International Journal of Computational Intelligence Systems
Vol. **12(2)**, 2019, *pp.* 809–821
DOI: https://doi.org/10.2991/ijcis.d.190712.001; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

Information Structures in an Incomplete Interval-Valued Information System

Jiasheng Zeng[1], Zhaowen Li[2,*], Meng Liu[3,*], Shimin Liao[3]

[1] *School of Mathematics and Statistics, Hunan University of Commerce, Changsha, Hunan, China*
[2] *Key Laboratory of Complex System Optimization and Big Data Processing in Department of Guangxi Education, Yulin Normal University, Yulin, Guangxi, China*
[3] *School of Science, Guangxi University for Nationalities, Nanning, Guangxi, China*

ARTICLE INFO

Article History
Received 02 Feb 2019
Accepted 10 Jul 2019

Keywords
Incomplete interval-valued information system
Granular computing
Information granule
Information structure
Dependence
Information distance

ABSTRACT

An incomplete interval-valued information system (IIVIS) is an information system (IS) in which the information values are interval numbers with missing values. This article researches information structures in an IIVIS. First, information structures in an IIVIS are obtained. In addition, the dependence and information distance are presented. The properties of information structures are investigated. Furthermore, group and lattice characterizations of information structures in an IIVIS are studied. Lastly, the θ-rough entropy is explored as an application of the proposed information structures.

1. INTRODUCTION

Granular computing (GrC), developed by Zadeh [1–4], is vital in solving multilevel and multi-perspective problems. GrC abstracts the ideas of granular processing in various disciplines, allowing further exploration of structured thinking, structured problem-solving, and structured information processing. Information granules are an essential notion of GrC. A granule refers to the blocks formed by some individuals through unclear relations, similar relations, adjacent relations, or functional relations. It is obvious that a collection of information granules can constitute a vector that is said to be a granular structure. Since the importance of GrC was highlighted by Lin [5,6] and Yao [7–9], people have recognized this concept.

Rough set theory was presented by Pawlak [10,11]. This theory is a significant approach for managing uncertainty. One of the advantages of a rough set is that it does not need any preliminary or additional data information, but it is directly based on the original data, so it is more objective and credible. Many applications of rough set theory are connected with information systems (ISs) [12–17].

Given an IS, an equivalence relation of each attribute subset can be determined, which is a special similarity between two objects and divides the object set into some disjoint classes; these classes are regarded as equivalence classes. Each equivalence class is looked upon as an information granule [12]. The collection of all these equivalence classes constitutes a vector that is referred to as an information structure.

In this regard, many scholars have made contributions. For instance, Zhang *et al.* [18] explored information structures in a fully fuzzy IS. Chen *et al.* [19] presented information structures in a lattice-valued IS. Qian *et al.* [20] discussed knowledge structures in a knowledge base. Li *et al.* [21] considered relationships between knowledge bases. Tang *et al.* [22] discussed information structures in a lattice-valued IS. Xia *et al.* [23] researched information structures in set-valued ISs from a GrC viewpoint. Xie *et al.* [24] investigated information structures and uncertainty measures in an incomplete probabilistic set-valued IS. Yu. [25] studied information structures in an incomplete IS. Their findings have been proven to be useful for knowledge discovery in ISs or knowledge bases [26]. Obviously, information structures or knowledge structures can be seen as granular structures under the significance of GrC.

An incomplete interval-valued information system (IIVIS) means an IS where its information values are interval numbers with missing values. Nevertheless, information structures in an IIVIS under a framework of GrC have not been investigated. Therefore, this article focuses on this topic. The work process is displayed in Figure 1.

Why do we consider information structures in an IIVIS? This is because information structures in an IIVIS are helpful for knowledge discovery from an IIVIS. Why do we consider information

*Corresponding authors. Email: 402503@gxun.edu.cn, lizhaowen8846@126.com

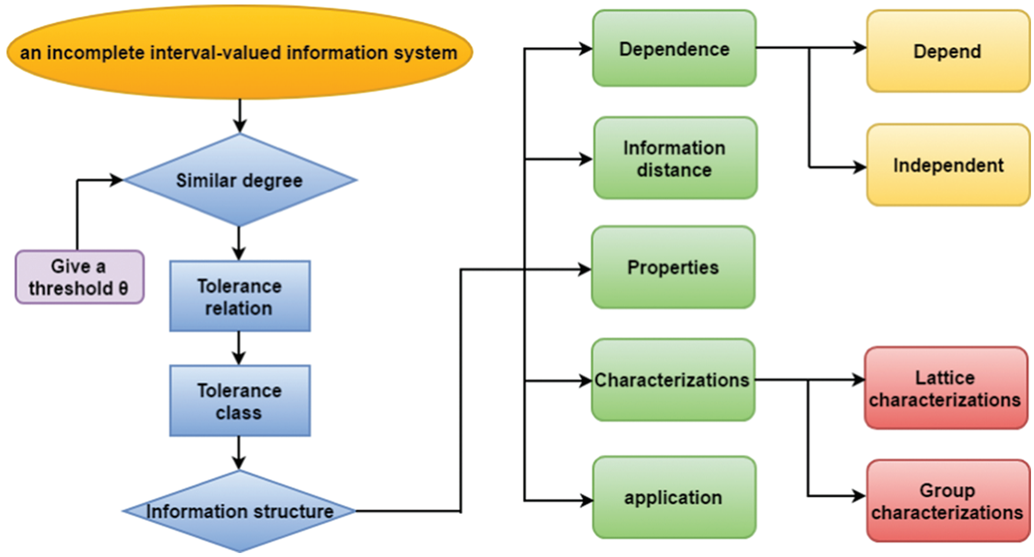

Figure 1 The work process of the article.

granules and information structures together? This is because information granules are constructed, and then information structures can be established.

The rest of this article is designed as follows. Section 2 reviews the essential notions of binary relations, interval numbers and an IIVIS. Section 3 gives the similarity degree and tolerance relations in an IIVIS. Section 4 obtains information structures and studies the dependence, information distance and properties of information structures in an IIVIS. Section 5 studies group and lattice characterizations of information structures in an IIVIS. Section 6 investigates the entropy measure of uncertainty for an IIVIS. Section 7 discusses and concludes this article.

2. PRELIMINARIES

In this section, the essential notions of binary relations, interval numbers and an IIVIS are reviewed.

In this article, U signifies the finite universe, 2^U expresses a set of all subsets of U, and $|X|$ means the number of elements in $X \in 2^U$.

Let
$$U = \{u_1, u_2, \cdots, u_n\},$$
$$\delta = U \times U, \Delta = \{(u,u) : u \in U\}.$$

2.1. Binary Relations

R is said to be a binary relation on U whenever $R \subseteq U \times U$. If $(u,v) \in R$, then xRy.

Let R be a binary relation on U. Then, R is regarded as a universal relation on U whenever $R = \delta$; R is regarded as an identity relation on U whenever $R = \Delta$.

Assume that R is a binary relation on U. Then, R is referred to as an equivalence relation on U if R meets the following requirements:

1. Reflexive: $\forall\, u \in U$, uRu;
2. Symmetric: $\forall\, u, v \in U$, uRv implies uRv;
3. Transitive: $\forall\, u, v, w \in U$, uRv and vRw imply uRw.

In addition, R is said to be a tolerance relation on U if R is reflexive and symmetric.

2.2. Interval-Valued Numbers

Let
$$[R] = \{m = [m^-, m^+] : m^-, m^+ \in R, m^- \leqslant m^+\}.$$

For any $m \in R$, express $\overline{m} = [m, m]$.

For any $m, n \in [R]$, define

1. $m = n \Leftrightarrow m^- = n^-, m^+ = n^+$.
2. $m \leqslant n \Leftrightarrow m^- \leqslant n^-, m^+ \leqslant n^+; m < n \Leftrightarrow m \leqslant n, m \neq n$.

Definition 2.1. [27,28] *Let $m, n \in [R]$. Then, the possible degree of m relative to n is defined as follows:*

$$p(m,n) = min\left\{1, max\left\{\frac{m^+ - n^-}{(m^+ - m^-) + (n^+ - n^-)}, 0\right\}\right\}.$$

Proposition 2.2. *[28,29] The following properties hold:*

1. $\forall m, n \in [R]$, $0 \leqslant p(m,n) \leqslant 1$;
2. $\forall m \in [R]$, $p(m,m) = 0.5$;
3. $\forall m, n \in [R]$, $p(m,n) + p(n,m) = 1$.

Definition 2.3. [30] *Let $m, n \in [R]$. Then, the similarity degree of m and n is defined as follows:*

$$v(m,n) = 1 - |p(m,n) - p(n,m)|.$$

Proposition 2.4. *[30] The following properties hold:*

1. $\forall\, m, n \in [R]$, $v(m,n) = v(n,m)$;
2. $\forall m, n \in [R]$, $0 \leqslant v(m,n) \leqslant 1$;
3. $\forall m, n \in [R]$, $v(m,n) = 1 \Leftrightarrow m = n$.

Example 2.5. [31] *Pick* $m = [5, 7]$ *and* $n = [6, 8]$. *Then,*

$$p(m, n) = min\left\{1, max\left\{\frac{7-6}{(7-5)+(8-6)}, 0\right\}\right\} = \frac{1}{4},$$

$$p(n, m) = min\left\{1, max\left\{\frac{8-5}{(7-5)+(8-6)}, 0\right\}\right\} = \frac{3}{4},$$

$$v(m, n) = 1 - |p(m, n) - p(n, m)| = 1 - \left|\frac{1}{4} - \frac{3}{4}\right| = 0.5.$$

2.3. An IIVIS

Definition 2.6. [32] *Let* U *be a finite set of objects.* A *expresses a finite set of attributes. Then, the ordered pair* (U, A) *is referred to as an IS if* $a \in A$ *is able to determine a function* $a : U \to V_a$, *where* $V_a = \{a(u) : u \in U\}$.

