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Matrix-Based Approaches for Updating Approximations in Multigranulation Rough Set While Adding and Deleting Attributes

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ABSTRACT

With advanced technology in medicine and biology, data sets containing information could be huge and complex that sometimes are difficult to handle. Dynamic computing is an efficient approach to solve some problems. Since multigranulation rough sets were proposed, many algorithms have been designed for updating approximations in multigranulation rough sets, but they are not efficient enough in terms of computational time. The purpose of this study is to further reduce the computational time of updating approximations in multigranulation rough sets. First, searching regions in data sets for updating approximations in multigranulation rough sets are shrunk. Second, matrix-based approaches for updating approximations in multigranulation rough set are proposed. The incremental algorithms for updating approximations in multigranulation rough sets are then designed. Finally, the efficiency and validity of the designed algorithms are verified by experiments.

1. INTRODUCTION

Since the rough set [1,2] was proposed by Pawlak in 1982, it has been widely used in various fields such as pattern recognition [3–10], machine learning [11,12], image processing [11,13–19], decision-making [20–22], data mining, and so on. A lot of extensions have been proposed to extend its application including covering based rough sets [23], variable precision rough sets [24], probabilistic rough sets [18], fuzzy rough sets [9,13,25,26], fuzzy variable precision rough sets [27], and so on.

Qian *et al.* proposed multigranulation rough sets (MGRSs) based on multiple equivalence relation in 2010, which include optimistic MGRSs and pessimistic MGRSs. In recent years, many models have been proposed based on two decision strategies: “Seeking common ground while reserving differences” and “Seeking common ground with eliminating differences.” For example, by popularizing the binary relation from equivalence relation to neighborhood relation, Lin *et al.* proposed neighborhood MGRSs. Lots of studies focus on deriving models by the same decision strategy. Huang *et al.* proposed intuitionistic fuzzy MGRSs [28]. Feng *et al.* proposed variable precision multigranulation decision-theoretic fuzzy rough sets [29]. Li *et al.* proposed three-way cognitive concept learning via multi-granularity [30]. There are research on MGRSs and their relative models, such as MGRSs theory over two universe [31], a comparative study of MGRSs and concept lattices via rule acquisition [32], and so on.

In an information explosion era, approximation computing becomes more and more difficult: the size of the data sometimes is too huge to handle, the structure of the data becomes more complex, and the granular structures often increase or decrease. The issue of computing and updating approximations in MGRSs and their derived models attracts much research interest. These studies are often categorized into four classes by scholars, namely, how to update approximations while varying attributes [33,34], how to update approximations while varying attribute values [35,36], how to update approximations while varying decision attribute values [33,37], and how to update approximations while varying object set [38,39].

No matter what variation is, there always exist two means to determine the relation between two sets: set operation or matrix product. By this viewpoint, we can classify those studies into two categories. One is based on set operation. Scholars use set operation to determine whether a set is contained in another set or not, or whether their intersection is empty or not (see Chuan Luo [20,21], Wenhao Shu [40], Guangming Lang [41], Mingjie Cai [42], Wei Wei [43], Xin Yang [44], etc.). When granules sizes are generally big, set operation is a time-consuming way because of its searching strategy: when we compute the intersection of two sets, we must confirm whether every object in a set is in another set or not. In the extreme case, when the two sets are both U (all samples are in one data set), then the time complexity of computing the intersection of them is $|U|^2$. The other is based on matrix. These studies are mostly based on matrix product or other operations. Scholars often

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change a set into a binary matrix, and then design algorithms based on properties of binary matrix (see Jingqian Wang [45], Chengxiang Hu [46], Yanyong Huang [47] Yunge Jing [48], etc.). Although the time complexity of determining the relation between two sets via matrix approach is a constant, they often consider all the objects in the universe without filtering.

We attempt to combine the two approaches to derive new approaches to overcome their defect. In other words, we concentrate on which part of the universe does not need to be considered while computing and updating approximations in MGRS. At the same time, we determine the relation between two sets by matrix product. Why we try to propose the approaches? Because in real life application, it is common to add and delete attributes when there is some new information and some expired information. Different granular structures have a great influence on approximations in MGRS, thus different granular structures induce different decision-making processes. Moreover, adding and deleting attributes exists in the whole attribute reduction process. In decision-making and attribute reduction process, calculating approximations of decisions is an important and necessary step, so it is important to compute approximations based on approximations we have computed, that is, updating approximations. We need to proposed approaches for updating approximations because that updating approximations could be more efficient than compute the approximations again.

The purpose of this paper is to derive algorithms for updating approximations while adding and deleting attributes. First, searching region while updating approximations in MGRS need to be shrunk. A shrunk searching region can reduce the executing time of the algorithms. Second, matrix-based approaches for updating approximations need to be proposed to make algorithms more efficient.

The rest of this paper is organized as follows: Some basic concepts of rough set and MGRS are introduced in Section 2, and so is matrix-based static algorithm to calculate approximation in MGRS. In Section 3, dynamic approaches for updating approximations in MGRS while adding and deleting attributes are proposed. Several algorithms are proposed in Section 4. Experimental evaluations are conducted in Section 5 to verify the efficiency and validity of the algorithms that we designed. Finally, some conclusions and future work are given in Section 6.

2. PRELIMINARIES

In this section, we review some main concepts in MGRSs as well as static algorithm for computing approximations in MGRS.

2.1. Multigranulation Rough Sets

In the past few years, many extensions of MGRSs [49] have been proposed. Since MGRS are our basic model, we review the main results in this subsection.

Definition 1. [1] Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $U = \{x_1, x_2, \cdots, x_n\}$ is a nonempty finite set of objects called the universe. $A = \{a_1, a_2, \cdots, a_r\}$ is a nonempty finite set of attributes. The element $A \in AT$ is called an attribute set. $V_{AT} = \cup_{A \in AT} V_A$ is a domain of attribute values, where V_A is the domain of attribute set A. $f : U \times AT \rightarrow V$ is a decision function such that $f(x, A) \in V_A, \forall A \in AT, x \in U$.

Definition 2. [49] Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, the optimistic multigranulation lower and upper approximations of X are denoted by $\underline{\sum_{k=1}^{m} A_k^O}(X)$ and $\overline{\sum_{k=1}^{m} A_k^O}(X)$, respectively.

$$\underline{\sum_{k=1}^{m} A_k^O}(X) = \left\{x \in U | [x]_{A_1} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\right\}. \tag{1}$$

$$\overline{\sum_{k=1}^{m} A_k^O}(X) = \sim \underline{\sum_{k=1}^{m} A_k^O}(\sim X), \tag{2}$$

where $[x]_{A_k}$ is the equivalence class of x in terms of the attribute set A_k, $\sim X$ is the complement of the set X.

Theorem 1. *[50] Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, since $[x]_{A_k} \subseteq X$, we have $x \in X$. The following result holds.*

$$\underline{\sum_{k=1}^{m} A_k^O}(X) = \left\{x \in X | [x]_{A_1} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\right\}. \tag{3}$$

Theorem 2. *[49] Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, the optimistic multigranulation upper approximation of X is denoted by $\overline{\sum_{k=1}^{m} A_k^O}(X)$, we have*

$$\overline{\sum_{k=1}^{m} A_k^O}(X) = \left\{x \in U | [x]_{A_1} \cap X \neq \varnothing \wedge \cdots \wedge x \in U | [x]_{A_m} \cap X \neq \varnothing\right\}. \tag{4}$$

Definition 3. [49] Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, the pessimistic multigranulation lower and upper approximation of X are denoted by $\underline{\sum_{k=1}^{m} A_k^P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^P}(X)$, respectively.

$$\overline{\sum_{k=1}^{m} A_k^P}(X) = \left\{x \in U | [x]_{A_1} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\right\}. \tag{5}$$

$$\overline{\sum_{k=1}^{m} A_k^P}(X) = \sim \underline{\sum_{k=1}^{m} A_k^P}(\sim X). \tag{6}$$

Theorem 3. *[50] Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, since $[x]_{A_k} \subseteq X$, we have $x \in X$. The following result holds.*

$$\underline{\sum_{k=1}^{m} A_k^O}(X) = \left\{x \in X | [x]_{A_1} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\right\}. \tag{7}$$

Theorem 4. *[49] Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, the* *optimistic multigranulation upper approximation of X is denoted by* $\overline{\sum_{k=1}^{m} A_k^O}(X)$, *we have*

$$\overline{\sum_{k=1}^{m} A_k^P}(X) = \left\{x \in U \middle| [x]_{A_1} \cap X \neq \varnothing \vee [x]_{A_2} \cap X \neq \varnothing \vee \cdots \vee [x]_{A_m} \cap X \neq \varnothing \right\}. \quad (8)$$

2.2. Matrix-Based Algorithm for Computing Approximations in MGRSs

Definition 4. [51] Let $U = \{x_1, x_2, \cdots, x_n\}$. For any $X \subseteq U$, the matrix representation of X is denoted by $V(X) = \left[v_1(X), \cdots, v_n(X)\right]$, where

$$v_i(X) = \begin{cases} 1 & x_i \in X \\ 0 & x_i \notin X \end{cases} \quad i \in \{1, 2, \cdots, n\}.$$

Lemma 5. [52] *Let* $U = \{x_1, x_2, \cdots, x_n\}$ *.* $\forall X, Y \subseteq U$, *if* $Y \subseteq X$, *then*

$$V(Y) \cdot V^t(\sim X) = 0,$$

"T" denotes the transpose operation, and "·" is matrix product.

