Journal of Statistical Theory and Applications
Vol. **18(2)**, June 2019, *pp.* 87–102
DOI: https://doi.org/10.2991/jsta.d.190306.003; ISSN 1538-7887
https://www.atlantis-press.com/journals/jsta

A New Generalized Two-Sided Class of the Distributions Via New Transmuted Two-Sided Bounded Distribution

O. Kharazmi and M. Zargar*

Department of Statistics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

ARTICLE INFO

Article History

Received 8 Aug 2017
Accepted 2 Feb 2018

Keywords

Hazard rate function
order statistics
maximum likelihood estimator
transmutation map
transmuted two-sided distribution

ABSTRACT

In the present paper, we first consider a generalization of the standard two-sided power distribution so-called the transmuted two-sided distribution, and then extend proposed idea to generalized two-sided class of distributions, introduced by Korkmaz and Genç [1]. Some statistical and reliability properties including explicit expressions for quantiles, hazard rate function, order statistics, and maximum likelihood estimation are obtained in general setting. Generalized transmuted two-sided exponential distribution is considered as a especial case and denoted with the name $TTSG-G$. A simulation study is presented to investigate the bias and mean square error of the maximum likelihood estimators. We use a real data set and obtain the maximum likelihood and parametric bootstrap estimator of the parameters of $TTSG-G$ distribution. Finally, the superiority of the new model to some common statistical distributions is shown through the different criteria of selection model including log-likelihood values, Akaike information criterion, and Kolmogorov–Smirnov test statistic values.

1. INTRODUCTION

The statistical distribution theory has been widely explored by researchers in recent years. Given the fact that the data from our surrounding environment follow various statistical models, it is necessary to extract and develop appropriate high-quality models. One of the most important models in the statistical theory is the change point models. In the distribution theory, the change point distributions are used in the different branch of sciences such as economic, engineering, agriculture, and so on. Van Dorp and Kotz [2] introduced a family of the change point distributions so-called two-sided power (TSP) distribution with the probability density function (pdf),

$$f(x;\alpha,\beta)=\begin{cases}\alpha\left(\dfrac{x}{\beta}\right)^{\alpha-1}, & 0<x\le\beta,\\[2ex] \alpha\left(\dfrac{1-x}{1-\beta}\right)^{\alpha-1}, & \beta\le x<1,\end{cases}\tag{1}$$

and with the cumulative distribution function $\left(cdf\right)$,

$$F(x;\alpha,\beta)=\begin{cases}\beta\left(\dfrac{x}{\beta}\right)^{\alpha}, & 0<x\le\beta,\\[2ex] 1-(1-\beta)\left(\dfrac{1-x}{1-\beta}\right)^{\alpha}, & \beta\le x<1,\end{cases}\tag{2}$$

where $0\le\beta\le1$ and $\alpha>0$. The parameter β is the location parameter called “turning point” and α is the shape parameter that control the shape of distribution on the left and right of β.

The *TSP* distribution for $\alpha>3$ can be used for modeling unimodal phenomena on a bounded domain when a peak in data is observed. Van Dorp and Kotz [3] introduced an extension of the three-parameter triangular distribution utilized in risk analysis. Their model includes the *TSP* distribution as a special case. Van Dorp and Kotz [4] considered a family of continuous distributions on a bounded interval generated by convolutions of the *TSP* distributions. In recent years, a number of researchers have studied some generalization of the *TSP* distribution such as Nadarajah [5], Oruç and Bairamov [6], Vicari *et al.* [7], Herrerías-Velasco *et al.* [8], and Soltani and Homei [9]. Korkmaz and Genç [10] proposed a new generalization of Weibull distribution by making use of a transformation of the standard TSP distribution. Also, Korkmaz

*Corresponding author. Email: zargar_m@vru.ac.ir

and Genç [11] considered the log transformation of the *TSP* distribution instead of uniform distribution and introduced a generalization of the exponential distribution. Korkmaz and Genç [1] extended the idea of two-sidedness to other ordinary distributions like normal and introduced the two-sided generalized normal distribution.

One of the interesting methods for constructing new distributions is the transmutation map approach. Recently, some new distributions have been generalized based on the transmutation method. The transmuted distribution based on the $G(x)$ is defined as

$$H(x) = (1+\lambda)\,G(x) - \lambda\,[G(x)]^2, \qquad |\lambda| \leq 1, \tag{3}$$

where $G(x)$ is the *cdf* of the parent distribution.

Aryal and Tsokos [12] generated a flexible family of probability distributions taking extreme value distribution as the base value distribution by using the quadratic rank transmutation map (QRTM). Aryal and Tsokos [13] generalized the two-parameter Weibull distribution using the QRTM. Aryal [14] introduced a generalization of the log-logistic distribution so-called the transmuted log-logistic distribution. Abd El Hady [15] introduced a new generalization of the two-parameter Weibull distribution by using the QRTM. This new distribution is named exponentiated transmuted Weibull (*ETW*) distribution. Elgarhy and Shawki [16] introduced a new generalized version of the quasi Lindley distribution which is called the transmuted generalized quasi Lindley (*TGQ*L) distribution.

The transmutation method, as an important method for developing statistical distributions, hasn't been used for the change point distributions, yet. The main motivation of the present paper is to apply the transmutation technique for increasing the flexibility and usefulness of the *TSP* distribution and generalized *TSP* class of distributions.

This paper organized as follows. In Section 2, we introduce a new distribution so-called transmuted two-sided distribution. In Section 3, we propose a generalization of the transmuted two-sided distribution and consider the hazard function, quantiles, and order statistics of this distribution. We consider the exponential distribution as a parent distribution and introduce transmuted two-sided generalized exponential (TSGE) distribution, in Section 4. In this section, we plot the shape of density function and hazard function. In Section 5, the estimation of parameters of the generalized transmuted two-sided distribution are obtained by using two methods maximum likelihood estimation (MLE) and bootstrap estimation. Also, we study the performance of MLEs of parameters of the transmuted TSGE distribution via a simulation study. In Section 6, the superiority of new model to some competitor statistical models is shown through the different criteria of selection model. Finally, the paper is concluded in Section 7.

2. TRANSMUTED TWO-SIDED DISTRIBUTION

In this section, we introduce the transmuted two-sided distribution and then we consider its shape for different values of parameters. The main motivation for introducing this new family is to provide the more flexibility for the *TSP* distribution by compounding two-sided distributions family and transmutation map approach.

Let the random variable X_1 have a beta distribution with *pdf*,

$$f_{X_1}(x) = ax^{\alpha-1}, \qquad 0 < x < 1, \alpha > 0.$$

Assume that X be a random variable associated with the truncated X_1 on the right at β, as

$$X \stackrel{d}{=} X_1 | 0 < X_1 \leq \beta,$$

where $\stackrel{d}{=}$ denotes identically distributed. The *cdf* of the random variable X is given by

$$F_X(x) = \left(\frac{a}{\beta}\right)^{\alpha}, \qquad 0 < x \leq \beta, \alpha > 0. \tag{4}$$

Using Eqs. (3) and (4) the *pdf* of the transmuted truncated beta distribution is given by

$$\begin{aligned} h(x) &= (1+\lambda) f_X(x) - 2\lambda f_X(x) F_X(x) \\ &= (1+\lambda)\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} - 2\lambda\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{2\alpha-1}, \qquad 0 < x \leq \beta. \end{aligned} \tag{5}$$

Now, suppose that Z be a random variable with the density function $h(x)$. By defining

$$Y = 1 - \frac{1-\beta}{\beta} Z,$$

the pdf of Y is given by

$$h_Y(y) = \frac{\alpha}{1-\beta}\left((1+\lambda)\left(\frac{1-y}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-y}{1-\beta}\right)^{2\alpha-1}\right), \qquad \beta \le y < 1. \tag{6}$$

According to relations Eqs. (5) and (6), a new distribution is defined by

$$f(x;\alpha,\beta,\lambda) = \begin{cases} \beta h(x), & 0 < x \le \beta, \\ (1-\beta)\, h_Y(x), & \beta \le x < 1. \end{cases} \tag{7}$$

So, based on Eq. (7), we have the following definition.