If (U, A) is an IS, given $B \subseteq A$, we can then define

$$ind(B) = \{(u, v) \in U \times U : \forall a \in P, a(u) = a(v)\}.$$

Evidently, $ind(B)$ is an equivalence relation on U, and $ind(B) = \bigcap_{a \in P} ind(\{a\})$.

Denote

$$[u]_B = \{v \in U : (u, v) \in ind(B)\}.$$

Then, $[u]_B$ is known as the equivalence class of the object u under the equivalence relation $ind(B)$.

Definition 2.7. [32] *Let* (U, A) *be an IS. Then,* (U, A) *is known as an incomplete IS if there are* $u \in U$ *and* $a \in A$ *such that* $a(u)$ *is missing.*

We call (U, A) an incomplete IS. Given $B \subseteq A$, then a binary relation on U can be defined as

$$sim(B) = \{(u, v) \in U \times U : \forall a \in P,$$

$$a(u) = a(v) \text{ or } a(u) = * \text{ or } a(v) = *\},$$

Here, $*$ is a missing value.

Evidently, $sim(B)$ is a tolerance relation on U, and $sim(B) = \bigcap_{a \in P} sim(\{a\})$.

For each $a \in A$, denote

$$V_a^* = V_a - \{a(u) : a(u) = *\}.$$

V_a^* means the set of all non-missing information values of the attribute a.

Definition 2.8. [33] *Assuming that* (U, A) *is an IS,* (U, A) *is called an interval-valued IS if for* $\forall a \in A$ *and* $u \in U$, $a(u)$ *is an interval.*

Definition 2.9. [33] *Assuming that* (U, A) *is an IS,* (U, A) *is called an IIVIS if* (U, A) *is both incomplete and interval-valued.*

If $B \subseteq A$, *then* (U, B) *is known as the subsystem of* (U, A).

Example 2.10. *Table 1 depicts an IIVIS* (U, A) *where* $U = \{u_1, u_2, \cdots, u_{10}\}$ *and* $A = \{a_1, a_2, \cdots, a_6\}$.

Table 1 An IIVIS.

	a_1	a_2	a_3	a_4	a_5	a_6
u_1	[2.17,2.86]	[1.78,2.98]	[6.37,10.28]	[3.01,3.84]	[7.24,10.47]	[2.54,3.12]
u_2	*	[1.42,2.09]	[5.32,7.23]	[3.41,5.28]	[7.12,11.26]	[3.24,4.70]
u_3	[2.17,2.86]	[1.78,2.98]	*	[3.06,4.65]	[7.24,10.47]	[2.54,3.12]
u_4	*	[1.78,2.98]	[5.32,7.23]	*	[7.24,10.47]	[2.06,2.79]
u_5	[2.17,2.86]	*	[6.37,10.28]	[3.06,4.65]	*	[2.06,2.79]
u_6	[2.17,2.86]	[1.42,2.09]	[5.32,7.23]	[3.06,4.65]	[7.12,11.26]	[2.54,3.12]
u_7	*	[1.78,2.98]	[5.32,7.23]	[3.01,3.84]	[7.12,11.26]	[3.24,4.70]
u_8	[2.17,2.86]	[1.78,2.98]	*	[3.01,3.84]	[7.24,10.47]	[2.54,3.12]
u_9	[2.17,2.86]	[1.78,2.98]	[6.37,10.28]	[3.01,3.84]	[7.24,10.47]	[3.24,4.70]
u_{10}	[1.83,2.70]	[1.42,2.09]	[6.37,10.28]	[3.06,4.65]	[7.12,11.26]	[2.06,2.79]

IIVIS, incomplete interval-valued information system.

Example 2.11. (*Continued from Example 2.10*)

$$\begin{aligned} V_{a_1}^* &= \{[2.17, 2.86], [1.83, 2.70]\}, \\ V_{a_2}^* &= \{[1.78, 2.98], [1.42, 2.09]\}, \\ V_{a_3}^* &= \{[6.37, 10.28], [5.32, 7.23]\}, \\ V_{a_4}^* &= \{[3.01, 3.84], [3.41, 5.28], [3.06, 4.65]\}, \\ V_{a_5}^* &= \{[7.24, 10.47], [7.12, 11.26]\}, \\ V_{a_6}^* &= V_{a_6} = \{[2.54, 3.12], [3.24, 4.70], [2.06, 2.79]\}. \end{aligned}$$

3. TOLERANCE RELATIONS IN AN IIVIS

In this section, the similarity degree between two information values on a given attribute in an IIVIS is constructed, and the tolerance relation induced by a given subsystem is given.

3.1. The Similarity Degree between two Information Values on a Given Attribute

Why do we consider the similarity degree between two information values? This is because we can introduce the tolerance relation in an IIVIS by the similarity degree. Based on the introduced tolerance relation, we can obtain the tolerance class of each object, and the tolerance class is looked upon as the information granule.

Definition 3.1. *Suppose that* (U, A) *is an IIVIS. Then,* $\forall u, v \in U$, $a \in A$, *the similarity degree between* $a(u)$ *and* $a(v)$ *is defined as follows:*

$$s(a(u), a(v)) = \begin{cases} 1 & u = v; \\ \frac{1}{|V_a^*|^2} & u \neq v, a(u) = *, a(v) = *; \\ \frac{1}{|V_a^*|} & u \neq v, a(u) \neq *, a(v) = *; \\ \frac{1}{|V_a^*|} & u \neq v, a(u) = *, a(v) \neq *; \\ 1 & u \neq v, a(u) \neq *, a(v) \neq *, \\ & a(u) = a(v); \\ v(a(u), a(v)) & u \neq v, a(u) \neq *, a(v) \neq *, \\ & a(u) \neq a(v). \end{cases}$$

For the convenience of expression, denote

$$s_{ij}^k = s\left(a_k(u_i), a_k(u_j)\right).$$

s_{ij}^k indicates the similarity degree between $a_k(u_i)$ and $a_k(u_j)$. This parameter also expresses the similarity degree between two objects u_i and u_j with respect to the attribute a_k.

Example 3.2. (*Continued from Example 2.10*) $\forall i, j, k$, s_{ij}^k *is obtained as follows (see Tables 2–7).*

3.2. Tolerance Relations in an IIVIS

Definition 3.3. *Consider that (U, A) is an IIVIS. Given $\theta \in (0, 1)$ and $B \subseteq A$, then $R_B^\theta \subseteq U \times U$ can be defined as shown below.*

$$R_B^\theta = \{(u, v) \in U \times U : \forall\, a \in B,\ s(a(u), a(v)) \geqslant \theta\}.$$

Obviously, $R_B^\theta \subseteq U \times U$ is a tolerance relation, and $R_B^\theta = \bigcap_{a \in B} R_a^\theta$.

In addition, if $B = \{a\}$, then $R_B^\theta = R_{\{a\}}^\theta$, which can be briefly expressed as $R_B^\theta = R_a^\theta$.

Proposition 3.4. *Let (U, A) be an IIVIS. We can obtain the following properties:*

Table 2 s_{ij}^1.

s_{ij}^1	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	0.68
u_2	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
u_3	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	0.68
u_4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
u_5	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	0.68
u_6	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	0.68
u_7	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
u_8	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	0.68
u_9	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	0.68
u_{10}	0.68	$\frac{1}{2}$	0.68	$\frac{1}{2}$	0.68	0.68	$\frac{1}{2}$	0.68	0.68	1

Table 3 s_{ij}^2.

s_{ij}^2	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	1	0.33	1	1	$\frac{1}{2}$	0.33	1	1	1	0.33
u_2	0.33	1	0.33	0.33	$\frac{1}{2}$	1	0.33	0.33	0.33	1
u_3	1	0.33	1	1	$\frac{1}{2}$	0.33	1	1	1	0.33
u_4	1	0.33	1	1	$\frac{1}{2}$	0.33	1	1	1	0.33
u_5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
u_6	0.33	1	0.33	0.33	$\frac{1}{2}$	1	0.33	0.33	0.33	1
u_7	1	0.33	1	1	$\frac{1}{2}$	0.33	1	1	1	0.33
u_8	1	0.33	1	1	$\frac{1}{2}$	0.33	1	1	1	0.33
u_9	1	0.33	1	1	$\frac{1}{2}$	0.33	1	1	1	0.33
u_{10}	0.33	1	0.33	0.33	$\frac{1}{2}$	1	0.33	0.33	0.33	1

Table 4 s_{ij}^3.

s_{ij}^3	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	1	0.30	$\frac{1}{2}$	0.30	1	0.30	0.30	$\frac{1}{2}$	1	1
u_2	0.30	1	$\frac{1}{2}$	1	0.30	1	1	$\frac{1}{2}$	0.30	0.30
u_3	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
u_4	0.30	1	$\frac{1}{2}$	1	0.30	1	1	$\frac{1}{2}$	0.30	0.30
u_5	1	0.30	$\frac{1}{2}$	0.30	1	0.30	0.30	$\frac{1}{2}$	1	1
u_6	0.30	1	$\frac{1}{2}$	1	0.30	1	1	$\frac{1}{2}$	0.30	0.30
u_7	0.30	1	$\frac{1}{2}$	1	0.30	1	1	$\frac{1}{2}$	0.30	0.30
u_8	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
u_9	1	0.30	$\frac{1}{2}$	0.30	1	0.30	0.30	$\frac{1}{2}$	1	1
u_{10}	1	0.30	$\frac{1}{2}$	0.30	1	0.30	0.30	$\frac{1}{2}$	1	1

Table 5 s_{ij}^4.

s_{ij}^4	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	1	0.32	0.64	$\frac{1}{3}$	0.64	0.64	1	1	1	0.64
u_2	0.32	1	0.72	$\frac{1}{3}$	0.72	0.72	0.32	0.32	0.32	0.72
u_3	0.64	0.72	1	$\frac{1}{3}$	1	1	0.64	0.64	0.64	1
u_4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
u_5	0.64	0.72	1	$\frac{1}{3}$	1	1	0.64	0.64	0.64	1
u_6	0.64	0.72	1	$\frac{1}{3}$	1	1	0.64	0.64	0.64	1
u_7	1	0.32	0.64	$\frac{1}{3}$	0.64	0.64	1	1	1	0.64
u_8	1	0.32	0.64	$\frac{1}{3}$	0.64	0.64	1	1	1	0.64
u_9	1	0.32	0.64	$\frac{1}{3}$	0.64	0.64	1	1	1	0.64
u_{10}	0.64	0.72	1	$\frac{1}{3}$	1	1	0.64	0.64	0.64	1

1. If $B_1 \subseteq B_2 \subseteq A$, then $\forall \theta \in [0, 1]$ and $u \in U$,

$$R_{B_2}^\theta(u) \subseteq R_{B_1}^\theta(u);$$

2. If $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then $\forall\, B \subseteq A$ and $u \in U$,

$$R_B^{\theta_2}(u) \subseteq R_B^{\theta_1}(u).$$

Proof. This proof is clear.