Example 1. Let $IS = \left(U, AT, V_{AT}, f\right)$ be an information system, as shown in Table 1, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $B = A \cup d$, and $A = \{a_1, a_2, a_3\}$. Let $X = \{x_2, x_4, x_5\}$. According to Definition 4 , we have $V(X) = [0, 1, 0, 1, 1, 0]$. Suppose $Y = \{x_2, x_4\}$, then $V(Y) = [0, 1, 0, 1, 0, 0]$. Obviously, $Y \subseteq X$. $V(\sim X) = [1, 0, 1, 0, 0, 1]$, $V(Y) \cdot V^t(\sim X) = [0, 1, 0, 1, 0, 0] \cdot [1, 0, 1, 0, 0, 1]^t = 0$.

Definition 5. [53] Let $IS = \left(U, AT, V_{AT}, f\right)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, the lower approximation character sets of X in MGRS can be calculated as

$$I_{A_k}^L(X) = \cup \left\{[x]_{A_k} \middle| [x]_{A_k} \subseteq X \wedge x \in X\right\}, \forall k = 1, 2, \cdots, m. \quad (9)$$

Lemma 6. [53] *Let* $IS = \left(U, AT, V_{AT}, f\right)$ *be an information system, where* $A_k \in AT$ *for any* $k \in \{1, 2, \cdots, m\}$. *For any* $X \subseteq U$, *the pessimistic and optimistic lower approximations in MGRS can be calculated by*

$$\underline{\sum_{k=1}^{m} A_k^P}(X) = \cap_{k=1}^{m} I_{A_k}^U(X), \quad (10)$$

$$\underline{\sum_{k=1}^{m} A_k^O}(X) = \cup_{k=1}^{m} I_{A_k}^U(X).$$

Example 2. (Continuation of Example 1) Suppose $A_1 = a_1, A_2 = a_2, A_3 = a_3$, by Definition 5, we have

Table 1 | A decision information system.

U	a_1	a_2	a_3	d
x_1	1	1	2	3
x_2	2	3	3	3
x_3	1	1	2	2
x_4	2	2	3	2
x_5	1	2	1	2
x_6	1	1	1	3

$$V\left([x_2]_{A_1}\right) \cdot V^t(\sim X) = V\left([x_4]_{A_1}\right) \cdot V^t(\sim X) = 0,$$

$$V\left([x_5]_{A_1}\right) \cdot V^t(\sim X) \neq 0, V\left(I_{A_1}^L(X)\right) = [0, 1, 0, 1, 0, 0];$$

$$V\left([x_2]_{A_2}\right) \cdot V^t(\sim X) = 0,$$

$$V\left([x_4]_{A_2}\right) \cdot V^t(\sim X) = V\left([x_5]_{A_2}\right) \cdot V^t(\sim X) = 0,$$

$$V\left(I_{A_2}^L(X)\right) = [0, 1, 0, 1, 1, 0];$$

$$V\left([x_2]_{A_3}\right) \cdot V^t(\sim X) = V\left([x_4]_{A_3}\right) \cdot V^t(\sim X) = 0,$$

$$V\left([x_5]_{A_3}\right) \cdot V^t(\sim X) \neq 0, V\left(I_{A_3}^L(X)\right) = [0, 1, 0, 1, 0, 0].$$

By Lemma 6,

$$V\left(\underline{\sum_{k=1}^{m} A_k^P}(X)\right) = \wedge_{k=1}^{m} V\left(\cap_{k=1}^{m} I_{A_k}^L(X)\right)$$

$$\begin{aligned} &= V\left(I_{A_1}^L(X)\right) \wedge V\left(I_{A_2}^L(X)\right) \wedge V\left(I_{A_3}^L(X)\right) \\ &= [0, 1, 0, 1, 0, 0] \wedge [0, 1, 0, 1, 1, 0] \wedge [0, 1, 0, 1, 0, 0] \\ &= [0, 1, 0, 1, 0, 0]. \end{aligned}$$

By Definition 4,

$$\underline{\sum_{k=1}^{m} A_k^P}(X) = \{x_2, x_4\}.$$

$$V\left(\underline{\sum_{k=1}^{m} A_k^O}(X)\right) = \vee_{k=1}^{m} V\left(I_{A_k}^L(X)\right)$$

$$\begin{aligned} &= V\left(I_{A_1}^L(X)\right) \vee V\left(I_{A_2}^L(X)\right) \vee V\left(I_{A_3}^L(X)\right) \\ &= [0, 1, 0, 1, 0, 0] \vee [0, 1, 0, 1, 1, 0] \vee [0, 1, 0, 1, 0, 0] \\ &= [0, 1, 0, 1, 0, 0]. \end{aligned}$$

By Definition 4, $\underline{\sum_{k=1}^{m} A_k^O}(X) = \{x_2, x_4, x_5\}$.

Definition 6. [53] Let $IS = \left(U, AT, V_{AT}, f\right)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. For any $X \subseteq U$, the upper approximation character sets of X in MGRS can be defined as

$$I_{A_k}^U(X) = \cup \left\{[x]_{A_k} \middle| x \in X\right\}, \forall k = 1, 2, \cdots, m. \quad (11)$$

Lemma 7. [53] *Let* $IS = \left(U, AT, V_{AT}, f\right)$ *be an information system, where* $A_k \in AT$ *for any* $k \in \{1, 2, \cdots, m\}$. *For any* $X \subseteq U$, *the pessimistic and optimistic upper approximations can be calculated by*

$$\overline{\sum_{k=1}^{m} A_k^O}(X) = \cap_{k=1}^{m} I_{A_k}^U(X), \quad (12)$$

$$\overline{\sum_{k=1}^{m} A_k^P}(X) = \cap_{k=1}^{m} I_{A_k}^U(X).$$

Example 3. Continuation of Example 2. From Table 1, we have

$$\begin{aligned} V\left([x_2]_{A_1}\right) &= V([x_4])_{A_1} = [0,1,0,1,0,0], \\ V\left([x_5]_{A_1}\right) &= [1,0,1,0,1,1]; \\ V\left([x_2]_{A_2}\right) &= [0,1,0,0,0,0], \\ V\left([x_4]_{A_2}\right) &= V\left([x_5]_{A_2}\right) = [0,0,0,1,1,0]; \\ V\left([x_2]_{A_3}\right) &= V\left([x_4]_{A_2}\right) = [0,1,0,1,0,0], \\ V\left([x_5]_{A_3}\right) &= [0,0,0,0,1,1]. \end{aligned}$$

By Definition 6,

$$\begin{aligned} V\left(I^U_{A_1}(X)\right) &= V\left([x_2]_{A_1}\right) \vee V\left([x_4]_{A_1}\right) \vee V\left([x_5]_{A_1}\right) \\ &= [1,1,1,1,1,1], \\ V\left(I^U_{A_2}(X)\right) &= V\left([x_2]_{A_2}\right) \vee V\left([x_4]_{A_2}\right) \vee V\left([x_5]_{A_2}\right) \\ &= [0,1,0,1,1,0], \\ V\left(I^U_{A_3}(X)\right) &= V([x_2]A_3) \vee V\left([x_4]_{A_3}\right) \vee V\left([x_5]_{A_3}\right) \\ &= [0,1,0,1,1,1]. \end{aligned}$$

By Lemma 7,

$$\overline{\sum_{k=1}^{m} A_k^O}(X) = V\left(\cap_{k=1}^{m} I^U_{A_k}(X)\right)$$

$$\begin{aligned} &= V\left(I^U_{A_1}(X)\right) \wedge V\left(I^U_{A_2}(X)\right) \wedge V\left(I^U_{A_3}(X)\right) \\ &= [1,1,1,1,1,1] \wedge [0,1,0,1,1,0] \wedge [0,1,0,1,1,1] \\ &= [0,1,0,1,1,0]. \end{aligned}$$

By Definition 4,

$$\overline{\sum_{k=1}^{m} A_k^P}(X) = \{x_2, x_3, x_4\}.$$

$$V\left(\overline{\sum_{k=1}^{m} A_k^P}(X)\right) = \vee_{k=1}^{m} V\left(I^U_{A_k}(X)\right)$$

$$\begin{aligned} &= V\left(I^U_{A_1}(X)\right) \vee V\left(I^U_{A_2}(X)\right) \vee V\left(I^U_{A_3}(X)\right) \\ &= [1,1,1,1,1,1] \vee [0,1,0,1,1,0] \vee [0,1,0,1,1,1] \\ &= [1,1,1,1,1,1]. \end{aligned}$$

By Definition 4,

$$\overline{\sum_{k=1}^{m} A_k^O}(X) = U.$$

Algorithm 1 [53] is a matrix-based algorithm for computing approximations in MGRS. The total time complexity of the algorithm is $O\left(m|X||U|\right)$. Steps 3–6 are to calculate $I^L_{A_k}$ and $I^U_{A_k}$ ($k \in \{1,2,\cdots,m\}$) whose time complexity is $O\left(m|X||U|\right)$. Steps 17–22 are to compute the approximations of MGRS whose time complexity is $O\left(m|U|\right)$.