Definition 2.1. A random variable X is said to be transmuted two-sided distribution if its *pdf* is given by

$$f(x;\alpha,\beta,\lambda) = \begin{cases} \alpha\left((1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{x}{\beta}\right)^{2\alpha-1}\right), & 0 < x \le \beta, \\ \alpha\left((1+\lambda)\left(\frac{1-x}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-x}{1-\beta}\right)^{2\alpha-1}\right), & \beta \le x < 1, \end{cases} \tag{8}$$

and its *cdf* is given by

$$F(x;\alpha,\beta,\lambda) = \begin{cases} \beta\left((1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha} - \lambda\left(\frac{x}{\beta}\right)^{2\alpha}\right), & 0 < x \le \beta, (13) \\ 1-(1-\beta)\left((1+\lambda)\left(\frac{1-x}{1-\beta}\right)^{\alpha} - \lambda\left(\frac{1-x}{1-\beta}\right)^{2\alpha}\right), & \beta \le x < 1, \end{cases} \tag{9}$$

where, α is a shape parameter, β is a scale parameter, and λ is a transmuted parameter.
We denote the transmuted two-sided distribution by $TTS(\alpha,\beta,\lambda)$.

Remark 2.1. If $\lambda = 0$, we have the *pdf* and *cdf* of *TSP* distribution in Eqs. (1) and (2), respectively.

2.1. Density Shape of the *TTS* Distribution

Here, we consider a discussion about the shape of the proposed density function. In the end points of the support, the behaviour of the *pdf* Eq. (8) is given as follows:

$$\lim_{x\to 0,1} f(x;\alpha,\beta,\lambda) = \begin{cases} \infty, & \alpha < 1, \\ 1+\lambda, & \alpha = 1, \\ 0, & \alpha > 1. \end{cases}$$

The derivative $f'(x;\alpha,\beta,\lambda)$ is

$$f'(x;\alpha,\beta,\lambda) = \begin{cases} \frac{\alpha\beta}{x^2}\left(\frac{x}{\beta}\right)^{\alpha}\left[(-4\alpha+2)\lambda\left(\frac{x}{\beta}\right)^{\alpha} + (1+\lambda)(\alpha-1)\right], & 0 < x \le \beta, \\ \frac{\alpha(\beta-1)}{(x-1)^2}\left(\frac{1-x}{1-\beta}\right)^{\alpha}\left[(-4\alpha+2)\lambda\left(\frac{1-x}{1-\beta}\right)^{\alpha} + (1+\lambda)(\alpha-1)\right], & \beta \le x < 1. \end{cases}$$

The right- and left-hand limits of f' at $x = \beta$ are given by

$$\begin{aligned} \lim_{x\to\beta^+} f'(x;\alpha,\beta,\lambda) &= \frac{\alpha^2(1-3\lambda) + \alpha(\lambda-1)}{\beta}, \\ \lim_{x\to\beta^-} f'(x;\alpha,\beta,\lambda) &= \frac{\alpha^2(1-3\lambda) + \alpha(\lambda-1)}{\beta-1}. \end{aligned}$$

These limits are different. So, $f'(\beta)$ does not exist and f has a corner at $x = \beta$.

When $\alpha > 1$, $\lambda > 0$, and $0 < x \le \beta$, if $f'(x) = 0$ then $x_1 = \beta\left(\frac{(1+\lambda)(\alpha-1)}{(4\alpha-2)\lambda}\right)^{1/\alpha}$ and when $\alpha > 1$, $\lambda > 0$, and $\beta \le x < 1$, if $f'(x) = 0$ then $x_2 = 1+(\beta-1)\left(\frac{(1+\lambda)(\alpha-1)}{(4\alpha-2)\lambda}\right)^{1/\alpha}$. So, when $\alpha > 1$ and $\lambda > 0$, the maximum of f occurs at points x_1 and x_2. It implies when $\alpha > 1$ and $\lambda > 0$, f is a bimodal *pdf*.

When $\alpha \geq 1, \lambda \leq 0$, and $0 < x \leq \beta$, we have $f'(x) > 0$ and when $\alpha \geq 1, \lambda \leq 0$, and $\beta \leq x < 1$, we have $f'(x) < 0$. So, for $\alpha \geq 1, \lambda \leq 0$, f is a unimodal *pdf*. The density shapes of the *TTS* distribution for different choices of the parameters are plotted in Fig. 1.

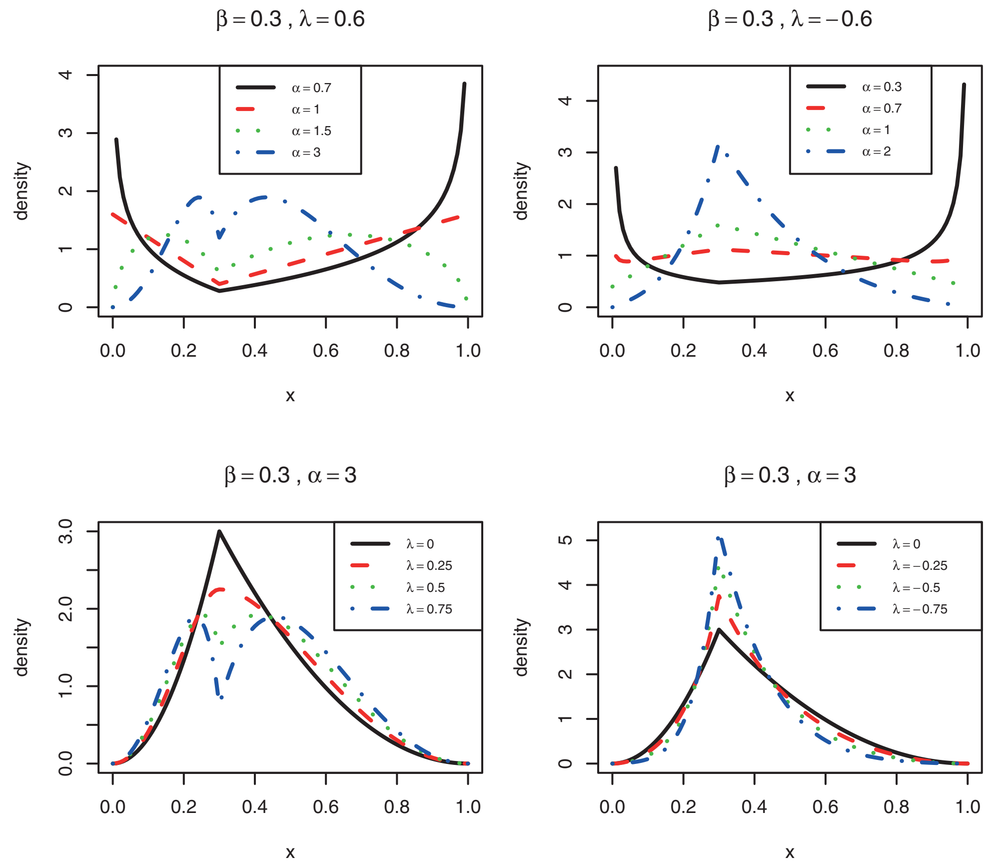

Figure 1 The graphs of the densities of the distribution.

3. TRANSMUTED TWO-SIDED GENERALIZED-G FAMILY OF THE DISTRIBUTIONS

Consider a continuous random variable with *cdf* $G(x;\xi)$ and *pdf* $g(x;\xi)$. Using Definition 2.1, we define a generalization of the transmuted two-sided distribution as follows:

Definition 3.1. A random variable X is said to be transmuted two-sided generalized-G (TSG-G) distribution if its *pdf* is given by

$$f(x;\alpha,\beta,\lambda,\xi) = \begin{cases} \alpha g(x;\xi)\left[(1+\lambda)\left(\frac{G(x;\xi)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{G(x;\xi)}{\beta}\right)^{2\alpha-1}\right], & -\infty < x \leq G^{-1}_{(x;\xi)}(\beta), \\ \alpha g(x;\xi)\left[(1+\lambda)\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{2\alpha-1}\right], & G^{-1}_{(x;\xi)}(\beta) \leq x < \infty, \end{cases}$$

and its *cdf* is given by

$$F(x;\alpha,\beta,\lambda,\xi) = \begin{cases} \beta\left[(1+\lambda)\left(\frac{G(x;\xi)}{\beta}\right)^{\alpha} - \lambda\left(\frac{G(x;\xi)}{\beta}\right)^{2\alpha}\right], & -\infty < x \leq G^{-1}_{(x;\xi)}(\beta), \\ 1-(1-\beta)\left[(1+\lambda)\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{\alpha} - \lambda\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{2\alpha}\right], & G^{-1}_{(x;\xi)}(\beta) \leq x < \infty, \end{cases}$$

where, $-\infty < x < \infty$, ξ is a parameter vector in the *cdf* $G(x;\xi)$ and $G^{-1}_{(x;\xi)}(.)$ is its inverse.

We denote transmuted TSG-G family of distributions by $TTSG-G(\alpha,\beta,\lambda,\xi)$.