Corollary 3.5. *Assume that (U, A) is an IIVIS. If $C \subseteq B \subseteq A$ and $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then $R_B^{\theta_2} \subseteq R_C^{\theta_1}$.*

Proof. This can be obtained from Proposition 3.4.

Proposition 3.6. *Suppose that (U, A) is an IIVIS. Then, $\forall\, B, C \subseteq A$ and $\theta \in (0, 1]$, $R_B^\theta \cap R_C^\theta = R_{B\,\cup\,C}^\theta$.*

Proof. This result can be obtained from Definition 3.3 and Proposition 3.4.

Table 6 s_{ij}^5.

s_{ij}^5	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	1	0.91	1	1	$\frac{1}{2}$	0.91	0.91	1	1	0.91
u_2	0.91	1	0.91	0.91	$\frac{1}{2}$	1	1	0.91	0.91	1
u_3	1	0.91	1	1	$\frac{1}{2}$	0.91	0.91	1	1	0.91
u_4	1	0.91	1	1	$\frac{1}{2}$	0.91	0.91	1	1	0.91
u_5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
u_6	0.91	1	0.91	0.91	$\frac{1}{2}$	1.	1	0.91	0.91	1
u_7	0.91	1	0.91	0.91	$\frac{1}{2}$	1	1	0.91	0.91	1
u_8	1	0.91	1	1	$\frac{1}{2}$	0.91	0.91	1	1	0.91
u_9	1	0.91	1	1	$\frac{1}{2}$	0.91	0.91	1	1	0.91
u_{10}	0.91	1	0.91	0.91	$\frac{1}{2}$	1	1	0.91	0.91	1

Table 7 s_{ij}^6.

s_{ij}^6	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	1	0	1	0.38	0.38	1	0	1	0	0.38
u_2	0	1	0	0	0	0	1	0	1	0
u_3	1	0	1	0.38	0.38	1	0	1	0	0.38
u_4	0.38	0	0.38	1	1	0.38	0	0.38	0	1
u_5	0.38	0	0.38	1	1	0.38	0	0.38	0	1
u_6	1	0	1	0.38	0.38	1	0	1	0	0.38
u_7	0	1	0	0	0	0	1	0	1	0
u_8	1	0	1	0.38	0.38	1	0	1	0	0.38
u_9	0	1	0	0	0	0	1	0	1	0
u_{10}	0.38	0	0.38	1	1	0.38	0	0.38	0	1

Definition 3.7. *Let (U, A) be an IIVIS. Given $\theta \in (0, 1]$ and $B \subseteq A$, then $\forall\, u \in U$, the tolerance class of u under R_B^θ is defined as*

$$R_B^\theta(u) = \left\{v \in U : (u, v) \in R_B^\theta\right\}.$$

Clearly, $R_B^\theta(u) = \bigcap_{a \in B} R_a^\theta(u)$.

Corollary 3.8. *Suppose that (U, A) is an IIVIS.*

1. *If $C \subseteq B \subseteq A$, then $\forall\, \theta \in (0, 1]$, $R_B^\theta \subseteq R_C^\theta$.*
2. *If $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then $\forall\, B \subseteq A$, $R_B^{\theta_2} \subseteq R_B^{\theta_1}$.*

Proof. This result can be obtained from Proposition 3.4.

Corollary 3.9. *Given that (U, A) is an IIVIS. If $C \subseteq B \subseteq A$, $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$ and $u \in U$, then $R_B^{\theta_2}(u) \subseteq R_C^{\theta_1}(u)$.*

Proof. This result can be obtained by Corollary 3.8.

Corollary 3.10. *Consider that (U, A) is an IIVIS. Then, $\forall\, B, C \subseteq A$, $\theta \in (0, 1]$ and $u \in U$, $R_B^\theta(u) \cap R_C^\theta(u) = R_{B \cup C}^\theta(u)$.*

Proof. This result can be obtained from Proposition 3.6.

Example 3.11. (*Continued from Example* 3.2)

Select $\theta = 0.4$. Then, for any i and $u \in U$, the tolerance class $R_{a_i}^\theta(u)$ of the object u is obtained (see Tables 8 and 9).

Thus, $R_A^\theta(u_1) = \{u_1, u_3, u_8\}$, $R_A^\theta(u_2) = \{u_2\}$,

$R_A^\theta(u_3) = \{u_1, u_3, u_8\}$, $R_A^\theta(u_4) = \{u_4\}$,

$R_A^\theta(u_5) = \{u_5, u_{10}\}$, $R_A^\theta(u_6) = \{u_6\}$,

$R_A^\theta(u_7) = \{u_7\}$, $R_A^\theta(u_8) = \{u_1, u_8\}$,

$R_A^\theta(u_9) = \{u_9\}$, $R_A^\theta(u_{10}) = \{u_5, u_{10}\}$.

An algorithm for computing the tolerance class is designed as follows:

4. INFORMATION STRUCTURES IN AN IIVIS

In this section, information structures in an IIVIS are presented.

4.1. Information Granules and Information Structures

Suppose that (U, A) is an IIVIS. Given $\forall\, B \subseteq A$ and $\theta \in (0, 1]$, a tolerance relation R_B^θ on U is obtained. For $\forall\, i$, $R_B^\theta(u_i)$ can be referred to as the information granule of the point u_i. Then, the concept of an information structure is presented as below.

Table 8 The tolerance class of each object under R_a^θ $(a = a_1, a_2, \theta = 0.4)$.

	$R_{a_1}^\theta(u)$	$R_{a_2}^\theta(u)$	$R_{a_3}^\theta(u)$
u_1	U	$U - \{u_2, u_6, u_{10}\}$	$U - \{u_2, u_4, u_6, u_7\}$
u_2	$U - \{u_4, u_7\}$	$\{u_2, u_5, u_6, u_{10}\}$	$U - \{u_1, u_5, u_9, u_{10}\}$
u_3	U	$U - \{u_2, u_6, u_{10}\}$	$U - \{u_8\}$
u_4	$U - \{u_2, u_7\}$	$U - \{u_2, u_6, u_{10}\}$	$U - \{u_1, u_5, u_9, u_{10}\}$
u_5	U	U	$U - \{u_2, u_4, u_6, u_7\}$
u_6	U	$\{u_2, u_5, u_6, u_{10}\}$	$U - \{u_1, u_5, u_9, u_{10}\}$
u_7	$U - \{u_2, u_4\}$	$U - \{u_2, u_6, u_{10}\}$	$U - \{u_1, u_5, u_9, u_{10}\}$
u_8	U	$U - \{u_2, u_6, u_{10}\}$	$U - \{u_3\}$
u_9	U	$U - \{u_2, u_6, u_{10}\}$	$U - \{u_2, u_4, u_6, u_7\}$
u_{10}	U	$\{u_2, u_5, u_6, u_{10}\}$	$U - \{u_2, u_4, u_6, u_7\}$

Table 9 The tolerance class of each object under R_a^θ $(a = a_4, a_5, a_6, \theta = 0.4)$.

	$R_{a_4}^\theta(u)$	$R_{a_5}^\theta(u)$	$R_{a_6}^\theta(u)$
u_1	$U - \{u_2, u_4\}$	U	$\{u_1, u_3, u_6, u_8\}$
u_2	$\{u_2, u_3, u_5, u_6, u_{10}\}$	U	$\{u_2, u_7, u_9\}$
u_3	$U - \{u_4\}$	U	$\{u_1, u_3, u_6, u_8\}$
u_4	$\{u_4\}$	U	$\{u_4, u_5, u_{10}\}$
u_5	$U - \{u_4\}$	U	$\{u_4, u_5, u_{10}\}$
u_6	$U - \{u_4\}$	U	$\{u_1, u_3, u_6, u_8\}$
u_7	$U - \{u_2, u_4\}$	U	$\{u_2, u_7, u_9\}$
u_8	$U - \{u_2, u_4\}$	U	$\{u_1, u_3, u_6, u_8\}$
u_9	$U - \{u_2, u_4\}$	U	$\{u_2, u_7, u_9\}$
u_{10}	$U - \{u_4\}$	U	$\{u_4, u_5, u_{10}\}$

Algorithm 1: : Computing $R_B^\theta(u)$.