Algorithm 1: Matrix-based algorithm for computing approximations in MGRS

Require: (1) An information system $IS = \left(U, AT, V_{AT}, f\right)$ (2) A target concept $X \subseteq U$(3) Equivalence classes $[x]_{A_k}, x \in U, k \in \{1,2,\cdots,m\}$.
Ensure: Approximations in MGRS.
1: $n \leftarrow |U|$
2: $m \leftarrow |X|$
3: **for** $i = 1 \to n$ **do**
4: **for** $k = 1 \to m$ **do**
5: **for** $j = 1 \to n$ **do**
6: **if** $v_i(X) = 1 \wedge V^t(\sim(X)) \cdot V\left([x_i]_{A_k}\right) = 0$ **then** $v_i\left(I^L_{A_k}\right) = 1$
7: **end if**
8: **if** $v_i(X) = 1 \wedge v_j\left([x_i]_{A_k}\right) = 1$ **then** $v_j\left(I^U_{A_k}\right) = 1$
9: **end if**
10: **end for**
11: **end for**
12: **end for**
13: $V\left(\overline{\sum_{k=1}^{m} A_k^O}(X)\right) \leftarrow V\left(I^U_{A_1}\right)$
14: $V\left(\overline{\sum_{k=1}^{m} A_k^P}(X)\right) \leftarrow V\left(I^U_{A_1}\right)$
15: $V\left(\underline{\sum_{k=1}^{m} A_k^P}(X)\right) \leftarrow V\left(I^L_{A_1}\right)$
16: $V\left(\underline{\sum_{k=1}^{m} A_k^P}(X)\right) \leftarrow V\left(I^L_{A_1}\right)$
17: **for** $k = 2 \to m$ **do**
18: $V\left(\underline{\sum_{k=1}^{m} A_k^O}(X)\right) \leftarrow V\left(\overline{\sum_{k=1}^{m} A_k^O}(X)\right) \vee V\left(I^L_{A_k}\right)$
19: $V\left(\overline{\sum_{k=1}^{m} A_k^O}(X)\right) \leftarrow V\left(\overline{\sum_{k=1}^{m} A_k^O}(X)\right) \wedge V\left(I^U_{A_k}\right)$
20: $V\left(\underline{\sum_{k=1}^{m} A_k^P}(X)\right) \leftarrow V\left(\underline{\sum_{k=1}^{m} A_k^P}(X)\right) \wedge V\left(I^L_{A_k}\right)$
21: $V\left(\overline{\sum_{k=1}^{m} A_k^P}(X)\right) \leftarrow V\left(\overline{\sum_{k=1}^{m} A_k^P}(X)\right) \vee V\left(I^U_{A_k}\right)$
22: **end for**
23: Return $\underline{\sum_{k=1}^{m} A_k^O}(X)$, $\overline{\sum_{k=1}^{m} A_k^O}(X)$, $\underline{\sum_{k=1}^{m} A_k^P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^P}(X)$

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3. MATRIX-BASED DYNAMIC APPROACHES FOR UPDATING APPROXIMATIONS IN MGRS WHILE ADDING AND DELETING ATTRIBUTES

3.1. Matrix-Based Dynamic Approaches for Updating Approximations While Adding Attributes

In this subsection, we present matrix-based dynamic approaches for updating approximations in MGRS, while adding attributes, let $IS^t = \left(U, AT^t, V_{AT^t}, f^t\right)$ be an information system at time t, $IS^{t+1} = \left(U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1}\right)$ be an information system at time $t+1$, and for all $A_k^t \in AT^t (k \leqslant m)$, exists $A_k^{t+1} \in AT^{t+1}$, such that $A_k^t \subseteq A_k^{t+1}$ for any $k \in \{1,2,\cdots,m\}$. Also, for all $x \in U$, we denote equivalence class of x at time t by $[x]^t_{A_k}$. Denote equivalence class of x at time $t+1$ by $[x]^{t+1}_{A_k}$. Denote pessimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$ at time

$t+1$, respectively. Denote pessimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^t}^O(X)$ and $\overline{\sum_{k=1}^{m} A_k^t}^O(X)$ at time $t+1$, respectively. Denote optimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$ at time t, respectively. Denote optimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$ at time $t+1$, respectively.

Lemma 8. [46] *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the following results hold:*

1. $\underline{\sum_{k=1}^{m} A_k^t}^O(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$;
2. $\overline{\sum_{k=1}^{m} A_k^t}^O(X) \supseteq \overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$.

Lemma 9. [46] *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the following results hold:*

1. $\underline{\sum_{k=1}^{m} A_k^t}^P(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$;
2. $\overline{\sum_{k=1}^{m} A_k^t}^P(X) \supseteq \overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$.

Lemmas 8 and 9 indicate the relations of lower and upper approximations in MGRS between time t and time $t+1$. However, Lemmas 8 and 9 are not clear enough for updating approximation in MGRS. The following theorem provides accurate approaches for updating approximations in MGRS from time t to $t+1$.

Theorem 10. *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, we have*

i. *If* $\Delta \underline{\sum_{k=1}^{m} A_k}^O(X) = \{x | \exists k \in \{1, 2, \cdots, m\}, [x]_{A_k}^{t+1} \subseteq X \wedge x \in \overline{\sum_{k=1}^{m} A_k^t}^P(X) - \underline{\sum_{k=1}^{m} A_k^t}^P(X)\}$, *then* $\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X) = \underline{\sum_{k=1}^{m} A_k^t}^O(X) \cup \Delta \underline{\sum_{k=1}^{m} A_k}^O(X)$

ii. *If* $\Delta \overline{\sum_{k=1}^{m} A_k}^O(X) = \{x | \exists k \in \{1, 2, \cdots, m,\ [x]_{A_k}^{t+1} \cap X = \varnothing \wedge x \in \overline{\sum_{k=1}^{m} A_k^t}^P(X) - \underline{\sum_{k=1}^{m} A_k^t}^P(X)\}$, *then* $\overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X) = \overline{\sum_{k=1}^{m} A_k^t}^O(X) - \Delta \overline{\sum_{k=1}^{m} A_k}^O(X)$

iii. *If* $\Delta \underline{\sum_{k=1}^{m} A_k}^P(X) = \{x | \forall k \in \{1, 2, \cdots, m\}, [x]_{A_k}^{t+1} \subseteq X \wedge x \in \overline{\sum_{k=1}^{m} A_k^t}^P(X) - \underline{\sum_{k=1}^{m} A_k^t}^P(X)\}$, *then* $\underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X) = \sum_{k=1}^{m} A_k^t{}^P(X) \cup \Delta \underline{\sum_{k=1}^{m} A_k}^P(X)$

iv. *If* $\Delta \overline{\sum_{k=1}^{m} A_k}^P(X) = \{x | \forall k \in 1, 2, \cdots, m\}, [x]_{A_k}^{t+1} \cap X = \varnothing \wedge x \in \overline{\sum_{k=1}^{m} A_k^t}^P(X) - \underline{\sum_{k=1}^{m} A_k^t}^P(X)\}$, *then* $\overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X) = \overline{\sum_{k=1}^{m} A_k^t}^P(X) - \Delta \overline{\sum_{k=1}^{m} A_k}^P(X)$.