Remark 3.1. If $\lambda = 0$, we have the *pdf* and *cdf* of the TSG-G family introduced by Korkmaz and Genç [1]. If $\alpha = 1$, $\beta = 1$, and $\lambda = 0$, we have the *pdf* of the base distribution.

In the next subsections, we study hazard function, random variate generation, order statistics and relative entropy of the $TTSG - G$ distribution.

3.1. Hazard Function

The hazard rate is a fundamental tools in reliability modelling for evaluation the ageing process. Knowing the shape of the hazard rate is important in reliability theory, risk analysis, and other disciplines. The concepts of increasing, decreasing, bathtub-shaped (first decreasing and then increasing) and upside-down bathtub-shaped (first increasing and then decreasing) hazard rate functions are very useful in reliability analysis. The lifetime distributions with these ageing properties are designated as the increasing failure rate (*IFR*), decreasing failure rate (*DFR*), bathtub-shape (*BUT*), and upside-down bathtub-shaped (*UBT*) distributions, respectively. The hazard function of the *TTSG-G* distribution is given by

$$
\begin{aligned}
r(x) &= \frac{f(x)}{1 - F(x)} \\
&= \begin{cases}
\dfrac{\alpha g(x;\xi)\left((1+\lambda)\left(\frac{G(x;\xi)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{G(x;\xi)}{\beta}\right)^{2\alpha-1}\right)}{1-\beta\left((1+\lambda)\left(\frac{G(x;\xi)}{\beta}\right)^{\alpha} - \lambda\left(\frac{G(x;\xi)}{\beta}\right)^{2\alpha}\right)}, & -\infty < x \le G^{-1}_{(x;\xi)}(\beta), \\[2ex]
\dfrac{\alpha g(x;\xi)\left((1+\lambda)\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{2\alpha-1}\right)}{(1-\beta)\left((1+\lambda)\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{\alpha} - \lambda\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{2\alpha}\right)}, & G^{-1}_{(x;\xi)}(\beta) \le x < \infty.
\end{cases}
\end{aligned}
$$

3.2. Random Variate Generation

For generating random variables from the $TTSG - G$ distribution, we use the inverse transformation method. The quantile of order q of the $TTSG - G$ distribution is

$$
x_q = F^{-1}(q;\alpha,\beta,\lambda,\xi) = \begin{cases}
G^{-1}_{(x;\xi)}\left[\beta\left(\dfrac{1+\lambda-\sqrt{(1+\lambda)^2 - \frac{4\lambda q}{\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}\right], & 0 < q \le \beta, \\[3ex]
G^{-1}_{(x;\xi)}\left[1-(1-\beta)\left(\dfrac{1+\lambda-\sqrt{(1+\lambda)^2 - \frac{4\lambda(1-q)}{1-\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}\right], & \beta \le q < 1.
\end{cases}
$$

Let U be a random variable generated from a uniform distribution on $(0, 1)$, then

$$
X = \begin{cases}
G^{-1}_{(x;\xi)}\left[\beta\left(\dfrac{1+\lambda-\sqrt{(1+\lambda)^2 - \frac{4\lambda U}{\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}\right], & 0 < U \le \beta, \\[3ex]
G^{-1}_{(x;\xi)}\left[1-(1-\beta)\left(\dfrac{1+\lambda-\sqrt{(1+\lambda)^2 - \frac{4\lambda(1-U)}{1-\beta}}}{2\lambda}\right)^{\frac{1}{\alpha}}\right], & \beta \le U < 1,
\end{cases}
$$

is a random variable generated from the $TTSG - G$ distribution by the probability integral transform.

3.3. Order Statistics

Order statistics play a vital role in the theory of probability and statistics. Let $X_1, X_2, \dots, X_n$ be a random sample from the *TTSG-G* distribution. Let $X_{i:n}$ denote the ith order statistics. Then the *pdf* of $X_{i:n}$ is given by

$$f_{i:n}(x)=\begin{cases}\sum_{t_1=0}^{n-i}\sum_{t_2=0}^{i-1+t_1}\binom{n-i}{t_1}\binom{i-1+t_1}{t_2}(-1)^{t_1}(-\lambda)^{i-1+t_1-t_2}(1+\lambda)^{t_2}\beta^{i-1+t_1}\\ \times g(x;\xi)\left[(1+\lambda)\left(\frac{G(x;\xi)}{\beta}\right)^{\alpha(2i+2t_1-t_2-1)-1}-2\lambda\left(\frac{G(x;\xi)}{\beta}\right)^{\alpha(2i+2t_1-t_2)-1}\right], & x\le G^{-1}_{(x;\xi)}(\beta),\\ \sum_{t_1=0}^{i-1}\sum_{t_2=0}^{n-t_1-1}\binom{i-1}{t_1}\binom{n-t_1-1}{t_2}(-1)^{i-1-t_1}(-\lambda)^{n-t_1-t_2-1}(1+\lambda)^{t_2}(1-\beta)^{n-t_1-1}\\ \times g(x;\xi)\left[(1+\lambda)\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{\alpha(2n-2t_1-t_2-1)-1}-2\lambda\left(\frac{1-G(x;\xi)}{1-\beta}\right)^{\alpha(2n-2t_1-t_2)-1}\right], & G^{-1}_{(x;\xi)}(\beta)\le x,\end{cases}$$

where $A=\frac{n!\alpha}{(i-1)!(n-i)!}$.

3.4. Kullback—Leibler Divergence

The Kullback–Leibler divergence (or relative entropy) is an informational measure for comparing the similarity between two pdfs. The Kullback–Leibler divergence between the proposed distribution $TTSG-G$ and the $TSG-G$ distribution, introduced by Korkmaz and Genç [1], is obtained as

$$\begin{aligned}Kl\left(f_{TTSG-G}||f_{TSG-G}\right) &= \int_{-\infty}^{\infty} f_{TTSG-G}(x)\log\frac{f_{TTSG-G}(x)}{f_{TSG-G}(x)}dx\\ &= \int_0^1 (1+\lambda-2\lambda u)\log(1+\lambda-2\lambda u)du\\ &= \frac{-(1-\lambda)^2\log(1-\lambda)+(1+\lambda)^2\log(1+\lambda)-2\lambda}{4\lambda}.\end{aligned} \tag{10}$$

On the other hand, if U be a uniform random variable on (0, 1) then the corresponding transmuted distribution is given as

$$f_U(u)=1+\lambda-2\lambda u, \qquad 0<u<1.$$

The Shannon entropy of transmuted uniform distribution is computed by

$$\begin{aligned}H(U) &= -\int_0^1 f_U(u)\log f_U(u)du\\ &= -\int_0^1 (1+\lambda-2\lambda u)\log(1+\lambda-2\lambda u)du\\ &= -\frac{-(1-\lambda)^2\log(1-\lambda)+(1+\lambda)^2\log(1+\lambda)-2\lambda}{4\lambda}.\end{aligned} \tag{11}$$

From the relations, Eqs. (10) and (11) we see that the Kullback–Leibler divergence between the *TTSG-G* and the *TSG-G* distributions is equal to $-H(U)$. So, this informational measure is free of the parent distribution G.

4. TRANSMUTED TSGE DISTRIBUTION

The *TTSG-G* distribution is specialized by taking G as the well-known distribution. We suppose that the base distribution G has an exponential distribution with *cdf* and inverse *cdf* function $G(x;\theta)=1-e^{-\frac{x}{\theta}}, x>0, \theta>0$ and $G^{-1}(x;\theta)=-\theta\log(1-x)$, respectively.

The *pdf* of the parent distribution is $g(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$. By considering this distribution, the *pdf* of the *TTSG-G* distribution can be given as

$$f(x;\alpha,\beta,\lambda,\theta) = \begin{cases} \alpha\frac{1}{\theta}e^{-\frac{x}{\theta}}\left((1+\lambda)\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{2\alpha-1}\right), & 0 < x \le -\theta\log(1-\beta), \\ \alpha\frac{1}{\theta}e^{-\frac{x}{\theta}}\left((1+\lambda)\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{2\alpha-1}\right), & -\theta\log(1-\beta) \le x < \infty, \end{cases}$$

and its *cdf* is given by

$$F(x;\alpha,\beta,\lambda,\theta) = \begin{cases} \beta\left((1+\lambda)\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{\alpha} - \lambda\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{2\alpha}\right), & 0 < x \le -\theta\log(1-\beta), \\ 1-(1-\beta)\left((1+\lambda)\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{\alpha} - \lambda\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{2\alpha}\right), & -\theta\log(1-\beta) \le x < \infty. \end{cases}$$

We call this distribution the transmuted TSGE distribution and denote by $TTSG-E(\alpha,\beta,\lambda,\theta)$.