Input: An IIVIS (U, A), a threshold $\theta \in [0,1]$, $B \subseteq A$ and $u \in U$.
Output: The tolerance class $R_B^\theta(u)$.

for $v \in U$; **do**
  for $v = u$, **do**
    $s(a(u), a(v)) = 1$.
  end
  for $v \neq u$ **do**
    if $a(u) = *$ *and* $a(v) = *$, **then**
      $s(a(u), a(v)) = \frac{1}{|V_a^*|^2}$;
    end
    if $a(u) = *$ *and* $a(v) \neq *$, **then**
      $s(a(u), a(v)) = \frac{1}{|V_a^*|}$;
    end
    if $a(u) \neq *$ *and* $a(v) = *$, **then**
      $s(a(u), a(v)) = \frac{1}{|V_a^*|}$;
    end
    if $a(u) \neq *$ *and* $a(v) \neq *$ *and* $a(u) = a(v)$, **then**
      $s(a(u), a(v)) = 1$;
    end
    if $a(u) \neq *$ *and* $a(v) \neq *$ *and* $a(u) \neq a(v)$, **then**
      $s(a(u), a(v)) = v(a(u), a(v))$.
    end
  end
end
Select a value for θ. Let $R_B^\theta = \{(u,v) \in U \times U : \forall\, a \in P,\ s(a(u), a(v)) \geqslant \theta\}$, $R_B^\theta(u) = \{v \in U : (u,v) \in R_B^\theta\}$.
Obtain $R_B^\theta(u)$.

Definition 4.1. *Let (U, A) be an IIVIS. Given $\theta \in [0,1]$ and $B \subseteq A$. Then,*

$$S^\theta(B) = \left(R_B^\theta(u_1), R_B^\theta(u_2), \cdots\cdots, R_B^\theta(u_n)\right)$$

Then, $S^\theta(B)$ is said to be the information structure of (U, B) in relation to θ.

Definition 4.2. *Let (U, A) be an IIVIS. Let $\theta_1, \theta_2 \in (0,1]$ and $B, C \subseteq A$. If $\forall$ i, $R_B^{\theta_1}(u_i) = R_C^{\theta_2}(u_i)$, then $S^{\theta_1}(B)$ and $S^{\theta_2}(C)$ are deemed to be the same. This expression can be written as $S^{\theta_1}(B) = S^{\theta_2}(C)$.*

Definition 4.3. *Let (U, A) be an IIVIS. Given $\theta \in [0,1]$, let*

$$\mathbf{S}^\theta(U, A) = \left\{S^\theta(B) : B \subseteq A\right\}.$$

Then, $\mathbf{S}^\theta(U, A)$ is said to be the θ-information structure base of (U, A).

4.2. Dependence between Information Structures

Definition 4.4. *Given that (U,A) is an IIVIS, suppose that $\theta_1, \theta_2 \in (0,1]$ and $B, C \subseteq A$.*

1. *$S^{\theta_2}(C)$ is referred to as depending on $S^{\theta_1}(B)$ if for each i, $R_B^{\theta_1}(u_i) \subseteq R_C^{\theta_2}(u_i)$; we can write $S^{\theta_1}(B) \preceq S^{\theta_2}(C)$. $S^{\theta_2}(C)$ is known to depend strictly on $S^{\theta_1}(B)$ if $S^{\theta_1}(B) \preceq S^{\theta_2}(C)$ and $S^{\theta_1}(B) \neq S^{\theta_2}(C)$; we can write $S^{\theta_1}(B) \prec S^{\theta_2}(C)$.*
2. *$S^{\theta_2}(C)$ is referred to as partially dependent on $S^{\theta_1}(B)$ if there exists i such that $R_B^{\theta_1}(u_i) \subseteq R_C^{\theta_2}(u_i)$; we can write $S^{\theta_1}(B) \sqsubseteq S^{\theta_2}(C)$. $S^{\theta_2}(C)$ is known as to depend strictly on $S^{\theta_1}(B)$ if $S^{\theta_1}(B) \sqsubseteq S^{\theta_2}(C)$ and $S^{\theta_1}(B) \neq S^{\theta_2}(C)$; we can write $S^{\theta_1}(B) \sqsubset S^{\theta_2}(C)$.*
3. *$S^{\theta_2}(C)$ is referred to as independent of $S^{\theta_1}(B)$ if for each i, $R_B^{\theta_1}(u_i) \nsubseteq R_C^{\theta_2}(u_i)$; we can write $S^{\theta_1}(B) \bowtie S^{\theta_2}(C)$.*

The following conclusions can be obtained clearly.

$$\begin{aligned} S^{\theta_1}(B) = S^{\theta_2}(C) &\Leftrightarrow S^{\theta_1}(B) \preceq S^{\theta_2}(C) \text{ and } S^{\theta_2}(C) \preceq S^{\theta_1}(B), \\ S^{\theta_1}(B) \preceq S^{\theta_2}(C) &\Rightarrow S^{\theta_1}(B) \sqsubseteq S^{\theta_2}(C), \\ S^{\theta_1}(B) \prec S^{\theta_2}(C) &\Rightarrow S^{\theta_1}(B) \sqsubset S^{\theta_2}(C). \end{aligned}$$

4.3. Information Distance between Information Structures

The information distance between information structures is presented below.

For $P, Q \in 2^U$,, denote

$$P \oplus Q = P \cup Q - P \cap Q.$$

Thus, $P \oplus Q$ is referred to as the symmetric difference between P and Q.

Clearly, $|P \oplus Q| = |P \cup Q| - |P \cap Q|$.

Theorem 4.5. *[18] Suppose $P, Q \in 2^U$. Then*

$$P = Q \Leftrightarrow |P \oplus Q| = 0.$$

Theorem 4.6. *[18] Assume $P, Q, L \in 2^U$. Then*

$$|P \oplus Q| + |Q \oplus L| \geqslant |P \oplus L|.$$

Theorem 4.7. *[18] Suppose $P, Q, L \in 2^U$. If $P \subseteq Q \subseteq L$ or $L \subseteq Q \subseteq P$, then*

$$|P \oplus Q| + |Q \oplus L| = |P \oplus L|.$$

Definition 4.8. *Let (U, A) be an IIVIS. Suppose that $B, C \subseteq A$, as well as $\theta \in [0,1]$. Then, the information distance between $S^\theta(B)$ and $S^\theta(C)$ is defined as*

$$\rho\left(S^\theta(B), S^\theta(C)\right) = \frac{1}{n^2} \sum_{i=1}^{n} |R_B^\theta(u_i) \oplus R_C^\theta(u_i)|.$$

Theorem 4.9. *Assuming that (U, A) is an IIVIS and $\theta \in [0,1]$, then $\left(S^\theta(U, A), \rho\right)$ is a distance space.*

Proof. Suppose that $B, C, D \subseteq A$ and $\theta \in [0,1]$. By Definition 4.8 ,

$$\rho\left(S^\theta(B), S^\theta(C)\right) \geqslant 0, \rho\left(S^\theta(B), S^\theta(C)\right) = \rho\left(S^\theta(C), S^\theta(B)\right)$$

By Theorem 4.14,

$$\rho\left(S^\theta(B), S^\theta(C)\right) = 0 \Leftrightarrow \forall i, |R_B^\theta(u_i) \oplus R_C^\theta(u_i)| = 0 \\ \Leftrightarrow \forall i, R_B^\theta(u_i) = R_C^\theta(u_i) \Leftrightarrow S^\theta(B) = S^\theta(C)$$

By Theorem 4.6,

$$|R_B^\theta(u_i) \oplus R_C^\theta(u_i)| + |R_C^\theta(u_i) \oplus R_D^\theta(u_i)| \geqslant |R_B^\theta(u_i) \oplus R_D^\theta(u_i)|.$$

Then,

$$\rho\left(S^\theta(B), S^\theta(C)\right) + \rho\left(S^\theta(C), S^\theta(D)\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}|R_B^\theta(u_i) \oplus R_C^\theta(u_i)| + \frac{1}{n^2}\sum_{i=1}^{n}|R_C^\theta(u_i) \oplus R_D^\theta(u_i)|$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_B^\theta(u_i) \oplus R_C^\theta(u_i)| + |R_C^\theta(u_i) \oplus R_D^\theta(u_i)|\right)$$
$$\geqslant \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_B^\theta(u_i) \oplus R_D^\theta(u_i)|\right) = \rho\left(S^\theta(B), S^\theta(D)\right).$$

Evidence must be obtained.

Proposition 4.10. *Suppose that (U, A) is an IIVIS and $\theta \in [0,1]$. Then, $\forall B, C \subseteq A$,*

1. $0 \leqslant \rho\left(S^\theta(B), S^\theta(C)\right) \leqslant 1 - \frac{1}{n}$;
2. *If $S^\theta(B) \preceq S^\theta(C)$ and R_D^θ is an identify relation on U, then* $\rho\left(S^\theta(B), S^\theta(D)\right) \leqslant \rho\left(S^\theta(C), S^\theta(D)\right)$;
3. *If $S^\theta(B) \preceq S^\theta(C)$, then* $\rho\left(S^\theta(B), S^\theta(\varnothing)\right) \geqslant \rho\left(S^\theta(C), S^\theta(\varnothing)\right)$.