Proof.

i.
- *If* $x \in \underline{\sum_{k=1}^{m} A_k^t}^P(X)$, $x \in \underline{\sum_{k=1}^{m} A_k^t}^P(X) \Leftrightarrow [x]_{A_k}^t \subseteq X$, *since we have* $\forall y \in [x]_{A_k}^t$, $[y]_{A_k}^{t+1} \subseteq [x]_{A_k}^t$, $[x]_{A_k}^t \subseteq X \Leftrightarrow [x]_{A_k}^{t+1} \subseteq X \Leftrightarrow x \in \underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$.
- *If* $x \in \overline{\sum_{k=1}^{m} A_k^t}^P(X) - \underline{\sum_{k=1}^{m} A_k^t}^P(X)$, $[x]_{A_k}^{t+1} \subseteq X \Leftrightarrow x \in \underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$.
- *If* $x \in U - \overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$, *from Definition 2,* $[x]_{A_k}^t \nsubseteq X$, *since we have* $\forall y \in [x]_{A_k}^t$, $[y]_{A_k}^{t+1} \subseteq [x]_{A_k}^t \Leftrightarrow [x]_{A_k}^{t+1} \nsubseteq X \Leftrightarrow x \notin \underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$.
- *From the above, we have* $\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X) = \underline{\sum_{k=1}^{m} A_k^t}^O(X) \cup \Delta \underline{\sum_{k=1}^{m} A_k}^O(X)$.

ii.
- *If* $x \in \underline{\sum_{k=1}^{m} A_k^t}^P(X)$, $x \in \underline{\sum_{k=1}^{m} A_k^t}^P(X) \Leftrightarrow \forall k \in \{1, 2, \cdots, m\}, [x]_{A_k}^t \subseteq X$, *Since we have* $\forall y \in [x]_{A_k}^t$, $[y]_{A_k}^{t+1} \subseteq [x]_{A_k}^t$, *then* $\forall k \in \{1, 2, \cdots, m\}, [x]_{A_k}^t \subseteq X \Leftrightarrow \forall k \in \{1, 2, \cdots, m\}, [x]_{A_k}^{t+1} \subseteq X \Leftrightarrow x \in \underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X) \subseteq \overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$, *thus we have* $x \in \underline{\sum_{k=1}^{m} A_k^t}^P(X) \Leftrightarrow x \in \overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$.
- *If* $x \in \overline{\sum_{k=1}^{m} A_k^t}^P(X) - \underline{\sum_{k=1}^{m} A_k^t}^P(X)$, $\forall k \in \{1, 2, \cdots, m\}$, $[x]_{A_k}^{t+1} \cap X = \varnothing \Leftrightarrow x \notin \overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$.
- *If* $x \in U - \overline{\sum_{k=1}^{m} A_k^t}^P(X)$, *from Theorem 2,* $x \in U - \overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X) \Leftrightarrow \forall k \in \{1, 2, \cdots, m\}, [x]_{A_k}^t \cap X = \varnothing$. *Since* $\forall y \in [x]_{A_k}^t$, $\forall k \in \{1, 2, \cdots, m\}, [y]_{A_k}^{t+1} \subseteq [x]_{A_k}^t$, $[x]_{A_k}^t \cap X = \varnothing \Leftrightarrow x \notin \overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$.
- *From the above, we have* $\overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X) = \overline{\sum_{k=1}^{m} A_k^t}^O(X) - \Delta \overline{\sum_{k=1}^{m} A_k}^O(X)$.

iii. *It is similar to i.*

iv. *It is similar to ii.*

Example 4. (Continuation of Example 1) Suppose $A_1^t = a_1$, $A_2^t = a_2$; $A_1^{t+1} = \{a_1, a_3\}$, $A_2^{t+1} = \{a_2, a_3\}$, $X = \{x_2, x_3, x_4\}$, thus we have

$$\begin{aligned}
[x_1]_{A_1}^t &= [x_3]_{A_1}^t = [x_5]_{A_1}^t = [x_6]_{A_1}^t \\
&= \{x_1, x_3, x_5, x_6\}, \\
[x_2]_{A_1}^t &= [x_4]_{A_1}^t = \{x_2, x_4\} \\
[x_1]_{A_1}^{t+1} &= [x_3]_{A_1}^{t+1} = \{x_1, x_3\}, \\
[x_2]_{A_1}^{t+1} &= [x_4]_{A_1}^{t+1} = \{x_2, x_4\}, \\
[x_5]_{A_1}^{t+1} &= [x_6]_{A_1}^{t+1} = \{x_5, x_6\}.
\end{aligned}$$

$$
\begin{aligned}
[x_1]^t_{A_2} &= [x_3]^t_{A_2} = [x_6]^t_{A_2} = \{x_1, x_3, x_6\},\\
[x_2]^t_{A_2} &= \{x_2\};\\
[x_4]^t_{A_2} &= [x_5]^t_{A_2} = \{x_4, x_5\};\\
[x_1]^{t+1}_{A_2} &= [x_3]^{t+1}_{A_2} = \{x_1, x_3\},\\
[x_2]^{t+1}_{A_2} &= \{x_2\}, [x_4]^{t+1}_{A_2} = \{x_4\},\\
[x_5]^{t+1}_{A_2} &= \{x_5\}, [x_6]^{t+1}_{A_2} = \{x_6\}.
\end{aligned}
$$

From Definitions 2 and 3 we have

$$\underline{\sum_{k=1}^{2} A_k^t}^{O}(X) = \{x_2, x_4\}, \overline{\sum_{k=1}^{2} A_k^t}^{O}(X) = U.$$

$$\underline{\sum_{k=1}^{2} A_k^t}^{P}(X) = \{x_2\}, \overline{\sum_{k=1}^{2} A_k^t}^{P}(X) = U.$$

By Theorem 10, we have

$$\overline{\sum_{k=1}^{2} A_k^t}^{P}(X) - \underline{\sum_{k=1}^{2} A_k^t}^{P}(X) = \{x_1, x_3, x_4, x_5, x_6\}.$$

Since

$$
\begin{gathered}
[x_1]^{t+1}_{A_1} \nsubseteq X, [x_3]^{t+1}_{A_1} \nsubseteq X, [x_4]^{t+1}_{A_1} \subseteq X,\\
[x_5]^{t+1}_{A_1} \nsubseteq X, [x_6]^{t+1}_{A_1} \nsubseteq X;\\
[x_1]^{t+1}_{A_2} \nsubseteq X, [x_3]^{t+1}_{A_2} \nsubseteq X, [x_4]^{t+1}_{A_2} \subseteq X,\\
[x_5]^{t+1}_{A_2} \nsubseteq X, [x_6]^{t+1}_{A_2} \nsubseteq X.\\
[x_1]^{t+1}_{A_1} \cap X \neq \varnothing, [x_3]^{t+1}_{A_1} \cap X \neq \varnothing, [x_4]^{t+1}_{A_1} \cap X \neq \varnothing,\\
[x_5]^{t+1}_{A_1} \cap X = \varnothing, [x_6]^{t+1}_{A_1} \cap X = \varnothing;\\
[x_1]^{t+1}_{A_2} \cap X \neq \varnothing, [x_3]^{t+1}_{A_2} \cap X \neq \varnothing, [x_4]^{t+1}_{A_2} \cap X \neq \varnothing,\\
[x_5]^{t+1}_{A_2} \cap X = \varnothing, [x_6]^{t+1}_{A_2} \cap X = \varnothing.
\end{gathered}
$$

Thus we have

$$
\begin{aligned}
\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) &= \underline{\sum_{k=1}^{2} A_k^t}^{O}(X) \cup \{x_4\} = \{x_2, x_4\};\\
\overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) &= \overline{\sum_{k=1}^{2} A_k^t}^{O}(X) - \{x_5, x_6\}\\
&= \{x_1, x_2, x_3, x_4\}.\\
\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) &= \underline{\sum_{k=1}^{2} A_k^t}^{P}(X) \cup \{x_4\} = \{x_2, x_4\},\\
\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) &= \overline{\sum_{k=1}^{2} A_k^t}^{P}(X) - \{x_5, x_6\}\\
&= \{x_1, x_2, x_3, x_4\}.
\end{aligned}
$$

Definition 7. Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the dynamic lower approximation character sets of X in MGRS while adding attributes can be defined as

$$\Delta I^L_{A_k}(X) = \cup \left\{ [x]_{A_k} \mid [x]^{t+1}_{A_k} \subseteq X \wedge x \in \overline{\sum_{k=1}^{m} A_k^t}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^t}^{P}(X) \right\}, \forall k = 1, 2, \cdots, m.$$

Definition 8. Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the dynamic upper approximation character sets of X in MGRS while adding attributes can be defined as

$$\Delta I^U_{A_k}(X) = \cup \left\{ [x]^{t+1}_{A_k} \mid [x]^{t+1}_{A_k} \cap X = \varnothing \wedge x \in \overline{\sum_{k=1}^{m} A_k^t}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^t}^{P}(X) \right\}, \forall k = 1, 2, \cdots, m.$$