Remark 3.1. If $\lambda = 0$, we have the *pdf* and *cdf* of the TSGE distribution introduced by Korkmaz and Genç [11]. If $\alpha = 1, \beta = 1$ and $\lambda = 0$, we have the *pdf* of the base distribution.

4.1. Density Shape of the *TTSG* — *E* Distribution

In the end points of the support, the behaviour of the *pdf* of the $TTSG-E$ distribution is given as follows:

$$\lim_{x\to 0} f(x;\alpha,\beta,\lambda,\theta) = \begin{cases} \infty, & \alpha < 1, \\ \frac{1+\lambda}{\theta}, & \alpha = 1, \\ 0, & \alpha > 1, \end{cases} \quad \lim_{x\to\infty} f(x;\alpha,\beta,\lambda,\theta) = 0, \quad \forall\alpha > 0.$$

The derivative $f'(x;\alpha,\beta,\lambda,\theta)$ is

$$f'(x;\alpha,\beta,\lambda,\theta) = \begin{cases} \frac{\alpha\left(1-e^{-\frac{x}{\theta}}\right)^{\alpha-2} e^{-\frac{x}{\theta}}}{\theta^2\beta^{\alpha-1}}\left[\left(2\lambda - 4\lambda\alpha e^{-\frac{x}{\theta}}\right)\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)\right. \\ \left.+\left[(1+\lambda)\left(\alpha e^{-\frac{x}{\theta}}-1\right)\right]\right], & 0 < x \le -\theta\log(1-\beta), \\ \frac{\alpha^2\left(e^{-\frac{x}{\theta}}\right)^{\alpha}}{\theta^2(1-\beta)^{\alpha-1}}\left[4\lambda\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{\alpha} - \lambda - 1\right], & -\theta\log(1-\beta) \le x < \infty. \end{cases}$$

The right- and left-hand limits of f' at $x = -\theta\log(1-\beta)$ are given by

$$\lim_{x\to-\theta\log(1-\beta)^+} f'(x;\alpha,\beta,\lambda,\theta) = \frac{\alpha(1-\beta)}{\theta^2\beta}\left(\alpha(\beta-1)(3\lambda-1)+\lambda-1\right),$$

$$\lim_{x\to-\theta\log(1-\beta)^-} f'(x;\alpha,\beta,\lambda,\theta) = \frac{\alpha^2(1-\beta)(3\lambda-1)}{\theta^2}.$$

These limits are not equal. So, $f'\left(-\theta\log(1-\beta)\right)$ does not exit and the $TTSG-E$ distribution has a corner at $x = -\theta\log(1-\beta)$. The shape of the $TTSG-E$ distribution for different values of the parameters are plotted in Figs. 2 and 3. Figures 2 and 3 indicate that for $\alpha > 1$ and $\lambda > 0$, the $TTSG-E$ distribution is a bimodal distribution and for $\alpha \ge 1$ and $\lambda \le 0$, the $TTSG-E$ distribution is a unimodal distribution.

In the next section, we consider the hazard shape of the $TTSG-E$ distribution.

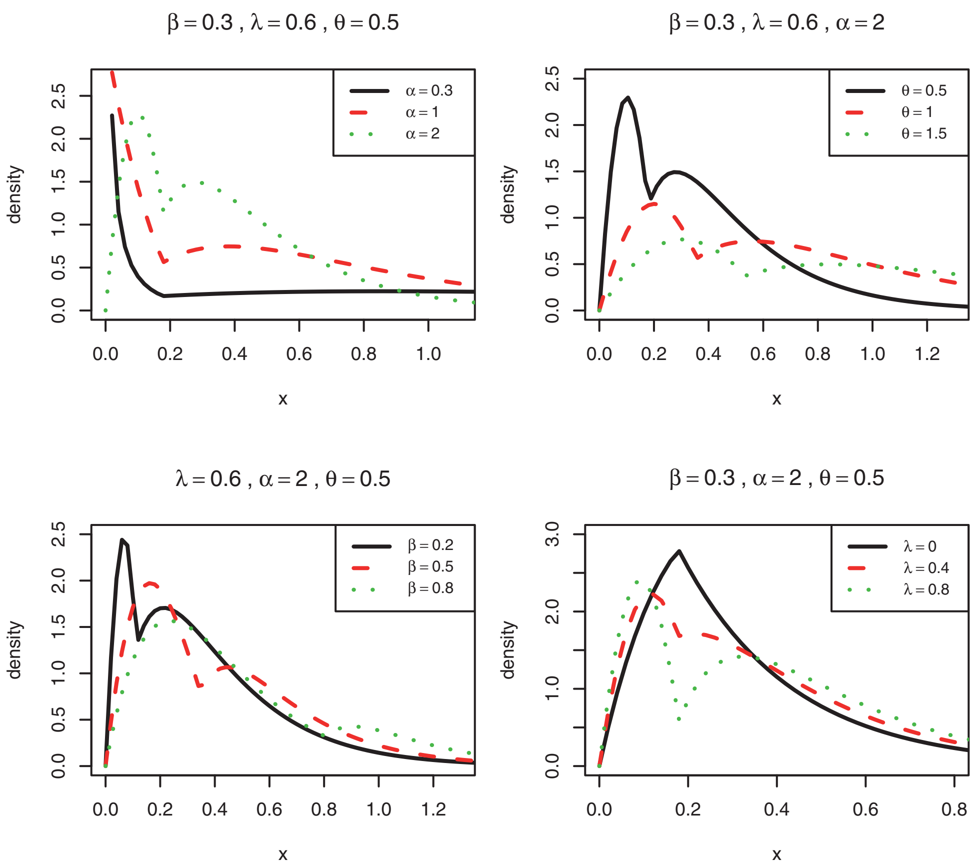

Figure 2 The graphs of the densities of the $TTSG-E$ distribution with $0 \leq \lambda \leq 1$.

4.2. Hazard Function of the *TTSG* — *E* Distribution

The hazard function of the $TTSG-E$ distribution is

$$
\begin{aligned}
r(x) &= \frac{f(x;\alpha,\beta,\lambda,\theta)}{1-F(x;\alpha,\beta,\lambda,\theta)} \\
&= \begin{cases}
\dfrac{\alpha\frac{1}{\theta}e^{-\frac{x}{\theta}}\left[(1+\lambda)\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{\alpha-1}-2\lambda\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{2\alpha-1}\right]}{1-\beta\left[(1+\lambda)\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{\alpha}-\lambda\left(\frac{1-e^{-\frac{x}{\theta}}}{\beta}\right)^{2\alpha}\right]}, & 0<x\leq-\theta\log(1-\beta), \\[2ex]
\dfrac{\alpha\frac{1}{\theta}e^{-\frac{x}{\theta}}\left[(1+\lambda)\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{\alpha-1}-2\lambda\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{2\alpha-1}\right]}{(1-\beta)\left[(1+\lambda)\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{\alpha}-\lambda\left(\frac{e^{-\frac{x}{\theta}}}{1-\beta}\right)^{2\alpha}\right]}, & -\theta\log(1-\beta)\leq x<\infty.
\end{cases}
\end{aligned}
$$

Because of complicated form of the hazard function, we couldn't explore this function analytically. We only consider the end points of the support. The behaviour of the hazard function in the end points is given as follows:

$$
\lim_{x\to 0} r(x) = \begin{cases} \infty, & \alpha<1, \\ \dfrac{1+\lambda}{\theta}, & \alpha=1, \\ 0, & \alpha>1, \end{cases} \qquad \lim_{x\to\infty} r(x) = \frac{\alpha}{\theta}, \quad \forall\alpha>0.
$$

Some shapes of the hazard function for the selected values of parameters is given in Figs. 4 and 5. Figures 4 and 5 show that the hazard rate function of the $TTSG-E$ distribution can be *IFR*, *DFR*, *BUT*, and *UBT*.