Proof. (1) It is obvious that $1 \leqslant |R_B^\theta(u_i) \cup R_C^\theta(u_i)| \leqslant n$ and $1 \leqslant |R_B^\theta(u_i) \cap R_C^\theta(u_i)| \leqslant n\ (i = 1, 2, \cdots, n)$. Then,

$$0 \leqslant |R_B^\theta(u_i) \oplus R_C^\theta(u_i)| \leqslant n - 1\ (i = 1, 2, \cdots, n).$$

By Definition 4.8,

$$0 \leqslant \rho\left(S^\theta(B), S^\theta(C)\right) \leqslant \frac{1}{n^2}\sum_{i=1}^{n}(n-1) = 1 - \frac{1}{n}.$$

(2) Because of $S^\theta(B) \preceq S^\theta(C)$, $\forall\ i$, $R_B^\theta(u_i) \subseteq R_C^\theta(u_i)$. By Definition 4.8,

$$\rho\left(S^\theta(B), S^\theta(D)\right) = \frac{1}{n^2}\sum_{i=1}^{n}|R_B^\theta(u_i) \oplus \{u_i\}|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_B^\theta(u_i) \cup \{u_i\}| - |R_B^\theta(u_i) \cap \{u_i\}|\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_B^\theta(u_i)| - 1\right) \leqslant \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_C^\theta(u_i)| - 1\right) \\ = \rho\left(S^\theta(C), S^\theta(D)\right)$$

(3) On account of $S^\theta(B) \preceq S^\theta(C)$, $\forall\ i$, $R_B^\theta(u_i) \subseteq R_C^\theta(u_i)$. By Definition 4.8,

$$\rho\left(S^\theta(B), S^\theta(\varnothing)\right) = \frac{1}{n^2}\sum_{i=1}^{n}|R_B^\theta(u_i) \oplus U|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_B^\theta(u_i) \cup U| - |R_B^\theta(u_i) \cap U|\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(n - |R_B^\theta(u_i)|\right) \geqslant \frac{1}{n^2}\sum_{i=1}^{n}\left(n - |R_C^\theta(u_i)|\right) \\ = \rho\left(S^\theta(C), S^\theta(\varnothing)\right)$$

Proposition 4.11. *Assuming that (U, A) is an IIVIS and $\theta \in [0,1]$, if R_D^θ is an identify relation on U, then $\forall\ B \subseteq A$,*

$$\rho\left(S^\theta(B), S^\theta(D)\right) + \rho\left(S^\theta(B), S^\theta(\varnothing)\right) = 1 - \frac{1}{n}.$$

Proof. By Definition 4.8, the following is apparent:

$$\rho\left(S^\theta(B), S^\theta(D)\right) + \rho\left(S^\theta(B), S^\theta(\varnothing)\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}|R_B^\theta(u_i) \oplus \{u_i\}| + \frac{1}{n^2}\sum_{i=1}^{n}|R_B^\theta(u_i) \oplus U|$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_B^\theta(u_i)| - 1\right) + \frac{1}{n^2}\sum_{i=1}^{n}\left(n - |R_B^\theta(u_i)|\right)$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}(n-1) = 1 - \frac{1}{n}.$$

Proposition 4.12. *Suppose that (U, A) is an IIVIS and $\theta \in [0,1]$. Then, $\forall\ B, C, D \subseteq A$. If $S^\theta(B) \preceq S^\theta(C) \preceq S^\theta(D)$ or $S^\theta(D) \preceq S^\theta(C) \preceq S^\theta(B)$, then*

$$\rho\left(S^\theta(B), S^\theta(C)\right) + \rho\left(S^\theta(C), S^\theta(D)\right) = \rho\left(S^\theta(B), S^\theta(D)\right).$$

Proof. Owing to $S^\theta(B) \preceq S^\theta(C) \preceq S^\theta(D)$ or $S^\theta(D) \preceq S^\theta(C) \preceq S^\theta(B)$, it can be obtained that $R_B^\theta(u_i) \subseteq R_C^\theta(u_i) \subseteq R_D^\theta(u_i)$ or $R_D^\theta(u_i) \subseteq R_C^\theta(u_i) \subseteq R_B^\theta(u_i)\ (i = 1, 2, \cdots, n)$.

By Theorem 4.17,

$$|R_B^\theta(u_i) \oplus R_C^\theta(u_i)| + |R_C^\theta(u_i) \oplus R_D^\theta(u_i)|$$

$$= |R_B^\theta(u_i) \oplus R_D^\theta(u_i)|\ (i = 1, 2, \cdots, n).$$

By Definition 4.8,

$$\rho\left(S^\theta(B), S^\theta(C)\right) + \rho\left(S^\theta(C), S^\theta(D)\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}|R_B^\theta(u_i) \oplus R_C^\theta(u_i)| + \frac{1}{n^2}\sum_{i=1}^{n}|R_C^\theta(u_i) \oplus R_D^\theta(u_i)|$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(|R_B^\theta(u_i) \oplus R_C^\theta(u_i)| + |R_C^\theta(u_i) \oplus R_D^\theta(u_i)|\right)$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}|R_B^\theta(u_i) \oplus R_D^\theta(u_i)| = \rho\left(S^\theta(B), S^\theta(D)\right)$$

4.4. Properties of Information Structures

Properties of information structures in an IIVIS are displayed below.

Theorem 4.13. *Assuming that (U, A) is an IIVIS, given $\theta_1, \theta_2 \in [0,1]$ and $B, C \subseteq A$, then*

$$S^{\theta_1}(B) = S^{\theta_2}(C) \Leftrightarrow R_B^{\theta_1} = R_C^{\theta_2}.$$

Proof. The proof is obvious.

Theorem 4.14. *Suppose that (U, A) is an IIVIS. Given $\theta_1, \theta_2 \in [0,1]$ and $B, C \subseteq A$. Then,*

$$S^{\theta_1}(B) \preceq S^{\theta_2}(C) \Leftrightarrow R_B^{\theta_1} \subseteq R_C^{\theta_2}.$$

Proof. Obviously.

Corollary 4.15. *Consider that (U, A) is an IIVIS. Given $\theta_1, \theta_2 \in [0,1]$ and $B, C \subseteq A$, then*

$$S^{\theta_1}(B) \prec S^{\theta_2}(C) \Leftrightarrow R_B^{\theta_1} \subset R_C^{\theta_2}.$$

Proof. This result can be obtained by Theorem 4.13 and Theorem 4.14.

Theorem 4.16. *Assuming that (U, A) is an IIVIS, then $\forall\, B, C \subseteq A$ and $\theta \in [0,1]$, these equations below are equivalent:*

1. $S^{\theta}(B) = S^{\theta}(C)$;
2. $R_B^{\theta}(u) = R_C^{\theta}(u)$;
3. $R_B^{\theta} = R_C^{\theta}$;
4. $\rho\left(S^{\theta}(B), S^{\theta}(C)\right) = 0$.

Proof. (1) $\Leftrightarrow$ (2). The proof is clear.

(1) $\Leftrightarrow$ (3). The result can be obtained from Theorem 4.14.

(2) $\Leftrightarrow$ (4). The proof is obvious.

Theorem 4.17. *Let (U, A) be an IIVIS.*

1. *If $B \subseteq C \subseteq A$, then $\forall\, \theta \in [0,1]$, $S^{\theta}(C) \preceq S^{\theta}(B)$.*
2. *If $0 < \theta_1 \leqslant \theta_2 \leqslant 1$, then $\forall\, B \subseteq A$, $S^{\theta_2}(B) \preceq S^{\theta_1}(B)$.*

Proof. This result can be obtained from Proposition 3.4 and Theorem 4.14.

Corollary 4.18. *Suppose that (U, A) is an IIVIS. Given $\theta_1, \theta_2 \in [0,1]$ and $B, C \subseteq A$, if $B \subseteq C$, $\theta_1 \leqslant \theta_2$, then*

$$S^{\theta_2}(C) \preceq S^{\theta_1}(C) \preceq S^{\theta_1}(B), S^{\theta_2}(C) \preceq S^{\theta_2}(B) \preceq S^{\theta_1}(B).$$

Proof. This result can be obtained from Theorem 4.17.

Definition 4.19. [34] *Let (U, A) be an IIVIS. Assuming that a mapping $D : \mathbf{S}^{\theta}(U, A) \times \mathbf{S}^{\theta}(U, A) \to [0,1]$ is said to be the inclusion degree on $\mathbf{S}^{\theta}(U, A)$, if $\forall\, B, C, D \subseteq A$.*

1. $0 \leqslant D\left(S^{\theta}(C)/S^{\theta}(B)\right) \leqslant 1$;
2. $S^{\theta}(B) \preceq S^{\theta}(C)$ means $D\left(S^{\theta}(C)/S^{\theta}(B)\right) = 1$;
3. $S^{\theta}(B) \sqsubseteq S^{\theta}(C) \sqsubseteq S^{\theta}(D)$ *means* $D\left(S^{\theta}(B)/S^{\theta}(D)\right) \leqslant D\left(S^{\theta}(B)/S^{\theta}(C)\right)$.

Definition 4.20. *Assuming that (U, A) is an IIVIS, then $\forall\, B, C \subseteq A$ and $\theta \in [0,1]$, define*

$$D\left(S^{\theta}(C)/S^{\theta}(B)\right) = \sum_{l=1}^{n} \frac{|R_C^{\theta}(u_l)|}{\sum_{i=1}^{n} |R_C^{\theta}(u_i)|} \chi_{R_C^{\theta}(u_l)}\left(R_B^{\theta}(u_l)\right),$$

where

$$\chi_{R_C^{\theta}(u_l)}\left(R_B^{\theta}(u_l)\right) = \begin{cases} 1, & \text{if } R_B^{\theta}(u_l) \subseteq R_C^{\theta}(u_l), \\ 0, & \text{if } R_B^{\theta}(u_l) \nsubseteq R_C^{\theta}(u_l). \end{cases}$$

Proposition 4.21. *D in Definition 4.20 is the inclusion degree under Definition 4.19.*

Proof. Let $\theta \in [0,1]$ and $B, C, D \subseteq A$.

1. Obviously, $0 \leqslant D\left(S^{\theta}(C)/S^{\theta}(B)\right) \leqslant 1$.
2. Suppose $S^{\theta}(B) \preceq S^{\theta}(C)$. Then, by Theorem 4.14, $R_B^{\theta} \subseteq R_C^{\theta}$. Thus, for each l, $R_B^{\theta}(u_l) \subseteq R_C^{\theta}(u_l)$. This result implies that

   $$\text{for each } l, \chi_{R_C^{\theta}(u_l)}\left(R_B^{\theta}(u_l)\right) = 1.$$

   Thus, $D\left(S^{\theta}(C)/S^{\theta}(B)\right) = 1$.
3. Suppose $S^{\theta}(B) \preceq S^{\theta}(C) \preceq S^{\theta}(D)$. Then, by Theorem 4.14, $R_B^{\theta} \subseteq R_C^{\theta} \subseteq R_D^{\theta}$. Thus, for each l, $R_B^{\theta}(u_l) \subseteq R_C^{\theta}(u_l) \subseteq R_D^{\theta}(u_l)$.