Example 5. (Continuation of Example 4)

$$\Delta I^L_{A_1}(X) = \cup \left\{ [x]_{A_1} \mid [x]^{t+1}_{A_1} \subseteq X \wedge x \in \overline{\sum_{k=1}^{2} A_2^t}^{P}(X) - \underline{\sum_{k=1}^{2} A_1^t}^{P}(X) \right\} = \{x_4\},$$

$$\Delta I^L_{A_2}(X) = \cup \left\{ [x]_{A_2} \mid [x]^{t+1}_{A_2} \subseteq X \wedge x \in \overline{\sum_{k=1}^{2} A_2^t}^{P}(X) - \underline{\sum_{k=1}^{2} A_2^t}^{P}(X) \right\} = \{x_4\};$$

$$\Delta I^U_{A_1}(X) = \cup \left\{ [x]^{t+1}_{A_1} \mid [x]^{t+1}_{A_1} \cap X = \varnothing, x \in \overline{\sum_{k=1}^{2} A_k^t}^{P}(X) - \underline{\sum_{k=1}^{2} A_k^t}^{P}(X) \right\} = \{x_5, x_6\},$$

$$\Delta I^U_{A_2}(X) = \cup \left\{ [x]^{t+1}_{A_2} \mid [x]^{t+1}_{A_2} \cap X = \varnothing, x \in \overline{\sum_{k=1}^{2} A_k^t}^{P}(X) - \underline{\sum_{k=1}^{2} A_k^t}^{P}(X) \right\} = \{x_5, x_6\}.$$

Theorem 11. *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, we have*

i. $\Delta \underline{\sum_{k=1}^{m} A_k}^{O}(X) = \cup\left\{\Delta I_{A_k}^{L}(X) \mid k = 1, 2, \cdots, m\right\}$.

ii. $\Delta \overline{\sum_{k=1}^{m} A_k}^{O}(X) = \cap\left\{\Delta I_{A_k}^{U}(X) \mid k = 1, 2, \cdots, m\right\}$.

iii. $\Delta \underline{\sum_{k=1}^{m} A_k}^{P}(X) = \cap\left\{\Delta I_{A_k}^{L}(X) \mid k = 1, 2, \cdots, m\right\}$.

iv. $\Delta \overline{\sum_{k=1}^{m} A_k}^{P}(X) = \cup\left\{\Delta I_{A_k}^{U}(X) \mid k = 1, 2, \cdots, m\right\}$.

Proof. *This theorem can be easily obtained by Theorem 10* and *Definitions 7 and 8.*

By Theorem 11 we can easily obtain a matrix-based approach for updating approximations in MGRS while adding attributes.

Corollary 12. *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, we have*

i. $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)\right) = V\left(\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)\right) \vee \left(\vee_{k=1}^{m} V\left(\Delta I_{A_k}^{L}(X)\right)\right)$,

ii. $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)\right) = V\left(\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)\right) \wedge \sim \left(\vee_{k=1}^{m} V\left(\Delta I_{A_k}^{U}(X)\right)\right)$,

iii. $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)\right) = V\left(\underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)\right) \vee \left(\wedge_{k=1}^{m} V\left(\Delta I_{A_k}^{L}(X)\right)\right)$,

iv. $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)\right) = V\left(\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)\right) \wedge \sim \left(\wedge_{k=1}^{m} V\left(\Delta I_{A_k}^{L}(X)\right)\right)$.

Proof. *This corollary is the matrix representation of Theorem 10.*

Example 6. (Continuation of Example 5)

1. $V\left(\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X)\right) = V\left(\underline{\sum_{k=1}^{2} A_k^{t}}^{O}(X)\right) \vee \left(\vee_{k=1}^{2} V\left(\Delta I_{A_k}^{L}(X)\right)\right)$
$[0,1,0,1,0,0] \vee [0,0,0,1,0,0] = [0,1,0,1,0,0]$,

2. $V\left(\overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X)\right) = V\left(\overline{\sum_{k=1}^{2} A_k^{t}}^{O}(X)\right) \wedge \sim \left(\vee_{k=1}^{2} V\left(\Delta I_{A_k}^{U}(X)\right)\right)$
$[1,1,1,1,1,1] \wedge [1,1,1,1,0,0] = [1,1,1,1,0,0]$,

3. $V\left(\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X)\right) = V\left(\underline{\sum_{k=1}^{2} A_k^{t}}^{P}(X)\right) \vee \left(\wedge_{k=1}^{2} V\left(\Delta I_{A_k}^{L}(X)\right)\right)$
$[0,1,0,1,0,0] \vee [0,0,0,1,0,0] = [0,1,0,1,0,0]$,

4. $V\left(\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X)\right) = V\left(\overline{\sum_{k=1}^{2} A_k^{t}}^{P}(X)\right) \wedge \sim \left(\wedge_{k=1}^{2} V\left(\Delta I_{A_k}^{U}(X)\right)\right)$
$[1,1,1,1,1,1] \wedge [1,1,1,1,0,0] = [1,1,1,1,0,0]$.

From Definition 4 we have that $\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_2, x_4\}$, $\overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_1, x_2, x_3, x_4\}$; $\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = \{x_2, x_4\}$, $\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = \{x_1, x_2, x_3, x_4\}$.

3.2. Matrix-Based Dynamic Approaches for Updating Approximations While Deleting Attributes

In this section, we present matrix-based dynamic approaches for updating approximations in MGRS, while deleting attributes, let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$, and for all $A_k^t \in AT^t (k \leqslant m)$, exists $A_k^{t+1} \in AT^{t+1}$, such that $A_k^{t+1} \subseteq A_k^t$ for any $k \in \{1, 2, \cdots, m\}$. Also, for all $x \in U$, we denote equivalence classof x at time t by $[x]_{A_k}^{t}$. Denote equivalence class of x at time $t+1$ by $[x]_{A_k}^{t+1}$. Denote pessimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)$ at time t, respectively. Denote pessimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$ at time $t+1$, respectively. Denote optimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)$ at time t, respectively. Denote optimistic lower and upper approximations of X by $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$ at time $t+1$, respectively. According to [46], we have the following results in this section:

Lemma 13. [46] *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the following results hold:*

1. $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)$;

2. $\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X) \subseteq \overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$.

Lemma 14. [46] *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the following results hold:*

1. $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)$;

2. $\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) \subseteq \overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$.

Lemmas 13 and 14 indicate the relation of lower and upper approximations in MGRS between time t and time $t+1$. However, Lemmas 13 and 14 are not clear enough for updating approximation in MGRS. The following theorem provides accurate approaches for updating approximations in MGRS from time t to $t+1$:

Theorem 15. *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, we have*

i. *If* $\underline{\nabla \sum_{k=1}^{m} A_k}^{O}(X) = \left\{x \middle| \forall k \in \{1,2,\cdots,m\},\ [x]_{A_k}^{t+1} \nsubseteq X \wedge x \in \underline{\sum_{m}^{k=1} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{m}^{k=1} A_k^t}^{O}(X)\right)\right\}$, *then* $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X) = \underline{\sum_{m}^{k=1} A_k^t}^{O}(X) - \nabla \underline{\sum_{m}^{k=1} A_k}^{O}(X)$.

ii. *If* $\overline{\nabla \sum_{k=1}^{m} A_k}^{O}(X) = \left\{x \middle| \forall k \in \{1,2,\cdots,m\},\ [x]_{A_k}^{t+1} \cap X \neq \varnothing \wedge x \in \underline{\sum_{m}^{k=1} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{m}^{k=1} A_k^t}^{O}(X)\right)\right\}$, *then* $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X) = \overline{\sum_{k=1}^{m} A_k^t}^{O}(X) \cup \overline{\nabla \sum_{k=1}^{m} A_k}^{O}(X)$.

iii. $\underline{\nabla \sum_{k=1}^{m} A_k}^{P}(X) = \left\{x \middle| \exists k \in \{1,2,\cdots,m\},\ [x]_{A_k}^{t+1} \nsubseteq X \wedge x \in \underline{\sum_{m}^{k=1} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{m}^{k=1} A_k^t}^{O}(X)\right)\right\}$, *then* $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X) = \underline{\sum_{m}^{k=1} A_k^t}^{P}(X) - \nabla \underline{\sum_{m}^{k=1} A_k}^{P}(X)$.

iv. If $\overline{\nabla \sum_{k=1}^{m} A_k}^{P}(X) = \left\{x \middle| \exists k \in \{1,2,\cdots,m\},\ [x]_{A_k}^{t+1} \cap X \neq \varnothing \wedge x \in \underline{\sum_{m}^{k=1} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{m}^{k=1} A_k^t}^{O}(X)\right)\right\}$, *then* $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X) = \overline{\sum_{m}^{k=1} A_k^t}^{P}(X) \cup \nabla \overline{\sum_{m}^{k=1} A_k}^{P}(X)$.

