\overline{F}^{-1}(u)}{\mu}, \quad 0 < u < 1.$$

It is worth pointing out that $h_F(u) = 0$ if either $u \geq 1$ or $u \leq 0$ and it has a decreasing probability density function. The aim of the present paper is to investigate some new properties of the MVF measure. Specifically, we provide some stochastic ordering properties and characterization results.

2. MAIN RESULTS

First, we observe that MVF is shift and scale dependent under linear transformation. In other terms, $E(v(X))$ has the same properties of the expectation of a random variable. First, we have the following lemma for CRE.

Lemma 2.1. *Let $a, b > 0$. It holds that*

$$\mathcal{E}(aX + b) = a\mathcal{E}(X). \tag{6}$$

Proof. The result follows by noting that $\overline{F}_{aX+b}(x) = \overline{F}\left(\frac{x-b}{a}\right)$, $x \geq 0$, and using (1).

From (4), Lemma 2.1 and the properties of the expected value we have the following proposition.

Proposition 2.1. *Let $a, b > 0$. It holds that*

$$E(v(aX + b)) = aE(v(X)) + b. \tag{7}$$

Let $X_\theta^\star$ be an absolutely continuous nonnegative random variable with survival function $\overline{F}_\theta^\star(x)$. The survival function of the proportional hazard rate model with proportionality constant $\theta > 0$, is defined as

$$\overline{F}_\theta^\star(x) = \left[\overline{F}(x)\right]^\theta, \qquad x \geq 0. \tag{8}$$

For more details on the applications and properties of proportional hazards rate model, see, for example, Gupta *et al.* [16], Gupta *et al.* [17], and Mudholkar *et al.* [18], and the references therein. In the next proposition, we provide an upper bound for the MVF of $X_\theta^\star$ depending on $E(v(x))$.

Proposition 2.2. *For $n = 1, 2, ...,$ we have*

$$E\left(v\left(X_\theta^\star\right)\right) \leq \theta\, E\left(v\left(X\right)\right)\ \text{if}\, \theta \geq 1,$$

and the inequality being reversed if $0 < \theta \leq 1$.

Proof. Recalling (1) and (8), we have

$$\mathcal{E}\left(X_\theta^\star\right) = -\theta \int_0^\infty \left[\overline{F}\left(x\right)\right]^\theta \log \overline{F}\left(x\right) dx.$$

Since $\left[\overline{F}\left(x\right)\right]^\theta \leq \overline{F}\left(x\right)$, when $\theta \geq 1$, we obtain

$$\mathcal{E}\left(X_\theta^\star\right) \leq \theta \mathcal{E}\left(X\right).$$

On the other hand, we obtain

$$E\left(X_\theta^\star\right) \leq E\left(X\right) \leq \theta E\left(X\right).$$

Hence representation (4) completes the proof. For $0 < \theta \leq 1$, we have $\overline{F}\left(x\right) \leq \left[\overline{F}\left(x\right)\right]^\theta$, and hence the desired result follows.

Hereafter, we obtain some stochastic ordering properties and characterization results of the MVF. For this purpose, we consider two absolutely continuous nonnegative random variables X and Y with cdfs F and G and pdfs f and g, respectively. First, we recall that a random variable X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if $E\left[\phi\left(X\right)\right] \leq E\left[\phi\left(Y\right)\right]$ for all increasing functions ϕ such that the expectations exist. A random variable X is said to be smaller than Y in the increasing convex order, denoted by $X \leq_{icx} Y$, if $E\left[\phi\left(X\right)\right] \leq E\left[\phi\left(Y\right)\right]$ for all increasing convex functions ϕ such that the expectations exist. A random variable X is said to be smaller than Y in the convex order, denoted by $X \leq_{cx} Y$, if $E\left[\phi\left(X\right)\right] \leq E\left[\phi\left(Y\right)\right]$ for all convex functions ϕ such that the expectations exist. Further properties and applications can be found in the book of Shaked and Shanthikumar [19]. Also, we recall the following definition given by Toomaj and Doostparast [15].