By Definition 4.20,

$$\begin{aligned} D\left(S^{\theta}(B)/S^{\theta}(D)\right) &= \sum_{l=1}^{n} \frac{|R_B^{\theta}(u_l)|}{\sum_{i=1}^{n} |R_B^{\theta}(u_i)|} \chi_{R_B^{\theta}(u_l)}\left(R_D^{\theta}(u_l)\right), \\ D\left(S^{\theta}(B)/S^{\theta}(C)\right) &= \sum_{l=1}^{n} \frac{|R_B^{\theta}(u_l)|}{\sum_{i=1}^{n} |R_B^{\theta}(u_i)|} \chi_{R_B^{\theta}(u_l)}\left(R_C^{\theta}(u_l)\right). \end{aligned}$$

If $R_C^{\theta}(u_l) \nsubseteq R_B^{\theta}(u_l)$, then $R_D^{\theta}(u_l) \nsubseteq R_B^{\theta}(u_l)$. This result illustrates that

$$\chi_{R_B^{\theta}(u_l)}\left(R_C^{\theta}(u_l)\right) = 0 \text{ implies } \chi_{R_B^{\theta}(u_l)}\left(R_D^{\theta}(u_l)\right) = 0.$$

Thus,

$$D\left(S^{\theta}(B)/S^{\theta}(D)\right) \leqslant D\left(S^{\theta}(B)/S^{\theta}(C)\right).$$

From the above, we know that D is the inclusion degree.

Example 4.22. (*Continued from Example 3.11*). *Suppose that (U, A) is an IIVIS. Let $B = \{a_1\}, C = \{a_5\}$ and $\theta = 0.4$. Then,*

$$
\begin{aligned}
D(K(B)/K(C)) = {} & \frac{|R_B(u_1)|}{\sum_{i=1}^{10}|R_B(u_i)|}\chi_{R_B(u_1)}(R_C(u_1)) \\
& + \frac{|R_C(u_2)|}{\sum_{i=1}^{10}|R_B(u_i)|}\chi_{R_B(u_2)}(R_C(u_2)) + \cdots \\
& + \frac{|R_B(u_{10})|}{\sum_{i=1}^{10}|R_B(u_i)|}\chi_{R_B(u_1)}(R_C(u_{10})) \\
= {} & \frac{35}{47},
\end{aligned}
$$

$$
\begin{aligned}
D(K(C)/K(B)) = {} & \frac{|R_C(u_1)|}{\sum_{i=1}^{10}|R_C(u_i)|}\chi_{R_C(u_1)}(R_B(u_1)) \\
& + \frac{|R_C(u_2)|}{\sum_{i=1}^{10}|R_C(u_i)|}\chi_{R_C(u_2)}(R_B(u_2)) \\
& + \frac{|R_C(u_3)|}{\sum_{i=1}^{10}|R_C(u_i)|}\chi_{R_C(u_3)}(R_B(u_3)) \\
& + \frac{|R_C(u_{10})|}{\sum_{i=1}^{10}|R_C(u_i)|}\chi_{R_C(u_{10})}(R_B(u_{10})) \\
= {} & \frac{|R_C(u_1)|}{\sum_{i=1}^{10}|R_C(u_i)|} + \frac{|R_C(u_2)|}{\sum_{i=1}^{10}|R_C(u_i)|} \\
& + \frac{|R_C(u_3)|}{\sum_{i=1}^{10}|R_C(u_i)|} + \frac{|R_C(u_{10})|}{\sum_{i=1}^{10}|R_C(u_i)|} \\
= {} & 1
\end{aligned}
$$

Consequently,

$$D(K(B)/K(C)) + D(K(C)/K(B)) \neq 1.$$

It can be obtained that the inclusion degree has the ability to quantify relationships by the theorem below.

Theorem 4.23. *Assuming that* (U, A) *is an IIVIS, then* $\forall\, B, C \subseteq A$ *and* $\theta \in [0,1]$.

1. $S^\theta(B) \preceq S^\theta(C) \Leftrightarrow D\left(S^\theta(C)/S^\theta(B)\right) = 1.$
2. $S^\theta(B) \bowtie S^\theta(C) \Leftrightarrow D\left(S^\theta(C)/S^\theta(B)\right) = 0.$
3. $S^\theta(B) \sqsubseteq S^\theta(C) \Leftrightarrow 0 < D\left(S^\theta(C)/S^\theta(B)\right) \leqslant 1.$

Proof. (1) "⇒" is evident. We prove "⇐." Suppose

$$|R_C^\theta(u_l)| = q_l, \quad \sum_{l=1}^{n}|R_C^\theta(u_l)| = q.$$

Then,

$$q = \sum_{l=1}^{n} q_l$$

Owing to $D\left(S^\theta(C)/S^\theta(B)\right) = 1$, it can be obtained that $\sum_{l=1}^{n} q_l\chi_{R_C^\theta(u_l)}\left(R_B^\theta(u_l)\right) = \sum_{l=1}^{n} q_l = q.$

Then,

$$q\left(1 - \chi_{R_C^\theta(u_l)}\left(R_B^\theta(u_l)\right)\right) = 0.$$

Consequently, $\forall l$,

$$1 - \chi_{R_C^\theta(u_l)}\left(R_B^\theta(u_l)\right) = 0.$$

Thus, it can be obtained that $\forall l$, $R_B^\theta(u_l) \subseteq R_C^\theta(u_l)$.

By Proposition 3.4 and Theorem 4.14, $S^\theta(B) \preceq S^\theta(C)$.

(2) "⇒." Owing to $S^\theta(B) \bowtie S^\theta(C)$, it can be obtained that $R_B^\theta(u_l) \nsubseteq R_C^\theta(u_l)$ $(\forall l)$. Then $\forall l$,

$$\chi_{R_C^\theta(u_l)}\left(R_B^\theta(u_l)\right) = 0.$$

By Definition 4.20, $D\left(S^\theta(C)/S^\theta(B)\right) = 0$.

"⇐." Owing to $D\left(S^\theta(C)/S^\theta(B)\right) = 0$, it can be obtained that $\forall l$, $\chi_{R_C^\theta(u_l)}\left(R_B^\theta(u_l)\right) = 0$.

Then, $\forall l$, $R_B^\theta(u_l) \nsubseteq R_C^\theta(u_l)$. By Definition 4.4, $S^\theta(B) \bowtie S^\theta(C)$.

(3) The result can be obtained from (1) and (2).

5. CHARACTERIZATIONS OF INFORMATION STRUCTURES IN AN IIVIS

This part will present group and lattice characterizations of information structures in an IIVIS.

5.1. Group Characterizations of Information Structures

Definition 5.1. [35] *Assume that S is a nonempty set and "·" a binary operation on S.*

1. *$(S,\cdot)$ is said to be a semigroup if $\forall\, p, q \in S$, $p\cdot q \in S$ and $\forall\, p, q, l \in S$, $(p\cdot q)\cdot l = p\cdot(q\cdot l)$.*
2. *$(S,\cdot)$ is said to be an exchangeable semigroup if it is a semigroup and if $\forall\, p, q \in S$, $p\cdot q = q\cdot p$.*
3. *$i \in S$ is said to be the identity element of S if $\forall\, p \in S$, $i\cdot p = p\cdot i = p$.*
4. *$(S,\cdot)$ is said to be a group if it is a semigroup and if every element has an inverse element.*

Suppose

$$
\begin{aligned}
S^\theta(B) &= \left(R_B^\theta(u_1), R_B^\theta(u_2), \cdots, R_B^\theta(u_n)\right),\\
S^\theta(C) &= \left(R_B^\theta(u_1), R_B^\theta(u_2), \cdots, R_B^\theta(u_n)\right).
\end{aligned}
$$

Define $S^\theta(B) \odot S^\theta(C)$

$$= \left(R_B^\theta(u_1) \cap R_C^\theta(u_1), R_B^\theta(u_2) \cap R_C^\theta(u_2), \cdots, R_B^\theta(u_n) \cap R_C^\theta(u_n)\right)$$

Theorem 5.2. *$\left(S^{\theta}(U,A),\odot\right)$ is an exchangeable semigroup with the identity element $S^{\theta}(\varnothing)$.*

Proof. Assume $B, C, D \subseteq A$. By Corollary 3.10, we have

$$S^{\theta}(B)\odot S^{\theta}(C)$$

$$\begin{aligned}&=\left(R_B^{\theta}(u_1)\cap R_C^{\theta}(u_1), R_B^{\theta}(u_2)\cap R_C^{\theta}(u_2),\cdots,R_B^{\theta}(u_n)\cap R_C^{\theta}(u_n)\right)\\&=\left(R^{\theta}_{B\cup C}(u_1), R^{\theta}_{B\cup C}(u_2),\cdots,R^{\theta}_{B\cup C}(u_n)\right)\\&=S^{\theta}(B\cup C).\end{aligned}$$

Similarly, $S^{\theta}(C)\odot S^{\theta}(B)=S^{\theta}(C\cup B)$.

Thus,

$$S^{\theta}(B)\odot S^{\theta}(C)=S^{\theta}(C)\odot S^{\theta}(B).$$

Owing to $\left(S^{\theta}(B)\odot S^{\theta}(C)\right)\odot S^{\theta}(D) = S^{\theta}(B\cup C)\odot S^{\theta}(D) = S^{\theta}(B\cup C\cup D)$,

$$\begin{aligned}S^{\theta}(B)\odot\left(S^{\theta}(C)\odot S^{\theta}(D)\right)&=S^{\theta}(B)\odot S^{\theta}(C\cup D)\\&=S^{\theta}(B\cup C\cup D),\end{aligned}$$

it can be obtained that

$$\left(S^{\theta}(B)\odot S^{\theta}(C)\right)\odot S^{\theta}(D)=S^{\theta}(B)\odot\left(S^{\theta}(C)\odot S^{\theta}(D)\right).$$

By Definition 5.1, $\left(S^{\theta}(U),\odot\right)$ is an exchangeable semigroup.