Proof.

i. *By* Lemma 13 *we have* $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)$, *thus we have* $\forall x \in \underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X), [x]_{A_k}^{t+1} \subseteq X \Leftrightarrow x \in \underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, *in other words,* $\forall x \in \underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X), [x]_{A_k}^{t+1} \nsubseteq X \Leftrightarrow x \notin \underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$.

ii. *By* Lemma 13 *we have* $\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, *thus we have* $\forall k \in \{1,2,\cdots,m\}, \forall x \in U - \underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X), [x]_{A_k}^{t+1} \cap X \neq \varnothing \Leftrightarrow x \in \overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$.

iii. *It is similar to i.*

iv. *It is similar to ii.*

Example 7. (Continuation of Example 1) Suppose $A_1^t = \{a_1, a_2\}$, $A_2^t = \{a_2, a_3\}$; $A_1^{t+1} = \{a_2\}$, $A_2^{t+1} = \{a_3\}$, $X = \{x_2, x_3, x_4\}$, thus we have

$$
\begin{aligned}
[x_1]_{A_1}^t &= [x_3]_{A_1}^t = [x_6]_{A_1}^t = \{x_1, x_3, x_6\},\\
[x_2]_{A_1}^t &= \{x_2\}, [x_4]_{A_1}^t = \{x_4\}, [x_5]_{A_1}^t = \{x_5\};\\
[x_1]_{A_1}^{t+1} &= [x_3]_{A_1}^{t+1} = [x_6]_{A_1}^{t+1} = \{x_1, x_3, x_6\},\\
[x_2]_{A_1}^{t+1} &= \{x_2\}. [x_4]_{A_1}^{t+1} = [x_5]_{A_1}^{t+1} = \{x_4, x_5\}.\\
[x_1]_{A_2}^t &= [x_3]_{A_2}^t = \{x_1, x_3\}, [x_2]_{A_2}^t = \{x_2\},\\
[x_4]_{A_2}^t &= \{x_4\}, [x_5]_{A_2}^t = \{x_5\}, [x_6]_{A_2}^t = \{x_6\};\\
[x_1]_{A_2}^{t+1} &= [x_3]_{A_2}^{t+1} = \{x_1, x_3\},\\
[x_2]_{A_2}^{t+1} &= [x_4]_{A_2}^{t+1} = \{x_2, x_4\};\\
[x_5]_{A_2}^{t+1} &= [x_6]_{A_2}^{t+1} = \{x_5, x_6\}.
\end{aligned}
$$

From Definitions 2 and 3 we have

$$\underline{\sum_{k=1}^{2} A_k^t}^{O}(X) = \{x_2, x_4\}, \overline{\sum_{k=1}^{2} A_k^t}^{O}(X) = \{x_1, x_2, x_3, x_4\}.$$

$$\underline{\sum_{k=1}^{2} A_k^t}^{P}(X) = \{x_2\}, \overline{\sum_{k=1}^{2} A_k^t}^{P}(X) = \{x_1, x_2, x_3, x_4, x_6\}.$$

By Theorem 15, we have

$$\underline{\sum_{k=1}^{2} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{2} A_k^t}^{O}(X)\right) = \{x_2, x_4, x_5, x_6\}.$$

Since

$$[x_2]_{A_1}^{t+1} \subseteq X, [x_4]_{A_1}^{t+1} \nsubseteq X, [x_5]_{A_1}^{t+1} \nsubseteq X, [x_6]_{A_1}^{t+1} \nsubseteq X.$$

$$[x_2]_{A_2}^{t+1} \subseteq X, [x_4]_{A_2}^{t+1} \subseteq X, [x_5]_{A_2}^{t+1} \nsubseteq X, [x_6]_{A_2}^{t+1} \nsubseteq X.$$

$$[x_2]_{A_1}^{t+1} \cap X \neq \varnothing, [x_4]_{A_1}^{t+1} \cap X \neq \varnothing, [x_5]_{A_1}^{t+1} \cap X \neq \varnothing,$$
$$[x_6]_{A_1}^{t+1} \cap X \neq \varnothing.$$

$$[x_2]_{A_2}^{t+1} \cap X \neq \varnothing, [x_4]_{A_2}^{t+1} \cap X \neq \varnothing, [x_5]_{A_2}^{t+1} \cap X = \varnothing,$$
$$[x_6]_{A_2}^{t+1} \cap X = \varnothing.$$

Thus we have

$$
\begin{aligned}
\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) &= \underline{\sum_{k=1}^{2} A_k^{t}}^{O}(X) - \{x_5, x_6\} = \{x_2, x_4\};\\
\overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) &= \overline{\sum_{k=1}^{2} A_k^{t}}^{O}(X) \cup \{x_2, x_4\}\\
&= \{x_1, x_2, x_3, x_4\}.\\
\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) &= \underline{\sum_{k=1}^{2} A_k^{t}}^{P}(X) - \{x_4, x_5, x_6\} = \{x_2\},\\
\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) &= \overline{\sum_{k=1}^{2} A_k^{t}}^{P}(X) \cup \{x_2, x_4, x_5, x_6\} = U.
\end{aligned}
$$

Definition 9. Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the dynamic upper approximation character sets of X in MGRS while deleting attributes can be defined as

$$\nabla I^U_{A_k}(X) = \cup\left\{[x]^{t+1}_{A_k} \mid [x]^{t+1}_{A_k} \nsubseteq X \wedge x \in \underline{\sum_{k=1}^{m} A^t_k}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A^t_k}^O(X)\right)\right\}, \forall k = 1, 2, \cdots, m. \quad (13)$$

Definition 10. Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, the dynamic lower approximation character sets of X in MGRS while deleting attributes can be defined as

$$\nabla I^U_{A_k}(X) = \cup\left\{[x]^{t+1}_{A_k} \mid [x]^{t+1}_{A_k} \cap X \neq \varnothing \wedge x \in \underline{\sum_{k=1}^{m} A^t_k}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A^t_k}^O(X)\right)\right\}, \forall k \in \{1, 2, \cdots, m\}. \quad (14)$$

Example 8. (Continuation of Example 7)

$$I^L_{A_1}(X) = \cup\left\{[x]_{A_1} \mid [x]^{t+1}_{A_1} \nsubseteq X \wedge x \in \underline{\sum_{k=1}^{m} A^t_k}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A^t_k}^O(X)\right)\right\} = \{x_4, x_5, x_6\}.$$

$$\nabla I^L_{A_2}(X) = \cup\left\{[x]_{A_2} \mid [x]^{t+1}_{A_2} \subseteq X \wedge x \in \underline{\sum_{k=1}^{m} A^t_k}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A^t_k}^O(X))\right)\right\} = \{x_5, x_6\}.$$

$$\nabla I^U_{A_1}(X) = \cup\left\{[x]_{A_1} \mid [x]^{t+1}_{A_1} \cap X = \varnothing \wedge x \in \underline{\sum_{k=1}^{m} A^t_k}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A^t_k}^O(X)\right)\right\} = \{x_2, x_4, x_5, x_6\}.$$

$$\nabla I^U_{A_2}(X) = \cup\left\{[x]_{A_1} \mid [x]^{t+1}_{A_2} \cap X = \varnothing \wedge x \in \underline{\sum_{k=1}^{m} A^t_k}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A^t_k}^O(X)\right)\right\} = \{x_2, x_4\}.$$

Theorem 16. *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, we have*

- $\nabla \underline{\sum_{k=1}^{m} A_k^P}(X) = \cup\left\{\nabla I^L_{A_k}(X) \mid k = 1, 2, \cdots, m\right\}$,
- $\nabla \overline{\sum_{k=1}^{m} A_k}^P(X) = \cup\left\{\nabla I^U_{A_k}(X) \mid k = 1, 2, \cdots, m\right\}$,
- $\nabla \underline{\sum_{k=1}^{m} A_k}^O(X) = \cap\left\{\nabla I^L_{A_k}(X) \mid k = 1, 2, \cdots, m\right\}$,
- $\nabla \overline{\sum_{k=1}^{m} A_k}^O(X) = \cap\left\{\nabla I^U_{A_k}(X) \mid k = 1, 2, \cdots, m\right\}$.

Proof. *This theorem can be easily obtained by Theorem 15* and Def*initions 9 and 10.*

By Theorem 16 we can easily obtain matrix-based approaches for updating approximations in MGRS while adding attributes.

Corollary 17. *Let $IS^t = (U, AT^t, V_{AT^t}, f^t)$ be an information system at time t, $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ be an information system at time $t+1$. For any $X \subseteq U$, we have*

- $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)\right) = V\left(\underline{\sum_{k=1}^{m} A_k^t}^O(X)\right) \wedge \sim \left(\wedge_{k=1}^{m} V\left(\nabla I^L_{A_k}(X)\right)\right)$,
- $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)\right) = V\left(\overline{\sum_{k=1}^{m} A_k^t}^O(X)\right) \vee \left(\wedge_{k=1}^{m} V\left(\nabla I^U_{A_k}(X)\right)\right)$.
- $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)\right) = V\left(\underline{\sum_{k=1}^{m} A_k^t}^P(X)\right) \wedge \sim \left(\wedge_{k=1}^{m} V\left(\nabla I^L_{A_k}(X)\right)\right)$,
- $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)\right) = V\left(\overline{\sum_{k=1}^{m} A_k^t}^P(X)\right) \vee \left(\vee_{k=1}^{m} V\left(\nabla I^U_{A_k}(X)\right)\right)$.