Definition 2.1. A random variable X is said to be smaller than Y in the MVF, denoted by $X \leq_{mvf} Y$, if $E\left(v\left(X\right)\right) \leq E\left(v\left(Y\right)\right)$ such that the expectations exist.

It is worth pointing out that the MVF order is not a stochastic order in a strict sense, since it does not satisfy the antisymmetric property, that is, $X =_{mvf} Y$ does not imply $X =_{st} Y$. Note that from Corollary 2.2 of Toomaj and Doostparast [15] we have that $X \leq_{st} Y$ implies $X \leq_{mvf} Y$. Therefore, $X \leq_{mvf} Y$ can be seen as a necessary condition for $X \leq_{st} Y$. Also, we show that $X \leq_{mvf} Y$ can be seen as a necessary condition for $X \leq_{icx} Y$ as well. First, consider the following proposition.

Proposition 2.3. *Let X be an absolutely continuous nonnegative random variable with $\mathcal{E}\left(X\right) < +\infty$. Then, we have*

$$\mathcal{E}\left(X\right) = E\left[g\left(X\right)\right], \tag{9}$$

where

$$g\left(x\right) = \int_0^x \Lambda\left(t\right) dt,\ x \geq 0. \tag{10}$$

Proof. From (1) and Fubini's theorem, we obtain

$$\mathcal{E}\left(X\right) = \int_0^\infty \left[\int_t^\infty f\left(x\right) dx\right] \Lambda\left(t\right) dt = \int_0^\infty f\left(x\right) \left[\int_0^x \Lambda\left(t\right) dt\right] dx,$$

which immediately follows (9) by using (10).

The next proposition shows that the increasing convex order implies the CRE order.

Proposition 2.4. *If X and Y are nonnegative random variables such that $X \leq_{icx} Y$, then it holds that $g\left(X\right) \leq_{icx} g\left(Y\right)$, where the function $g\left(\cdot\right)$ is defined in* Eq. (10). *In particular, $X \leq_{icx} Y$ implies $\mathcal{E}\left(X\right) \leq \mathcal{E}\left(Y\right)$.*

Proof. Since the function $g\left(\cdot\right)$ is an increasing convex function, it follows by Theorem 4.A.8. of Shaked and Shantikumar [19] that $g\left(X\right) \leq_{icx} g\left(Y\right)$. In particular, recalling the definition of increasing convexorder, we have $\mathcal{E}\left(X\right) \leq \mathcal{E}\left(Y\right)$.

Proposition 2.5. *Let X and Y be two nonnegative random variables with cdfs F and G, respectively. If* $X \leq_{icx} Y$, *then* $X \leq_{mvf} Y$.

Proof. From Eq. (4), it is sufficient to prove that $\mathcal{E}(X) + E(X) \leq \mathcal{E}(X) + E(X)$. First, the condition $X \leq_{icx} Y$, implies $E(X) \leq E(Y)$ due to relation 4.A.2 of Shaked and Shanthikumar [19]. Also, Proposition 2.4 yields $\mathcal{E}(X) \leq \mathcal{E}(Y)$ and hence the desired result follows.

Proposition 2.6. *Let X and Y be two nonnegative random variables with cdfs F and G, respectively. Then*

i. If $X \leq_{cx} Y$, then $X \geq_{mvf} Y$.

ii. If $Z_F \leq_{cx} Z_G$, then $X \leq_{mvf} Y$.

Proof. It is known that $X \leq_{cx} Y$ implies $\mu = E(X) = E(Y)$. Now, since $\psi(v)$ is a decreasing function of v, hence the proof of (i) can be obtained from Lemma 2.3 of Navarro and Rychlik [20], expression Eq. (5) and the definition of usual stochastic order. The proof of (ii) is an immediate consequence of Eq. (5) and the definition of the convex order.

Proposition 2.7. *Let* $X_{1:n}$ *and* $Y_{1:n}$ *be two lifetimes of series systems with i.i.d. components having the common cdfs F and G, respectively. If* $X \leq_{mvf} Y$, *then* $X_{1:n} \leq_{mvf} Y_{1:n}$.