Evidently, $S^{\theta}(\varnothing)$ is the identity element.

5.2. Lattice Characterizations of Information Structures

Theorem 5.3. *Assuming that (U,A) is an IIVIS, then*

1. *$L=\left(S^{\theta}(U,A),\preceq\right)$ is a lattice with $1_L=S^{\theta}(\varnothing)$, $0_L=S^{\theta}(A)$;*
2. *If $B_1, B_2\subseteq A$ and $\mathcal{A}(B_1,B_2)=\left\{B: B\subseteq A, R^{\theta}_{B_1}\cup R^{\theta}_{B_2}\subseteq R^{\theta}_B\right\}$, then*

$$S^{\theta}(B_1)\wedge S^{\theta}(B_2)=S^{\theta}(B_1)\odot S^{\theta}(B_2)=S^{\theta}(B_1\cup B_2),$$

$$S^{\theta}(B_1)\vee S^{\theta}(B_2)=\bigodot_{B\in\mathcal{A}(B_1,B_2)}S^{\theta}(B)=S^{\theta}(\cup\mathcal{A}(B_1,B_2)).$$

Proof. Clearly, $\forall B\subseteq A$, $S^{\theta}(B)\preceq S^{\theta}(B)$.

Assuming that $S^{\theta}(B)\preceq S^{\theta}(C)$ and $S^{\theta}(C)\preceq S^{\theta}(B)$, by Theorem 4.14, $R^{\theta}_B\subseteq R^{\theta}_C$, $R^{\theta}_C\subseteq R^{\theta}_B$. Then, $R^{\theta}_B=R^{\theta}_C$. Consequently, $S^{\theta}(B)=S^{\theta}(C)$.

Assuming that $S^{\theta}(B)\preceq S^{\theta}(C)$ and $S^{\theta}(C)\preceq S^{\theta}(D)$, by Theorem 4.14, $R^{\theta}_B\subseteq R^{\theta}_C$, $R^{\theta}_C\subseteq R^{\theta}_D$. Then, $R^{\theta}_B\subseteq R^{\theta}_D$). By Theorem 4.14, $S^{\theta}(B)\preceq S^{\theta}(D)$.

Accordingly, $\left(S^{\theta}(U,A),\preceq\right)$ is a partial order set.

By proving Theorem 5.2, it can be obtained that

$$S^{\theta}(B_1)\odot S^{\theta}(B_2)=S^{\theta}(B_1\cup B_2),$$

$$\bigodot_{B\in\mathcal{A}(B_1,B_2)}S^{\theta}(B)=S^{\theta}(\cup\mathcal{A}(B_1,B_2)).$$

Owing to $B_1, B_2\subseteq B_1\cup B_2$, by Proposition 4.17, it can be obtained that $S^{\theta}(B_1\cup B_2)\preceq S^{\theta}(B_1)$ *and* $S^{\theta}(B_1\cup B_2)\preceq S^{\theta}(B_2)$. Accordingly, $S^{\theta}(B_1\cup B_2)$ is the lower bound of $\left\{S^{\theta}(B_1), S^{\theta}(B_2)\right\}$.

Assuming that $S^{\theta}(B)$ is the lower bound of $\left\{S^{\theta}(B_1), S^{\theta}(B_2)\right\}$ with $B\subseteq A$, *then* $S^{\theta}(B)\preceq S^{\theta}(B_1)$ and $S^{\theta}(B)\preceq S^{\theta}(B_2)$. By Theorem 4.14, it can be obtained that $R^{\theta}_B\subseteq R^{\theta}_{B_1}$ and $R^{\theta}_B\subseteq R^{\theta}_{B_2}$. Then, $R^{\theta}_B\subseteq R^{\theta}_{B_1}\cap R^{\theta}_{B_2}=R^{\theta}_{B_1\cup B_2}$. By Theorem 4.14, $S^{\theta}(B)\preceq S^{\theta}(B_1\cup B_2)$.

Thus,

$$S^{\theta}(B_1)\wedge S^{\theta}(B_2)=S^{\theta}(B_1\cup B_2).$$

$\forall B\in\mathcal{A}(B_1,B_2)$, owing to $R^{\theta}_{B_1}\cup R^{\theta}_{B_2}\subseteq R^{\theta}_B$, it can be obtained that $R^{\theta}_{B_1}, R^{\theta}_{B_2}\subseteq R^{\theta}_B$. Then,

$$R^{\theta}_{B_1}, R^{\theta}_{B_2}\subseteq\bigcap_{B\in\mathcal{A}(B_1,B_2)}R^{\theta}_B=R^{\theta}_{\bigcup_{B\in\mathcal{A}(B_1,B_2)}B}=R^{\theta}_{\cup\mathcal{A}(B_1,B_2)}.$$

By Theorem 4.14, $S^{\theta}(B_1), S^{\theta}(B_2)\preceq S^{\theta}(\cup\mathcal{A}(B_1,B_2))$. Then, $S^{\theta}(\cup\mathcal{A}(B_1,B_2))$ is the upper bound of $\left\{S^{\theta}(B_1), S^{\theta}(B_2)\right\}$.

Assuming that $S^{\theta}(S)$ is the upper bound of $\left\{S^{\theta}(B_1), S^{\theta}(B_2)\right\}$ with $S\subseteq A$, then $S^{\theta}(B_1)\preceq S^{\theta}(S)$, $S^{\theta}(B_2)\preceq S^{\theta}(S)$. By Theorem 4.14, $R^{\theta}_{B_1}\subseteq R^{\theta}_S$ and $R^{\theta}_{B_2}\subseteq R^{\theta}_S$. Then, $S\in\mathcal{A}(B_1,B_2)$. Therefore, $S\subseteq\cup\mathcal{A}(B_1,B_2)$. By Theorem 4.17 (1), $S^{\theta}(\cup\mathcal{A}(B_1,B_2))\preceq S^{\theta}(S)$.

Thus, $S^{\theta}(B_1)\vee S^{\theta}(B_2)=S^{\theta}(\cup\mathcal{A}(B_1,B_2))$.

Therefore, L is a lattice.

It is obvious that $1_L=S^{\theta}(\varnothing)$, $0_L=S^{\theta}(A)$.

Corollary 5.4. *$\left(S^{\theta}(U,A),\wedge\right)$ is an exchangeable semigroup with the identity element $S^{\theta}(\varnothing)$.*

Example 5.5. *(Continued from Example 4.22) $\left(S^{\theta}(U,A),\preceq\right)$ is not a distributive lattice.*

By Theorem 5.3, $L=\left(S^{\theta}(U,A),\preceq\right)$ is a lattice with $1_L=S^{\theta}(\varnothing)$ and $0_L=S^{\theta}(A)$.

The following equations can be obtained:

$$\begin{aligned}\mathcal{A}(B_1,B_2\cup B_3)&=\mathcal{A}(B_1,B_6)=\left\{B\subseteq A: R^{\theta}_{B_1}\cup R^{\theta}_{B_6}\subseteq R^{\theta}_B\right\}\\&=\left\{B\subseteq A: R^{\theta}_{B_1}\subseteq R^{\theta}_B\right\}=\{\varnothing, B_1, B_2, B_4\},\end{aligned}$$

$\mathcal{A}(B_1,B_2)=\left\{B\subseteq A: R^{\theta}_{B_1}\cup R^{\theta}_{B_2}\subseteq R^{\theta}_B\right\}=\left\{B\subseteq A: R^{\theta}_{B_2}\subseteq R^{\theta}_B\right\}=\{\varnothing, B_2\}$,

$\mathcal{A}(B_1,B_3)=\left\{B\subseteq A: R^{\theta}_{B_1}\cup R^{\theta}_{B_3}\subseteq R^{\theta}_B\right\}=\{\varnothing\}$.

Then,

$$\cup\mathcal{A}(B_1,B_2\cup B_3)=\{a_1,a_2\},$$

$$(\cup\mathcal{A}(B_1,B_2))\cup(\cup\mathcal{A}(B_1,B_3))=\{a_2\}.$$

By Theorem 4.16,

$$S^{\theta}(\cup\mathcal{A}(B_1, B_2 \cup B_3)) \neq S^{\theta}((\cup\mathcal{A}(B_1, B_2)) \cup (\cup\mathcal{A}(B_1, B_3))).$$

By Theorem 5.3,

$$\begin{aligned} & S^{\theta}(B_1) \vee \left(S^{\theta}(B_2) \wedge S^{\theta}(B_3)\right) \\ & = S^{\theta}(B_1) \vee S^{\theta}(B_2 \cup B_3) \\ & = S^{\theta}((\cup A(B_1, B_2 \cup B_3))), \\ & \left(S^{\theta}(B_1) \vee S^{\theta}(B_2)\right) \wedge \left(S^{\theta}(B_1) \vee S^{\theta}(B_3)\right) \\ & = S^{\theta}((\cup\mathcal{A}(B_1, B_2))) \cup S^{\theta}((\cup\mathcal{A}(B_1, B_3))) \\ & = S^{\theta}((\cup\mathcal{A}(B_1, B_2)) \cup (\cup\mathcal{A}(B_1, B_3))). \end{aligned}$$

Thus,

$$\begin{aligned} & S^{\theta}(B_1) \vee \left(S^{\theta}(B_2) \wedge S^{\theta}(B_3)\right) \\ & \neq \left(S^{\theta}(B_1) \vee S^{\theta}(B_2)\right) \wedge \left(S^{\theta}(B_1) \vee S^{\theta}(B_3)\right). \end{aligned}$$

Accordingly, $\left(S^{\theta}(U, A), \preceq\right)$ is not a distributive lattice.