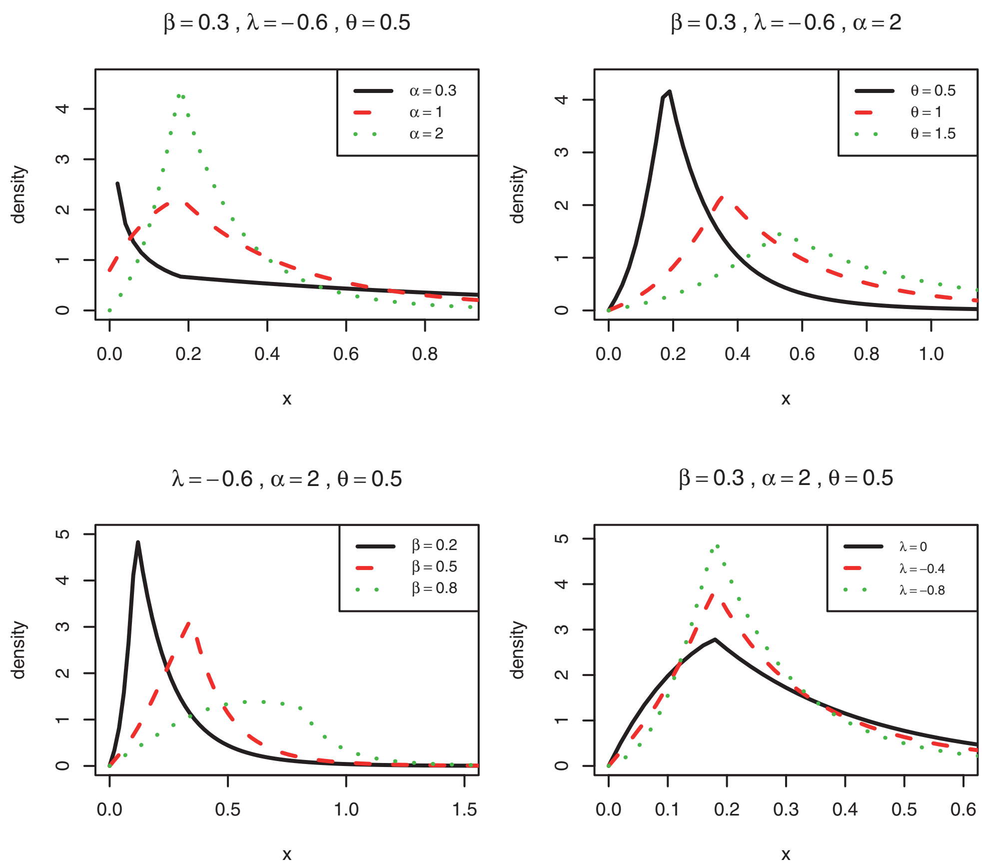

Figure 3 | The graphs of the densities of the $TTSG-E$ distribution with $-1 \leq \lambda \leq 0$.

5. ESTIMATION OF THE PARAMETERS OF THE *TTSG* — *G* DISTRIBUTION

In this section, we obtain the estimation of parameters the $TTSG-G$ distribution by using two methods: MLE and bootstrap estimation. Also, a simulation study is conducted for MLEs of parameters of the $TTSG-E$ distribution.

5.1. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the $TTSG-G$ distribution and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the corresponding order statistics. The log-likelihood function is given by

$$\begin{aligned} \ell\left(\alpha,\beta,\lambda,\xi\right) &= n\log\alpha + \sum_{i=1}^{n}\log\left(g\left(x_i;\xi\right)\right) \\ &+ \log\left\{\prod_{i=1}^{r}\left[(1+\lambda)\left(\frac{G\left(x_{i:n};\xi\right)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{G\left(x_{i:n};\xi\right)}{\beta}\right)^{2\alpha-1}\right]\right. \\ &\left.\times\prod_{i=r+1}^{n}\left[(1+\lambda)\left(\frac{1-G\left(x_{i:n};\xi\right)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-G\left(x_{i:n};\xi\right)}{1-\beta}\right)^{2\alpha-1}\right]\right\}, \end{aligned}$$

where $x_{r:n} \leq G^{-1}_{(x;\xi)}(\beta) < x_{r+1:n}$ for $r = 1, 2, \ldots, n$ and $X_{0:n} = -\infty$, $X_{n+1:n} = \infty$.

For estimating the parameters, we obtain the partial derivatives of the log-likelihood function with respect to the parameters. At the corner point β, the log-likelihood function for the $TTSG-G$ distribution is not differentiable and we can not find the estimate of β in a regular way. According to Van Dorp and Kotz [2], we can find the *MLE* of parameters. We first consider the *MLE*s of α and β when the parameters λ and ξ are known. Without loss of generality, we assume that $\lambda = 0$. So, the log-likelihood function will be

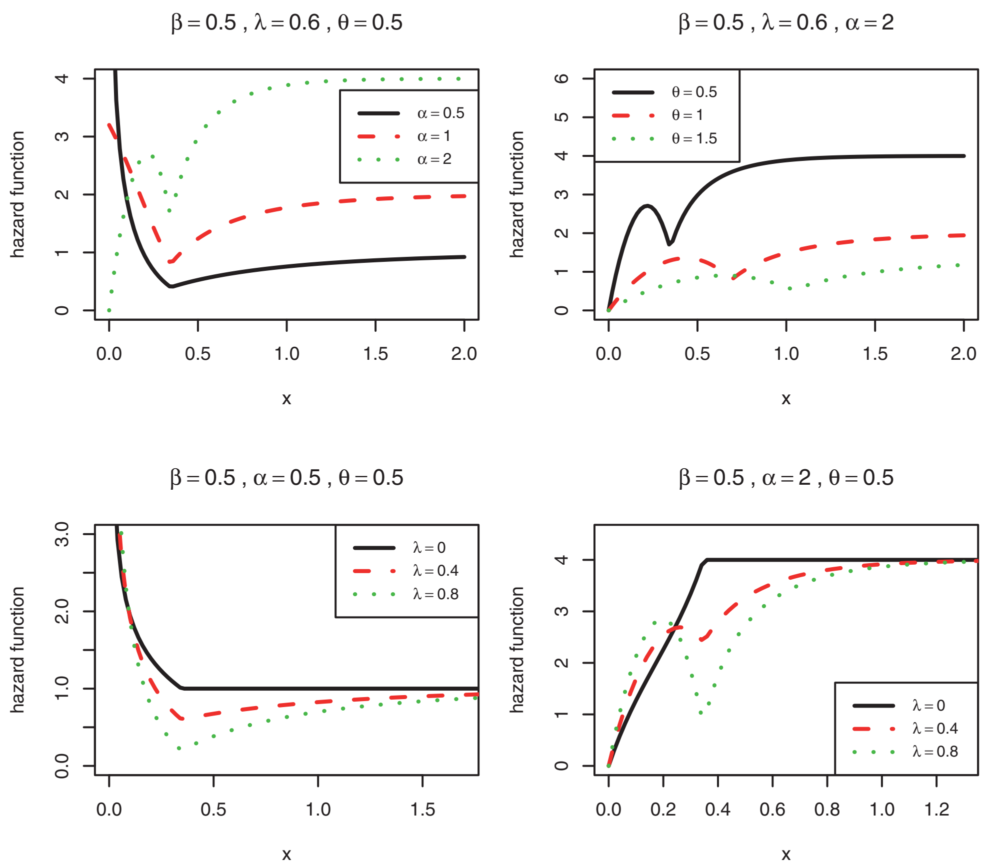

Figure 4 The graphs of the hazard function of the $TTSG-E$ distribution with $0 \leq \lambda \leq 1$.

$$
\begin{aligned}
\ell\left(\alpha,\beta,\lambda,\xi\right) &= n\log\alpha + \sum_{i=1}^{n}\log\left(g(x_i);\xi\right) + \log\left\{\prod_{i=1}^{r}\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{\alpha-1}\prod_{i=r+1}^{n}\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{\alpha-1}\right\} \\
&= n\log\alpha + \sum_{i=1}^{n}\log\left(g(x_i;\xi)\right) + (\alpha-1)\log\left\{\frac{\prod_{i=1}^{r}G\left(x_{i:n};\xi\right)\prod_{i=r+1}^{n}\left(1-G\left(x_{i:n};\xi\right)\right)}{\beta^{r}\left(1-\beta\right)^{n-r}}\right\}.
\end{aligned}
$$

According to Van Dorp and Kotz [2] and Korkmaz and Genç [1], the *MLE*s of α and β are as follows:

$$
\hat{\alpha} = -\frac{n}{\log M\left(\hat{r},\xi\right)}, \quad \hat{\beta} = G\left(x_{\hat{r}:n};\xi\right),
$$

where $\hat{r} = argmaxM\left(r,\xi\right), r \in \{1,2,\ldots,n\}$ with $M\left(r,\xi\right) = \prod_{i=1}^{r-1}\frac{G(x_{i:n};\xi)}{G(x_{r:n};\xi)}\prod_{r+1}^{n}\frac{1-G(x_{i:n};\xi)}{1-G(x_{r:n};\xi)}$.