Proof. Since $X \leq_{mvf} Y$, from, we have

$$E(v(X)) - E(v(Y)) = \int_0^1 \psi(u)\left(\overline{F}^{-1}(u) - \overline{G}^{-1}(u)\right) du \leq 0. \tag{11}$$

On the other hand, we have

$$E(v(X_{1:n})) - E(v(Y_{1:n})) = n\int_0^1 u^{n-1}\psi(u)\left(\overline{F}^{-1}(u) - \overline{G}^{-1}(u)\right) du,$$

$$\leq n\int_0^1 \psi(u)\left(\overline{F}^{-1}(u) - \overline{G}^{-1}(u)\right) du \leq 0. \tag{12}$$

The first inequality is obtained by noting that $u^{n-1} \leq 1,\ 0 < u < 1$, while the second inequality is obtained from Eq. (11).

Example 2.1.

Let us consider a Weibull distribution with survival function

$$\overline{F}(x) = e^{-(\lambda x)^{\alpha}},\ x > 0,$$

where $\alpha > 0$, and $\lambda > 0$ are scale and shape parameters, respectively. It is easy to verify that (see Baratpour [5]), $E(X) = \frac{1}{\lambda}\Gamma\left(1+\frac{1}{\alpha}\right)$ and $\mathcal{E}(X) = \frac{1}{\alpha\lambda}\Gamma\left(1+\frac{1}{\alpha}\right)$ and hence we have

$$E(v(X)) = \frac{\alpha+1}{\alpha\lambda}\Gamma\left(1+\frac{1}{\alpha}\right).$$

It is not hard to verify that

$$E(v(X_{1:n})) = \frac{\alpha+1}{n^{\frac{1}{\alpha}}\alpha\lambda}\Gamma\left(1+\frac{1}{\alpha}\right).$$

On the other hand, we have $E(X_{1:n}) = \frac{1}{n^{\frac{1}{\alpha}}\lambda}\Gamma\left(1+\frac{1}{\alpha}\right)$ and thus

$$\frac{E(v(X_{1:n}))}{E(X_{1:n})} = \frac{\alpha+1}{\alpha}.$$

Therefore, it shows that in the Weibull family this ratio is constant for all n.

In the next theorem, we show that, in all distributions, only in Weibull distribution the ratio $\frac{E(v(X_{1:n}))}{E(X_{1:n})}$ is constant analogue Theorem 2.1 of Baratpour [5]. First, we recall the following lemma due to *Müntz-Szász* Theorem (see Kamps [21]) which is used in the proofs of this paper. Recently, many authors applied the *Müntz-Szász* Theorem giving characterization results by recurrence relations for moments of order statistics; see for example Khan and Zia [22].

Lemma 2.2. *For any increasing sequence of positive integers* $\{n_j, j \geq 1\}$, *the sequence of polynomials* $\{x^{n_j}\}$ *is complete on* $L(0,1)$, *if and only if,*

$$\sum_{j=1}^{+\infty} n_j^{-1} = +\infty, \ 0 < n_1 < n_2 < \cdots.$$

Theorem 2.1. *Let* $X_1, \cdots, X_n$ *be n iid absolutely continuous nonnegative random variables with the common pdf f and cdf F. Then F belong to Weibull family, if and only if*

$$\frac{E(\nu(X_{1:n}))}{E[X_{1:n}]} = k, \ k > 1, \tag{13}$$

for all $n = n_j, j \geq 1$, *such that* $\sum_{j=1}^{+\infty} n_j^{-1} = +\infty$.

Proof. The necessity is trivial. Hence it remains to prove the sufficiency part. Since $E(\nu(X_{1:n})) = \mathcal{E}(X_{1:n}) + E(X_{1:n})$, hence it is sufficient to prove that $\frac{\mathcal{E}(X_{1:n})}{E[X_{1:n}]} = c$, where $c = k-1, \ k > 1$. Therefore, Theorem 2.1 of Baratpour [5] completes the proof.

In the next theorems, we characterize the distributions based on the MVF of the first-order statistics.