Proof. *This corollary is the matrix representation of Theorem 15.*

Example 9. (Continuation of Example 8)

i. $$\begin{aligned} V\left(\underline{\sum_{k=1}^{2} A_k^{t+1}}^O(X)\right) &= V\left(\underline{\sum_{k=1}^{2} A_k^t}^O(X)\right) \wedge \sim \\ \left(\wedge_{k=1}^{2} V\left(\nabla I^L_{A_2}(X)\right)\right) &= [0,1,0,1,0,0] \wedge [1,1,1,1,0,0] \\ &= [0,1,0,1,0,0], \end{aligned}$$

ii. $$\begin{aligned} V\left(\overline{\sum_{k=1}^{2} A_k^{t+1}}^O(X)\right) &= V\left(\overline{\sum_{k=1}^{2} A_k^t}^O(X)\right) \vee \\ \left(\wedge_{k=1}^{2} V\left(\nabla I^U_{A_k}(X)\right)\right) &= [1,1,1,1,0,0] \vee [1,1,0,0,0,0] \\ &= [1,1,1,1,0,0]. \end{aligned}$$

iii. $$\begin{aligned} V\left(\underline{\sum_{k=1}^{2} A_k^{t+1}}^P(X)\right) &= V\left(\underline{\sum_{k=1}^{2} A_k^t}^P(X)\right) \wedge \\ \left(\sim \wedge_{k=1}^{2} V\left(\nabla I^L_{A_k}(X)\right)\right) &= [0,1,0,0,0,0] \wedge [1,1,1,0,0,0] \\ &= [0,1,0,0,0,0], \end{aligned}$$

iv. $V\left(\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X)\right) = V\left(\overline{\sum_{k=1}^{2} A_k^{t}}^{P}(X)\right) \vee \left(\vee_{k=1}^{2} V\left(\nabla I_{A_k}^{U}(X)\right)\right) = [1,1,1,1,0,1] \vee [0,1,0,1,1,1] = [1,1,1,1,1,1].$

From Definition 4 we have that $\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_2, x_4\}$, $\overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_1, x_2, x_3, x_4\}$, $\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = \{x_2\}$, $\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = U$.

Algorithm 2: Matrix-based algorithm for updating approximations in MGRS while adding attributes.

Require: (1) $IS^t = (U, AT^t, V_{AT^t}, f^t)$ (2) $IS^{t+1} = (U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1})$ (3) A target concept $X \subseteq U$ (4) $\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)$. (4) Equivalence classes $[x]_{A_k}^{t+1}, x \in U, k \in \{1, 2, \cdots, m\}$.

Ensure: $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$), $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$), and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$)

1: $n \leftarrow |U|$
2: $t \leftarrow \left|\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)\right|$
3: **for** $i = 1 \rightarrow t$ **do**
4: **for** $k = 1 \rightarrow m$ **do**
5: **if** $v_i(\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)) = 1 \wedge V(\sim (X)) \cdot V^t([x_i]_{A_k}^{t+1}) = 0$ **then** $v_i\left(\Delta I_{A_k}^{L}\right) = 1$
6: **end if**
7: **if** $v_i\left(\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)\right) = 1 \wedge V(X) \cdot V^t\left([x_i]_{A_k}^{t+1}\right) = 0$ **then** $v_j\left(\Delta I_{A_k}^{U}\right) = 1$
8: **end if**
9: **end for**
10: **end for**
11: $V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\Delta I_{A_1}^{L}\right)$
12: $V\left(\Delta \overline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\Delta I_{A_1}^{U}\right)$
13: $V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\Delta I_{A_1}^{L}\right)$
14: $V\left(\Delta \overline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\Delta I_{A_1}^{U}\right)$
15: **for** $k = 2 \rightarrow m$ **do**
16: $V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \vee V\left(\Delta I_{A_k}^{L}\right)$
17: $V\left(\Delta \overline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\Delta \overline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \vee V\left(\Delta I_{A_k}^{U}\right)$
18: $V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \wedge V\left(\Delta I_{A_k}^{L}\right)$
19: $V\left(\Delta \overline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\Delta \overline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \wedge V\left(\Delta I_{A_k}^{U}\right)$
20: **end for**
21:
22: $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)\right) \leftarrow V\left(\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)\right) \vee V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right)$
23: $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)\right) \leftarrow V\left(\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)\right) \wedge \sim V\left(\overline{\Delta \sum_{k=1}^{m} A_k}^{O}(X)\right)$
24: $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)\right) \leftarrow V\left(\underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)\right) \vee V\left(\Delta \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right)$
25: $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)\right) \leftarrow V\left(\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)\right) \wedge \sim V\left(\overline{\Delta \sum_{k=1}^{m} A_k}^{P}(X)\right)$
26: Return $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$

4. MATRIX-BASED DYNAMIC ALGORITHMS FOR UPDATING APPROXIMATIONS WHILE ADDING AND DELETING ATTRIBUTES

Based on Corollary 12, we propose matrix-based Algorithm 2 for updating approximations in MGRS while adding attributes. The total time complexity of Algorithm 2 is $O\left(m|\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)||U|\right)$. Steps 3–12 are to calculate $\Delta I_{A_k}^{L}$ and $\Delta I_{A_k}^{U}$ $(k \in \{1, 2, \cdots, m\})$ with time complexity $O\left(m|\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)||U|\right)$. Steps 17–22 are to compute $\Delta \underline{\sum_{k=1}^{m} A_k}^{O}(X), \Delta \overline{\sum_{k=1}^{m} A_k}^{O}(X), \Delta \underline{\sum_{k=1}^{m} A_k}^{P}(X)$ and $\overline{\Delta \sum_{k=1}^{m} A_k}^{P}(X)$ with time complexity $O\left(m|U|\right)$. Steps 24–27 are to update the approximations of MGRS while increasing granular structures with time complexity $O\left(|U|\right)$.

Since the time complexity of Algorithm 1 is $O\left(m|X||U|\right)$, and in general, $O\left(m|\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)||U|\right) \leqslant O\left(m|X||U|\right)$ does not hold. Algorithm 3 is proposed to make sure the total time complexity is no more than $O\left(m|X||U|\right)$. In other words, when $|\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)| > |X|$ we call Algorithm 1; otherwise, we call Algorithm 2.

Based on Corollary 17, we propose matrix-based Algorithm 4 for updating approximations in MGRS while deleting attributes. The total time complexity of Algorithm 4 is $O\left(m|\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)\right)||U|\right)$. Steps 3–12 are to calculate $\nabla I_{A_k}^{L}$ and $\nabla I_{A_k}^{U}$ $(k \in \{1, 2, \cdots, m\})$ with time complexity $O\left(m|\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)\right)||U|\right)$. Steps 17–22

Algorithm 3: Ensure total time complexity of updating approximations in MGRS while adding attributes is no more than $O\left(|X||U|\right)$.

Require: (1) $IS^t = \left(U, AT^t, V_{AT^t}, f^t\right)$ (2) $IS^{t+1} = \left(U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1}\right)$ (3) A target concept $X \subseteq U$ (4) $\underline{\sum_{k=1}^{m} A_k^t}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k^t}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^t}^{P}(X)$, and $\overline{\sum_{k=1}^{m} A_k^t}^{P}(X)$. (4) Equivalence classes $[x]_{A_k}^{t+1}, x \in U, k \in \{1, 2, \cdots, m\}$.

Ensure: $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$

1: **if** $|\overline{\sum_{k=1}^{m} A_k^t}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^t}^{P}(X)| \leqslant |X|$ **then** Call Algorithm 2

2: **end if**

3: **if** $|\overline{\sum_{k=1}^{m} A_k^t}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^t}^{P}(X)| > |X|$ **then** Call Algorithm 1

4: **end if**

5: Return $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$

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are to compute $\nabla \underline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\nabla \overline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\nabla \underline{\sum_{k=1}^{m} A_k}^{P}(X)$ and $\nabla \overline{\sum_{k=1}^{m} A_k}^{P}(X)$ with time complexity $O\left(m|U|\right)$. Steps 24–27 are to update the approximations of MGRS while increasing granular structures with time complexity $O\left(m|U|\right)$.