By taking the derivative of the log-likelihood function with respect to parameter vector ξ and parameter λ, the *MLE*s of parameters ξ and λ are obtained by equating it to zero. These derivatives are given as

$$
\begin{aligned}
\frac{\partial\ell\left(\alpha,\beta,\lambda,\xi\right)}{\partial\lambda} &= \sum_{i=1}^{\hat{r}}\frac{\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{\alpha-1} - 2\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{2\alpha-1}}{(1+\lambda)\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{2\alpha-1}} \\
&+ \sum_{i=\hat{r}+1}^{n}\frac{\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{\alpha-1} - 2\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{2\alpha-1}}{(1+\lambda)\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{2\alpha-1}},
\end{aligned}
$$

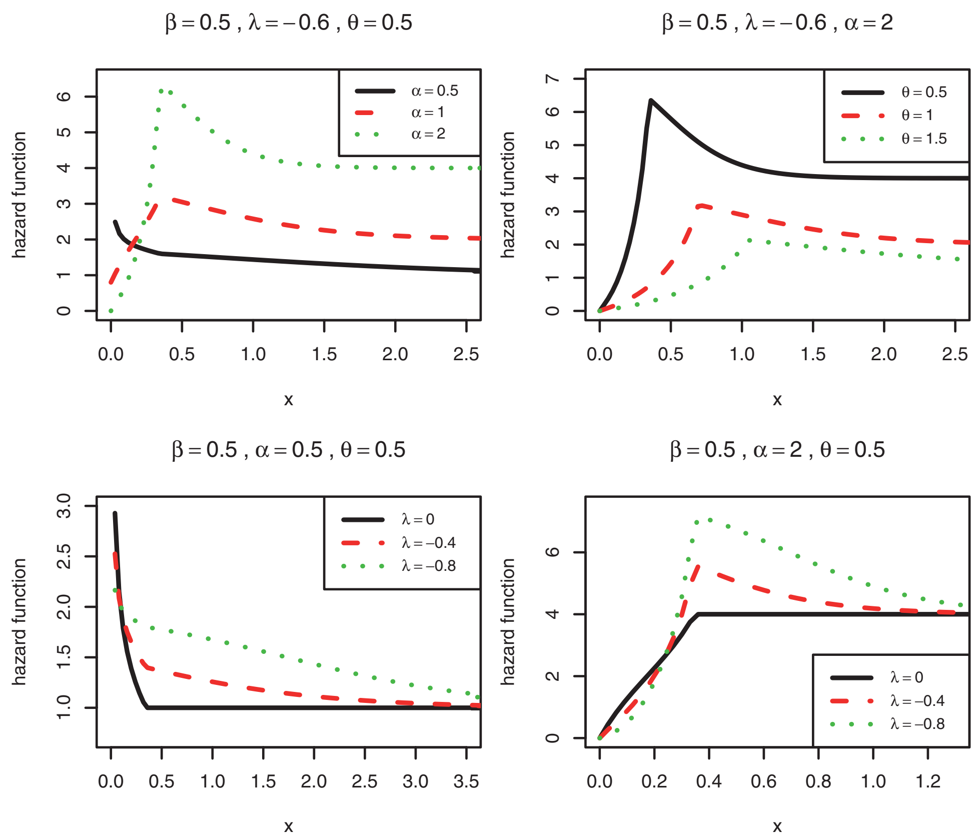

Figure 5 The graphs of the hazard function of the $TTSG-E$ distribution with $-1 \le \lambda \le 0$.

$$\begin{aligned}
\frac{\partial \ell\left(\alpha,\beta,\lambda,\xi\right)}{\partial \xi_k} &= \sum_{i=1}^{n} \frac{g'(x_{i:n};\xi)}{g(x_{i:n};\xi)} \\
&+ \sum_{i=1}^{\hat{r}} \frac{G'(x_{i:n};\xi)}{\beta} \cdot \frac{(1+\lambda)(\alpha-1)\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{\alpha-2} - 2\lambda(2\alpha-1)\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{2\alpha-2}}{(1+\lambda)\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{G(x_{i:n};\xi)}{\beta}\right)^{2\alpha-1}} \\
&+ \sum_{i=\hat{r}+1}^{n} \frac{-G'(x_{i:n};\xi)}{1-\beta} \cdot \frac{(1+\lambda)(\alpha-1)\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{\alpha-2} - 2\lambda(2\alpha-1)\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{2\alpha-2}}{(1+\lambda)\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{\alpha-1} - 2\lambda\left(\frac{1-G(x_{i:n};\xi)}{1-\beta}\right)^{2\alpha-1}},
\end{aligned}$$

where $g'\left(t;\xi\right) = \frac{\partial g(t;\xi)}{\partial \xi_k}$ and $G'\left(t;\xi\right) = \frac{\partial G(t;\xi)}{\partial \xi_k}$.

However, these equations are nonlinear and there are no explicit solutions. Thus, they have to be solved numerically. So, the *optim* package is used for estimating the parameters in *R* software.

5.2. Bootstrap Estimation

The parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) bootstrap resampling (see Efron and Tibshirani [17]). The parametric bootstrap procedure is described as follows:

Parametric bootstrap procedure:

1. Estimate θ (vector of unknown parameters), say $\hat{\theta}$, by using the *MLE* procedure based on a random sample.
2. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$ using $\hat{\theta}$ and obtain the bootstrap estimate of θ, say $\widehat{\theta}^*$, from the bootstrap sample based on the *MLE* procedure.
3. Repeat Step 2 *NBOOT* times.
4. Order $\widehat{\theta}^*{}_1, \dots, \widehat{\theta}^*{}_{NBOOT}$ as $\widehat{\theta}^*{}_{(1)}, \dots, \widehat{\theta}^*{}_{(NBOOT)}$. Then obtain γ-quantiles and $100\,(1-\alpha)\,\%$ confidence intervals (CIs) of parameters.

In case of the $TTSG-G$ distribution, the parametric bootstrap estimators (PBs) of α, β, λ, and ξ, say $\hat{\alpha}_{PB}, \hat{\beta}_{PB}, \hat{\lambda}_{PB}$, and $\hat{\xi}_{PB}$, respectively.

5.3. Simulation

Here, we assess the performance of the *MLE*s of the parameters with respect to sample size n for the $TTSG-E$ distribution. The assessment of performance is based on a simulation study by using the Monte Carlo method. Let $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$, and $\hat{\theta}$ be the *MLE*s of the parameters α, β, λ, and θ, respectively. We calculate the mean square error (*MSE*) and bias of the *MLE*s of the parameters, α, β, λ and θ based on the simulation results of 3000 independence replications. results are summarised in Table 1 for different values of, $n, \alpha, \beta, \lambda$ and θ. From Table 1 the results verify that *MSE* of the *MLE*s of the parameters decrease with respect to sample size n for all parameters. So, we can see the *MLE*s of α, β, λ, and θ are consistent estimators.

Table 1 MSE and bias (values in parentheses) of the MLEs of the parameters α, β, λ, and θ.