Proposition 2.8. *Let X and Y be two nonnegative random variables with pdfs* $f(x)$ *and* $g(x)$ *and absolutely continuous cdfs* $F(x)$ *and* $G(x)$, *respectively. Then F and G belong to the same family of distributions if and only if*

$$E(\nu(X_{1:n})) = E(\nu(Y_{1:n})),$$

for $n = n_j, \ j \geq 1$ *such that* $\sum_{j=1}^{\infty} n_j^{-1}$ *is infinite.*

Proof. The necessity is trivial and hence it remains to prove the sufficiency part. By using the probability integral transformation $U = \overline{F}(X)$, we have

$$E(\nu(X_{1:n})) = n\int_0^1 \overline{F}^{-1}(u)\, u^{n-1}\psi(u)\, du.$$

For $E(\nu(Y_{1:n}))$ can be obtain in a similar way. Since $E(\nu(X_{1:n})) = E(\nu(Y_{1:n}))$, we obtain

$$\int_0^1 u^{n-1}\psi(u)\left[\overline{F}^{-1}(u) - \overline{G}^{-1}(u)\right] du = 0. \tag{14}$$

If Eq. (14) holds for $n = n_j, \ j \geq 1$, such that $\sum_{j=1}^{\infty} n_j^{-1} = \infty$, then from Lemma 2.2 we can conclude that $\overline{F}^{-1}(u) = \overline{G}^{-1}(u), \ 0 < v < 1$, and this completes the proof.

Proposition 2.9. *Let X and Y be two nonnegative random variables with pdfs* $f(x)$ *and* $g(x)$ *and absolutely continuous cdfs* $F(x)$ *and* $G(x)$, *respectively. Then F and G belong to the same family of distributions, but for a change in location, if and only if*

$$\frac{E(\nu(X_{1:n}))}{E[X]} = \frac{E(\nu(Y_{1:n}))}{E[Y]}, \tag{15}$$

for $n = n_j, \ j \geq 1$ *such that* $\sum_{j=1}^{\infty} n_j^{-1}$ *is infinite.*

Proof. The necessity is trivial and hence it remains to prove the sufficiency part. By using Eq. (4) it is sufficient to prove that $\frac{\mathcal{E}(X_{1:n})}{E[X]} = \frac{\mathcal{E}(Y_{1:n})}{E[Y]}$. We have

$$\frac{\mathcal{E}(X_{1:n})}{E[X]} = \frac{\int_0^1 u^n \psi(u)/f\left(\overline{F}^{-1}(u)\right) du}{\int_0^1 \overline{F}^{-1}(u)\, du}.$$

If Eq. (15) holds, then we get

$$\frac{\int_0^1 u^n \psi(u) / f\left(\overline{F}^{-1}(u)\right) du}{\int_0^1 \overline{F}^{-1}(u)\, du} = \frac{\int_0^1 u^n \psi(u) / g\left(\overline{G}^{-1}(u)\right) du}{\int_0^1 \overline{G}^{-1}(u)\, du}. \tag{16}$$

Let us suppose that $c = \int_0^1 \overline{G}^{-1}(u)\, du / \int_0^1 \overline{F}^{-1}(u)\, du$, then Eq. (16) can be expressed as

$$\int_0^1 u^n \psi(u) \left(\frac{1}{f\left(\overline{F}^{-1}(u)\right)} - \frac{1}{cg\left(\overline{G}^{-1}(u)\right)} \right) du = 0. \tag{17}$$

If Eq. (17) holds for $n = n_j$, $j \geq 1$, such that $\sum_{j=1}^{\infty} n_j^{-1} = \infty$, then from Lemma 2.2 we can conclude that $f\left(\overline{F}^{-1}(u)\right) = cg\left(\overline{G}^{-1}(u)\right)$, $0 < v < 1$, and hence it follows that $\overline{F}^{-1}(u) = \overline{G}^{-1}(u) + d$. Since X and Y have a common support $[0, +\infty]$, we can conclude that $d = 0$, which means that F and G belong to the same family of distributions, but for a change in scale.

3. CONCLUSIONS

Some new properties of the MVF are investigated. It is explored that under what conditions the MVF of the first-order statistics can uniquely determines the parent distribution. It is shown that the Weibull family is characterized through ratio of the MVF of the first-order statistics to its expectation.

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