Example 5.6. *(Continued from Example 5.5) $\left(S^{\theta}(U, A), \sqsubseteq\right)$ is not a partial order set.*

It can be obtained that $S^{\theta}(B_1) \sqsubseteq S^{\theta}(B_2)$ and $S^{\theta}(B_2) \sqsubseteq S^{\theta}(B_1)$. However, $S^{\theta}(B_1) \neq S^{\theta}(B_2)$. This means that $\sqsubseteq$ is not antisymmetric.

Therefore, $\left(S^{\theta}(U, A), \sqsubseteq\right)$ is not a partial order set.

6. AN APPLICATION

As an application of information structures in an IIVIS, the θ-rough entropy is presented in this section.

Why do we study measures of uncertainty for an IIVIS? This is because an IIVIS itself has uncertainty. Why do we use information structures to measure the uncertainty of an IIVIS? This is because it is difficult to compare the size of measure values of uncertainty for an IIVIS. Moreover, if the dependence between two information structures are obtained, then the size of the measure values of uncertainty for an IIVIS can be compared by means of the dependence.

Definition 6.1. *Let (U, A) be an IIVIS. Given $B \subseteq A$ and $\theta \in [0, 1]$, the θ-rough entropy of subsystem (U, B) is defined as*

$$E_r^{\theta}(B) = -\sum_{i=1}^{n} \frac{|R_B^{\theta}(u_i)|}{n} \log_2 \frac{1}{|R_B^{\theta}(u_i)|},$$

where $\frac{|R_B^{\theta}(u_i)|}{n}$ represents the probability of $R_B^{\theta}(u_i)$.

Proposition 6.2. *Assume that (U, A) is an IIVIS. Given $B \subseteq A$ and $\theta \in [0, 1]$, then*

$$0 \leqslant E_r^{\theta}(B) \leqslant n \log_2 n.$$

Moreover, if $R_B^{\theta} = \triangle$, then $E_r^{\theta}(min) = 0$; if $R_B^{\theta} = \delta$, then $E_r^{\theta}(max) = n \log_2 n$.

Proof. Note that $\forall i$, $1 \leqslant |R_B^{\theta}(u_i)| \leqslant n$, we have

$$0 \leqslant -\log_2 \frac{1}{|R_B^{\theta}(u_i)|} = \log_2 |R_B^{\theta}(u_i)| \leqslant \log_2 n,$$

$$0 \leqslant \frac{|R_B^{\theta}(u_i)|}{n} \leqslant 1.$$

Then,

$$0 \leqslant \frac{|R_B^{\theta}(u_i)|}{n} \log_2 \frac{1}{|R_B^{\theta}(u_i)|} \leqslant \log_2 n.$$

By Definition 6.1,

$$0 \leqslant E_r^{\theta}(B) \leqslant n \log_2 n.$$

If $R_B^{\theta} = \triangle$, then $\forall i$, $|R_B^{\theta}(u_i)| = 1$. So $E_r^{\theta}(B) = 0$.

If $R_B^{\theta} = \delta$, then $\forall i$, $|R_B^{\theta}(u_i)| = n$. So $E_r^{\theta}(B) = n \log_2 n$.

Theorem 6.3. *Let (U, A) be an IIVIS. Assume $B, C \subseteq A$ and $\theta_1, \theta_2 \in [0, 1]$.*

1. *If $S^{\theta_1}(B) \preceq S^{\theta_2}(C)$, then $E_r^{\theta_1}(B) \leqslant E_r^{\theta_2}(C)$.*
2. *If $S^{\theta_1}(B) \prec S^{\theta_2}(C)$, then $E_r^{\theta_1}(B) < E_r^{\theta_2}(C)$.*

Proof. (1) This proof is obvious.

(2) By Definition 6.1,

$$\begin{aligned} E_r^{\theta_1}(B) &= -\sum_{i=1}^{n} \frac{|R_B^{\theta_1}(u_i)|}{n} \log_2 \frac{1}{|R_B^{\theta_1}(u_i)|}, \\ E_r^{\theta_2}(C) &= -\sum_{i=1}^{n} \frac{|R_C^{\theta_2}(u_i)|}{n} \log_2 \frac{1}{|R_C^{\theta_2}(u_i)|}. \end{aligned}$$

Note that $S^{\theta_1}(B) \prec S^{\theta_2}(C)$. Then, $\forall i$, $R_B^{\theta_1}(u_i) \subseteq R_C^{\theta_2}(u_i)$ and $\exists j$, $R_B^{\theta_1}(u_j) \subsetneq S_C^{\theta_2}(u_j)$. Thus, $\forall i$, $|R_B^{\theta_1}(u_i)| \leqslant |R_C^{\theta_2}(u_i)|$ and $\exists j$, $|R_B^{\theta_1}(u_j)| < |S_C^{\theta_2}(u_j)|$.

Hence, $\forall i$,

$$\begin{aligned} -|R_B^{\theta_1}(u_i)| \log_2 \frac{1}{|R_B^{\theta_1}(u_i)|} &= |R_B^{\theta_1}(u_i)| \log_2 |R_B^{\theta_1}(u_i)| \\ \leqslant |R_C^{\theta_2}(u_i)| \log_2 |R_C^{\theta_2}(u_i)| &= -|R_C^{\theta_2}(u_i)| \log_2 \frac{1}{|R_C^{\theta_2}(u_i)|}, \exists j \\ -|R_B^{\theta_1}(u_j)| \log_2 \frac{1}{|R_B^{\theta_1}(u_j)|} &= |R_B^{\theta_1}(u_j)| \log_2 |R_B^{\theta_1}(u_j)| \\ < |S_C^{\theta_2}(u_j)| \log_2 |S_C^{\theta_2}(u_j)| &= -|S_C^{\theta_2}(u_j)| \log_2 \frac{1}{|S_C^{\theta_2}(u_j)|}. \end{aligned}$$

Hence, $E_r^{\theta_1}(B) < E_r^{\theta_2}(C)$.

This proposition clarifies that the θ-rough entropy value increases as the available information becomes more uncertain. Therefore, it can be concluded that the θ-rough entropy presented in Definition 6.1 can evaluate the uncertainty of an IIVIS.

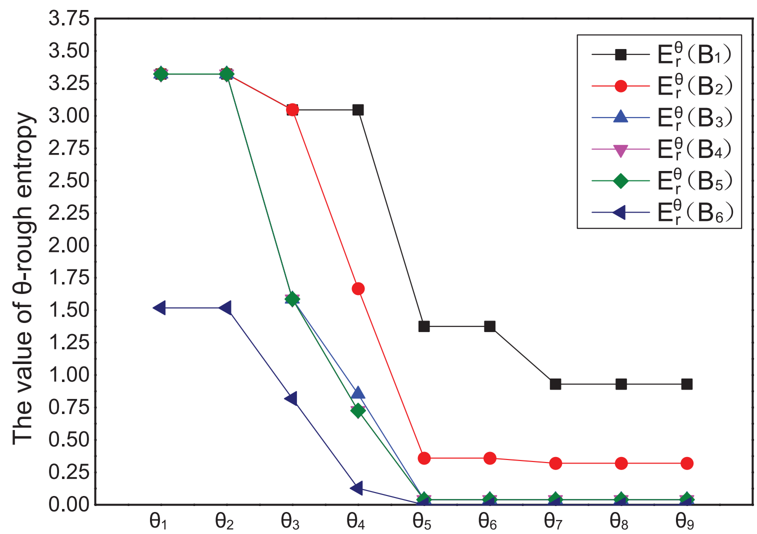

Figure 2 The change of the θ-rough entropy with different θ.

Proposition 6.4. *Let (U, A) be an IIVIS.*

1. *If* $B \subseteq C$ *with* $B, C \subseteq A$, *then for any* $\theta \in [0,1]$, $E_r^\theta(C) \preceq E_r^\theta(B)$.
2. *If* $0 < \theta_1 \leqslant \theta_2 \leqslant 1$, *then for any* $B \subseteq A$, $E_r^{\theta_2}(B) \preceq E_r^{\theta_1}(B)$.

Proof. This result follows from Proposition 3.4 and Theorem 6.3.

Corollary 6.5. *Let* (U, A) *be an IIVIS. Given* $\theta_1, \theta_2 \in [0,1]$ *and* $B, C \subseteq A$. *If* $B \subseteq C, \theta_1 \leqslant \theta_2$, *then*

$$E_r^{\theta_2}(C) \preceq E_r^{\theta_1}(C) \preceq E_r^{\theta_1}(B), E_r^{\theta_2}(C) \preceq E_r^{\theta_2}(B) \preceq E_r^{\theta_1}(B).$$

Proof. This result follows from Proposition 6.4.

Select $\theta_1 = 0.1, \theta_2 = 0.2, \cdots, \theta_9 = 0.9$ and $B_1 = \{a_1\}$, $B_2 = \{a_1, a_2\}$, $\cdots$, $B_8 = \{a_1, a_2 \ldots, a_6\} = A$. We compare $E^\theta(B_i)$ with θ under the same object, as shown in Figure 2.

7. CONCLUSIONS

This article has investigated information structures in an IIVIS. The concept of information structures has been introduced. The dependence and information distance between information structures in an IIVIS have been presented. In addition, group and lattice characterizations of information structures have been obtained. As an application, the θ-rough entropy of uncertainty for information structures in an IIVIS has been shown. We plan to study other applications of the information structures.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

AUTHOR CONTRIBUTIONS

All authors contribute equally in this work in all parts and in all steps.

Funding Statement

Natural Science Foundation of Guangxi (2018GXNSFDA294003, 2018GXNSFAA294134), Key Laboratory of Software Engineering in Guangxi University for Nationali-ties (2018-18XJSY-03), and Engineering Project of Undergraduate Teaching Reform of Higher Education in Guangxi (2017JGA179).

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