		$\alpha = 0.5$	$\beta = 0.3$	$\lambda = 0.5$	$\theta = 0.5$
n	30	0.0274 (0.0316)	0.0165 (0.0099)	0.0855 (0.0481)	0.0272 (0.0229)
	50	0.0153 (0.0269)	0.0092 (0.0034)	0.0721 (0.0502)	0.0141 (0.0147)
	100	0.0081 (0.0174)	0.0045 (0.0030)	0.0514 (0.0451)	0.0063 (0.0071)
	200	0.0035 (0.0098)	0.0020 (0.0018)	0.0243 (0.0245)	0.0033 (0.0046)
		$\alpha = 0.5$	$\beta = 0.3$	$\lambda = 0.5$	$\theta = 1.5$
n	30	0.0286 (0.0357)	0.0181 (0.0093)	0.0870 (0.0552)	0.2462 (0.0742)
	50	0.0156 (0.0294)	0.0097 (0.0057)	0.0720 (0.0552)	0.1245 (0.0412)
	100	0.0079 (0.0202)	0.0042 (0.0011)	0.0484 (0.0470)	0.0632 (0.0352)
	200	0.0035 (0.0097)	0.0019 (−0.0007)	0.0264 (0.0299)	0.0285 (0.0138)
		$\alpha = 2$	$\beta = 0.3$	$\lambda = 0.75$	$\theta = 0.5$
n	30	32.5268 (0.6036)	0.0291 (0.0479)	0.0914 (−0.0422)	2.9212 (0.1673)
	50	13.0018 (0.3243)	0.0167 (0.0174)	0.0449 (0.0192)	1.1566 (0.0768)
	100	0.1474 (0.0976)	0.0063 (0.0019)	0.0236 (0.0454)	0.0132 (0.0142)
	200	0.0412 (0.0524)	0.0022 (−0.0016)	0.0138 (0.0327)	0.0025 (0.0072)
		$\alpha = 2$	$\beta = 0.3$	$\lambda = 0.75$	$\theta = 1.5$
n	30	13.8834 (0.5349)	0.0291 (0.0320)	0.0946 (−0.0477)	16.7689 (0.4470)
	50	26.9195 (0.4725)	0.0174 (0.0109)	0.0460 (0.0255)	12.5857 (0.3298)
	100	2.0210 (0.1251)	0.0067 (−0.0021)	0.0249 (0.0396)	1.5966 (0.0752)
	200	0.0467 (0.0624)	0.0023 (−0.0025)	0.0150 (0.0388)	0.0256 (0.0241)
		$\alpha = 0.5$	$\beta = 0.3$	$\lambda = -0.5$	$\theta = 0.5$
n	30	16.0098 (0.6477)	0.0589 (0.0013)	0.8003 (0.8346)	3.5771 (0.2223)
	50	0.5018 (0.2976)	0.0619 (0.0477)	0.7082 (0.7889)	0.2463 (0.0554)
	100	0.2836 (0.2171)	0.0600 (0.0730)	0.4518 (0.6418)	0.1114 (0.0304)
	200	0.0460 (0.1789)	0.0595 (0.0878)	0.3221 (0.5549)	0.0172 (0.0254)
		$\alpha = 0.5$	$\beta = 0.3$	$\lambda = -0.5$	$\theta = 1.5$
n	30	5.7339 (0.5136)	0.0664 (0.0481)	0.8871 (0.8801)	13.4951 (0.4215)
	50	2.1495 (0.3433)	0.0707 (0.0714)	0.6986 (0.7841)	8.0247 (0.2459)
	100	0.0741 (0.2105)	0.0666 (0.0826)	0.4685 (0.6529)	0.2054 (0.0803)
	200	0.0449 (0.1787)	0.0615 (0.0871)	0.3225 (0.5558)	0.1457 (0.0802)
		$\alpha = 2$	$\beta = 0.3$	$\lambda = -0.75$	$\theta = 0.5$
n	30	83.3071 (3.0259)	0.0518 (0.0820)	0.8762 (0.8905)	1.5715 (0.1644)
	50	24.7184 (1.9937)	0.0413 (0.0725)	0.9042 (0.9040)	0.4906 (0.0413)
	100	4.6772 (1.5489)	0.0344 (0.0633)	0.9048 (0.9032)	0.0876 (0.0049)
	200	2.0712 (1.3450)	0.0255 (0.0435)	0.7545 (0.8364)	0.0120 (−0.0076)

MLE, maximum likelihood estimation; MSE, mean square error.

6. APPLICATION OF THE *TTSG* — *E* DISTRIBUTION

To investigate the advantage of the proposed distribution, we consider a real data set provided by Bjerkedal [18]. This real data set consists of survival times of 72 guinea pigs injected with different amount of tubercle. This species of guinea pigs are known to have high susceptibility of human tuberculosis, which is one of the reasons for choosing. We consider only the study in which animals in a single cage are under the same regimen. The data represents the survival times of guinea pigs in days. The data are given below:

12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376.

6.1. Bootstrap Inference for Parameters of the *TTSG* — *E* Distribution

In this section, we obtain point and %95 CI estimation of parameters of the $TTSG - E$ distribution by parametric bootstrap method for the real data set. We provide results of bootstrap estimation based on 10,000 bootstrap replicates in Table 2. It is interesting to look at the joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between parameters (see Fig. 6).

Table 2 Parametric bootstrap point and interval estimation of the parameters α, β, λ, and θ.

	Point estimation	*CI*
α	2.223	(1.354, 4.117)
β	0.298	(0.157, 0.505)
λ	−0.307	(−0.846, 0.734)
θ	161.075	(111.722, 268.019)

CI, confidence interval.

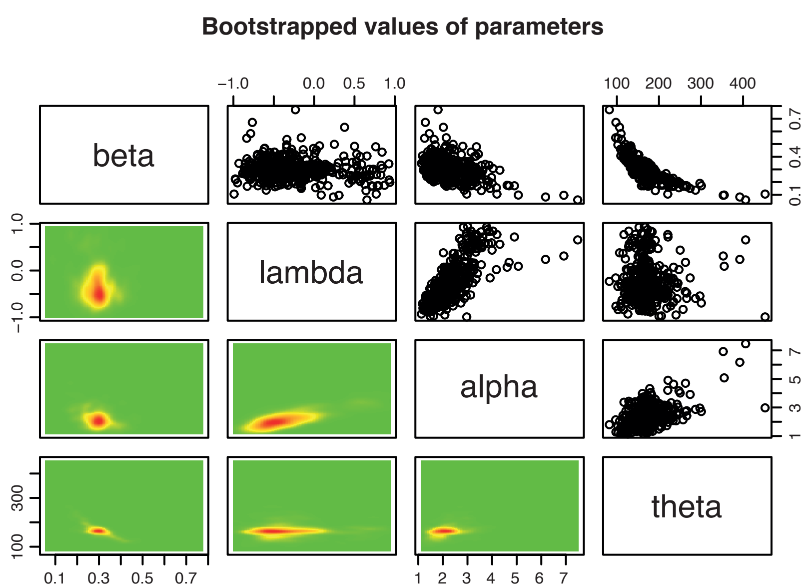

Figure 6 Parametric bootstrapped values of parameters of the *TTSG* —*E* distribution for the real data.

6.2. *MLE* Inference and Comparing with Other Models

We fit the proposed distribution to the real data set by *MLE* method and compare the results with the gamma, Weibull, TSGE, generalized exponential (*GE*), and weighted exponential (*WE*) distributions with respective densities

$$f_{gamma}(x) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}, \qquad x > 0$$

$$f_{Weibull}(x) = \frac{\beta}{\lambda^{\beta}}x^{\beta-1}e^{-\left(\frac{x}{\lambda}\right)^{\beta}}, \qquad x > 0$$

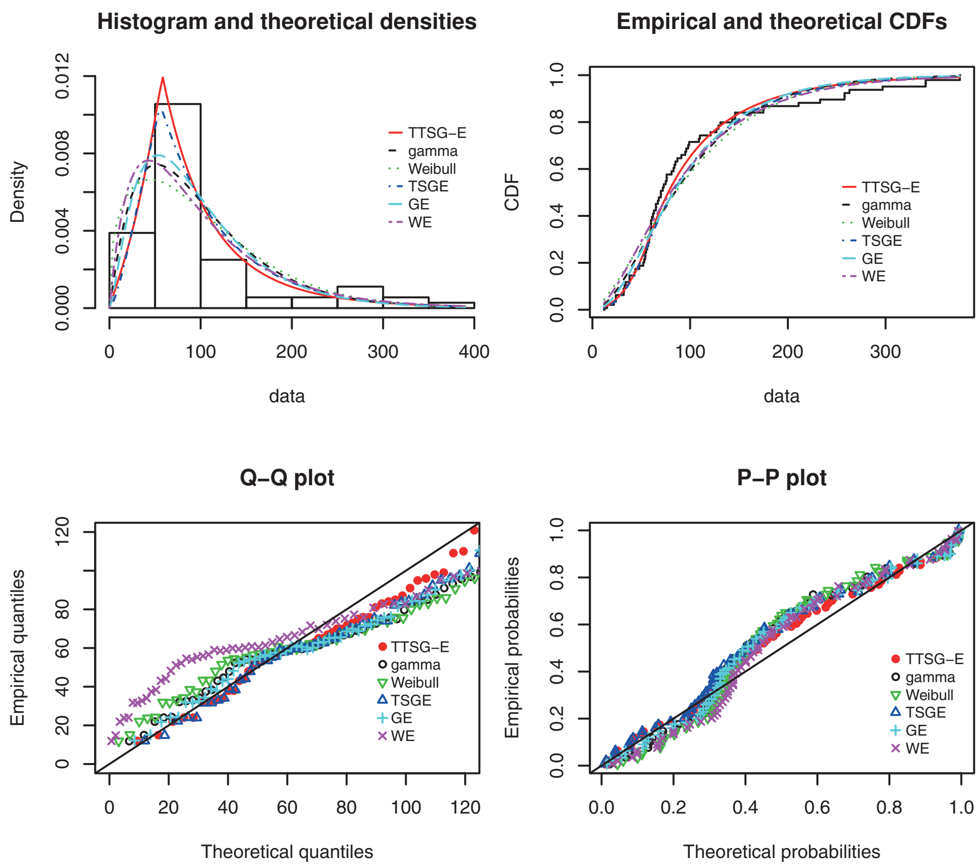

Figure 7 Histogram and fitted density plots, the plots of empirical and fitted *cdfs*, $P-P$ plots and $Q-Q$ plots for the real data set.

$$f_{TSGE}(x) = \begin{cases} \alpha \dfrac{1}{\theta} e^{-\frac{x}{\theta}} \left(\dfrac{1 - e^{\frac{-x}{\theta}}}{\beta} \right)^{\alpha-1}, & 0 < x \le -\theta \log(1-\beta), \\ \alpha \dfrac{1}{\theta} e^{-\frac{x}{\theta}} \left(\dfrac{e^{\frac{-x}{\theta}}}{1-\beta} \right)^{\alpha-1}, & -\theta \log(1-\beta) \le x < \infty, \end{cases}$$

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}, \qquad x > 0$$

$$f_{WE}(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1 - e^{-\alpha\lambda x}\right), \qquad x > 0.$$

For each model, Table 3 includes the *MLE*'s of parameters, Kolmogorov–Smirnov ($K-S$) distance between the empirical distribution and the fitted model, its corresponding p-value, log-likelihood, and Akaike information criterion (AIC) for the real data set. We fit the $TTSG-E$ distribution to the real data set and compare it with the mentioned distributions. The selection criterion is that the lowest AIC and $K-S$ statistic corresponding to the best fitted model. The $TTSG-E$ distribution provides the best fit for the data set as it has lower AIC and $K-S$ statistic than the other competitor models. The histogram of data set, fitted *pdf* of the $TTSG-E$ distribution and fitted *pdf*s of other competitor distributions for the real data set are plotted in Fig. 7. Also, the plots of empirical and fitted *cdf*s functions, $P-P$ plots and $Q-Q$ plots for the $TTSG-E$ and other fitted distributions are displayed in Fig. 7. These plots also support the results in Table 3. The asymptotic covariance matrix of *MLE*s for $TTSG-E$ model parameters which is the inverse of the Fisher information matrix, is given by

$$\begin{pmatrix} 0.15826 & 0.00006 & 0.08247 & 0.00072 \\ 0.00006 & 0.00011 & 0.00005 & -0.00013 \\ 0.08247 & 0.00005 & 0.07136 & 0.00007 \\ 0.00072 & -0.00013 & 0.00007 & 0.08243 \end{pmatrix}$$

and the 95% two-sided asymptotic CIs for α, β, λ, and θ are given by 1.950 ± 0.7797, 0.303 ± 0.0205, -0.423 ± 0.5236, and 160.67 ± 0.5627, respectively.

Table 3 | The MLEs of parameters for real data set.

Model	Estimation	Log-likelihood	*AIC*	$K-S$ statistic	p-value
$TTSG-E$	$\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta}\right)=(1.950,0.303,-0.423,160.67)$	−388.063	784.127	0.097	0.508
gamma	$\left(\hat{\alpha},\hat{\lambda}\right)=(2.812,0.020)$	−394.247	792.495	0.138	0.127
Weibull	$\left(\hat{\beta},\hat{\lambda}\right)=(1.392,110.529)$	−397.147	798.295	0.146	0.091
TSGE	$\left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)=(2.561, 0.270,177.911)$	−389.549	785.099	0.130	0.171
WE	$\left(\hat{\alpha},\hat{\lambda}\right)=(1.626,0.0138)$	−393.568	791.138	0.117	0.274
GE	$\left(\hat{\alpha},\hat{\lambda}\right)=(2.476, 0.017)$	−393.110	790.220	0.133	0.159

AIC, Akaike information criterion; GE, generalized exponential; K – S, Kolmogorov – Smirnov; MLE, maximum likelihood estimation; TSGE, two-sided generalized exponential, WE, weighted exponential.

6.3. Likelihood Ratio Test

We use the likelihood ratio test (*LRT*) for testing the null hypothesis that the *TSGE* distribution, proposed by Korkmaz and Genç [11], is equally close to the pig data against the alternative hypothesis that the $TTSG-E$ distribution is closer. That is, we wish to test

$$\begin{cases} H_0 : X \sim TTSG-E(\alpha,\beta,0,\theta) \equiv TSGE(\alpha,\beta,\theta) \\ H_1 : X \sim TTSG-E(\alpha,\beta,\lambda,\theta), \end{cases}$$

and equivalently, by considering the estimated value of the parameter λ in Table 3 we should test a one-tailed test as

$$\begin{cases} H_0 : \lambda = 0 \\ H_1 : \lambda < 0. \end{cases}$$

According to the LRT, the test statistic is given by

$$\Lambda(x) = \frac{\sup_{H_0} \ell(\alpha,\beta,\lambda,\theta)}{\sup_{H_1} \ell(\alpha,\beta,\lambda,\theta)} = \frac{\ell\left(\hat{\alpha}_0,\hat{\beta}_0,0,\hat{\theta}_0\right)}{\ell\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta}\right)},$$

where $\ell(\alpha,\beta,\lambda,\theta)$ is the log-likelihood function of $TTSG-E$ distribution. Based on Table 3, $-2\log\Lambda(x) = 2.972$. Since $-2\log\Lambda(x)$ is distributed asymptotically chi-squared distribution with 1 degrees of freedom, we can conclude that the null hypothesis is rejected in significance level $\alpha = 0.1$. Also, the p-value is 0.085.

7. CONCLUDING REMARKS

In this paper, we propose a new family of distributions that is a compounding of two-sided distributions family and transmuted technique. The proposed model generalizes TSP distribution and generalized two-sided family of distributions and contains these distributions as its submodels. Some reliability and statistical properties of the proposed family of distribution are discussed through the paper. Estimation and inference procedure for distribution parameters are investigated by two well-known maximum likelihood and bootstrap methods in general setting. The $TTSG-E$ distribution considered as a special case of this family. One of the advantage of this new distribution is that it can be fitted to the data sets with one or two modes. Data analysis shows that the $TTSG-E$ distribution provides the best fit and the best performance. The proposed distribution may be a better alternative than the other well-known distributions commonly used in literature for fitting statistical data.

ACKNOWLEDGMENTS

The authors would like to thank the associate editor and referees for their constructive comments that improved presentation of the paper.

REFERENCES

1. M.C. Korkmaz, A.İ. Genç, Commun. Stat. Simul. Comput. 46(2) (2017), 1441–1460.
2. J.R. Van Dorp, S. Kotz, Am. Stat. 56(2) (2002), 90–99.
3. J.R.Van Dorp, S. Kotz, J.R. Stat. Soc. Ser. D Statistician. 51(1) (2002), 63–79.

4. J.R. Van Dorp, S. Kotz, Commun. Stat. Theory Methods. 32(9) (2003), 1703–1723.
5. S. Nadarajah, Metrika. 61(3) (2005), 309–321.
6. Ö.E. Oruç, I. Bairamov, Commun. Stat. Theory Methods. 34(5) (2005), 1009–1017.
7. D. Vicari, J.R. Van Dorp, S. Kotz, J. Appl. Stat. 35(10) (2008), 1115–1129.
8. J.M. Herrerías-Velasco, R. HerrerÍas-Pleguezuelo, J.R. Van Dorp, J. Appl. Stat. 36(5) (2009), 573–587.
9. A.R. Soltani, H. Homei, Statistics. 43(6) (2009), 611–620.
10. M.C. Korkmaz, A.İ. Genç, J. Stat. Theory Appl. 14(3) (2015), 265–280.
11. M.C. Korkmaz, A.İ. Genç, Commun. Stat. Theory Methods. 44(23) (2015), 5049–5070.
12. G.R. Aryal, C.P. Tsokos, Nonlinear Anal. Theory Methods Appl. 71(12) (2009), e1401–e1407.
13. G.R. Aryal, C.P. Tsokos, Eur. J. Pure Appl. Math. 4(2) (2011), 89–102.
14. G.R. Aryal, J. Stat. Appl. Prob. 2(1) (2013), 11–20.
15. N.E. Abd El Hady, Int. J. Math Comput. Phys, Electrical Comput. Eng. 8(6) (2014), 903–911.
16. M. Elgarhy, M.R.A. Shawki, Int. J. Math. Trends Technol. 29(2) (2016), 145–154.
17. B. Efron, R.J. Tibshirani, An Introduction to the Bootstrap, CRC Press, Boca Raton, Florida, 1994.
18. T. Bjerkedal, Am. J. Hyg. 72(1) (1960), 130–48.