Algorithm 4: Matrix-based algorithm for updating approximations in MGRS while decreasing attributes

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Require: (1) $IS^t = \left(U, AT^t, V_{AT^t}, f^t\right)$ (2) $IS^{t+1} = \left(U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1}\right)$ (3) A target concept $X \subseteq U$ (4) $\underline{\sum_{k=1}^{m} A_k^t}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k^t}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^t}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^t}^{P}(X)$. (4) Equivalence classes $[x]_{A_k}^{t+1}, x \in U, k \in \{1, 2, \cdots, m\}$.

Ensure: $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$), $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$) and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$)

1: $n \leftarrow |U|$

2: $t \leftarrow \left|\underline{\sum_{k=1}^{m} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^t}^{O}(X)\right)\right|$

3: **for** $i = 1 \to t$ **do**

4: **for** $k = 1 \to m$ **do**

5: **if** $v_i\left(\underline{\sum_{k=1}^{m} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^t}^{O}(X)\right)\right) = 1 \wedge V(\sim X) \cdot V^t\left([x_i]_{A_k}^{t+1}\right) \neq= 0$ **then** $v_i\left(\nabla I_{A_k}^L\right) = 1$

6: **end if**

7: **if** $v_i\left(\underline{\sum_{k=1}^{m} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^t}^{O}(X)\right)\right) = 1 \wedge v_i(X) = 1$ **then** $v_i\left(\nabla I_{A_k}^U\right) = 1$

8: **end if**

9: **end for**

10: **end for**

11: $V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\nabla I_{A_1}^L\right)$

12: $V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\nabla I_{A_1}^U\right)$

13: $V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\nabla I_{A_1}^L\right)$

14: $V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\nabla I_{A_1}^U\right)$

15: **for** $k = 2 \to m$ **do**

16: $V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \wedge V\left(\nabla I_{A_k}^L\right)$

17: $V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \leftarrow V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{O}(X)\right) \wedge V\left(\nabla I_{A_k}^U\right)$

18: $V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \vee V\left(\nabla I_{A_k}^L\right)$

19: $V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \leftarrow V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{P}(X)\right) \vee V\left(\nabla I_{A_k}^U\right)$

20: **end for**

21:

22: $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)\right) \leftarrow V\left(\underline{\sum_{k=1}^{m} A_k^t}^{O}(X)\right) \wedge \sim V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{O}(X)\right)$

23: $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)\right) \leftarrow V\left(\overline{\sum_{k=1}^{m} A_k^t}^{O}(X)\right) \vee V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{O}(X)\right)$

24: $V\left(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)\right) \leftarrow V\left(\underline{\sum_{k=1}^{m} A_k^t}^{P}(X)\right) \wedge \sim V\left(\nabla \underline{\sum_{k=1}^{m} A_k}^{P}(X)\right)$

25: $V\left(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)\right) \leftarrow V\left(\overline{\sum_{k=1}^{m} A_k^t}^{P}(X)\right) \vee V\left(\nabla \overline{\sum_{k=1}^{m} A_k}^{P}(X)\right)$

26: Return $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$

Since the time complexity of Algorithm 1 is $O\left(m|X||U|\right)$, and in general, $O\left(m|\underline{\sum_{k=1}^{m} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^t}^{O}(X)\right)||U|\right) \leqslant O\left(m|X||U|\right)$ does not hold, Algorithm 5 is proposed to make sure the total time complexity is no more than $O\left(m|X||U|\right)$. In other words, when $|\underline{\sum_{k=1}^{m} A_k^t}^{O}(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^t}^{O}(X)\right)| > |X|$ we call Algorithm 1; otherwise, we call Algorithm 4.

5. EXPERIMENTAL EVALUATIONS

In this section, several experiments were conducted to evaluate the effectiveness and the efficiency of Algorithm 3 (DMB) and Algorithm 5 (DMB). Three algorithms were chosen to compare, namely,

Algorithm 5: Ensure total time complexity of updating approximations in MGRS while deleting attributes is no more than $O\left(|X||U|\right)$

Require: (1) $IS^t = \left(U, AT^t, V_{AT^t}, f^t\right)$ (2) $IS^{t+1} = \left(U, AT^{t+1}, V_{AT^{t+1}}, f^{t+1}\right)$ (3) A target concept $X \subseteq U$ (4) $\underline{\sum_{k=1}^{m} A_k^t}^O(X)$, $\overline{\sum_{k=1}^{m} A_k^t}^O(X)$, $\underline{\sum_{k=1}^{m} A_k^t}^P(X)$ and $\overline{\sum_{k=1}^{m} A_k^t}^P(X)$. (4) Equivalence classes $[x]_{A_k}^{t+1}, x \in U, k \in \{1, 2, \cdots, m\}$.

Ensure: $\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$), $\underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$) and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$)

1: **if** $|\underline{\sum_{k=1}^{m} A_k^t}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^t}^O(X)\right)| \leqslant |X|$ **then** Call Algorithm 4
2: **end if**
3: **if** $|\underline{\sum_{k=1}^{m} A_k^t}^O(X) \cup \left(U - \overline{\sum_{k=1}^{m} A_k^t}^O(X)\right)| > |X|$ **then** Call Algorithm 1
4: **end if**
5: Return $\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)$

matrix-based static algorithm (MB) [53], relation matrix-based static algorithm (RMB) [46], and relation matrix-based dynamic algorithm (DRMB) [46]. Six data sets were chosen from UCI machine learning repository. The details of the data sets are listed in Table 2. We can see that the sizes of data sets range from 194 to 1000, the attribute numbers range from 5 to 59. All the experiments were carried out on a personal computer with 64-bit windows 10, Inter(R) Core(TM) i7 6700HQ CPU @2.60 GHz, and 16GB memory. The program language was Matlab r2015b.

5.1. Comparison of Computational Time Using Data Sets with Different Size

The computational time were compared among the four algorithms in MGRSs while adding and deleting attributes when the size of data sets increases. First of all, we construct three granular structures. We randomly chose an attribute set $\hat{A}$ containing at least two attributes in the data set and divided the rest into three parts randomly to contribute to three granular structures respectively. While adding attributes, we added the attributes in $\hat{A}$ into the three granular structures at the same time. While deleting attributes, we combined $\hat{A}$ with each granular structure and deleted the attributes in $\hat{A}$ from the granular structures at the same time. We randomly divided each data set U into 10 subsets $\{U_1, U_2, \cdots, U_{10}\}$. Then U_1 was chosen as the first temporary data set. After that, some samples of temporary data set were randomly selected to contribute to the target concept X. The size of target concept X wasabout 0.85 times the size of each temporary data set. We calculated the four approximations in MGRS by the four algorithms 10 times and compared the averages. Then made $U_1 \cup U_2$ the second temporary data set and repeat the whole process was repeated.

When the size of data sets increases, results of the four algorithms while adding and deleting attributes in MGRS are shown in Figures 1 and 2. We can see that DMB is the most efficient algorithm when the size of data set increases gradually. From Figure 1 we can see that DMB is effective and it reduces the computational time.

Table 2 Details of data sets.

No.	Data Sets	Samples	Attributes
1	Blood Transfusion	748	5
2	Dermatology	366	20
3	Extention of ZAlizadehsani	303	59
4	Facebookmetrics	500	19
5	Flags	194	30
6	German Credit Data	1000	21

5.2. Comparison of Computational Time Using Target Concept with Different Size

Similarly, instead of construct temporary data sets, we construct temporary target concepts. The process of constructing and varying the three granular structures is similar to Section 5.1. We randomly divided each data set into ten subsets $\{X_1, X_2, \cdots, X_{10}\}$. And then X_1 was chosen as the first temporary target concept. Finally, we calculated the four approximations in MGRS by the the four algorithms 10 times and compared the averages. Then made $X_1 \cup X_2$ the second temporary target concept and the whole process was repeated.

When the size of target concept increases, results of MB, DMB, RMB, DRMB are shown in Figures 3 and 4. When the size of target concept increasing gradually, RMB is always the most time-consuming algorithmn the four algorithms. DMB and MB are more efficient than RMB and DRMB. The computational time of DMB is always less than or equal to MB, so DMB is more efficient than any other algorithms. Sometimes the computational time of DMB is a little more than the MB while deleting attributes, which is due to additional computation in DMB and it is within the expected range.

6. CONCLUSION

Data sets in real life application sometimes are complex and huge, which is difficult to handle. In addition, the granular structures often increase and decrease in some data sets. It is important to design algorithms to update approximations in MGRS while adding and deleting attributes. In this paper, four algorithms have been proposed to ensure that the time complexity of the incremental algorithm is less than or equal to the static algorithm. Experimental results show that the computational time of the DMB is no more than the other algorithms in most of the situations.

Approximation computation is a basic process of attribute reduction. In the future, we will further investigate attribute reduction algorithm using the approaches we proposed.

CONFLICT OF INTEREST

There are no conflicts of interest.

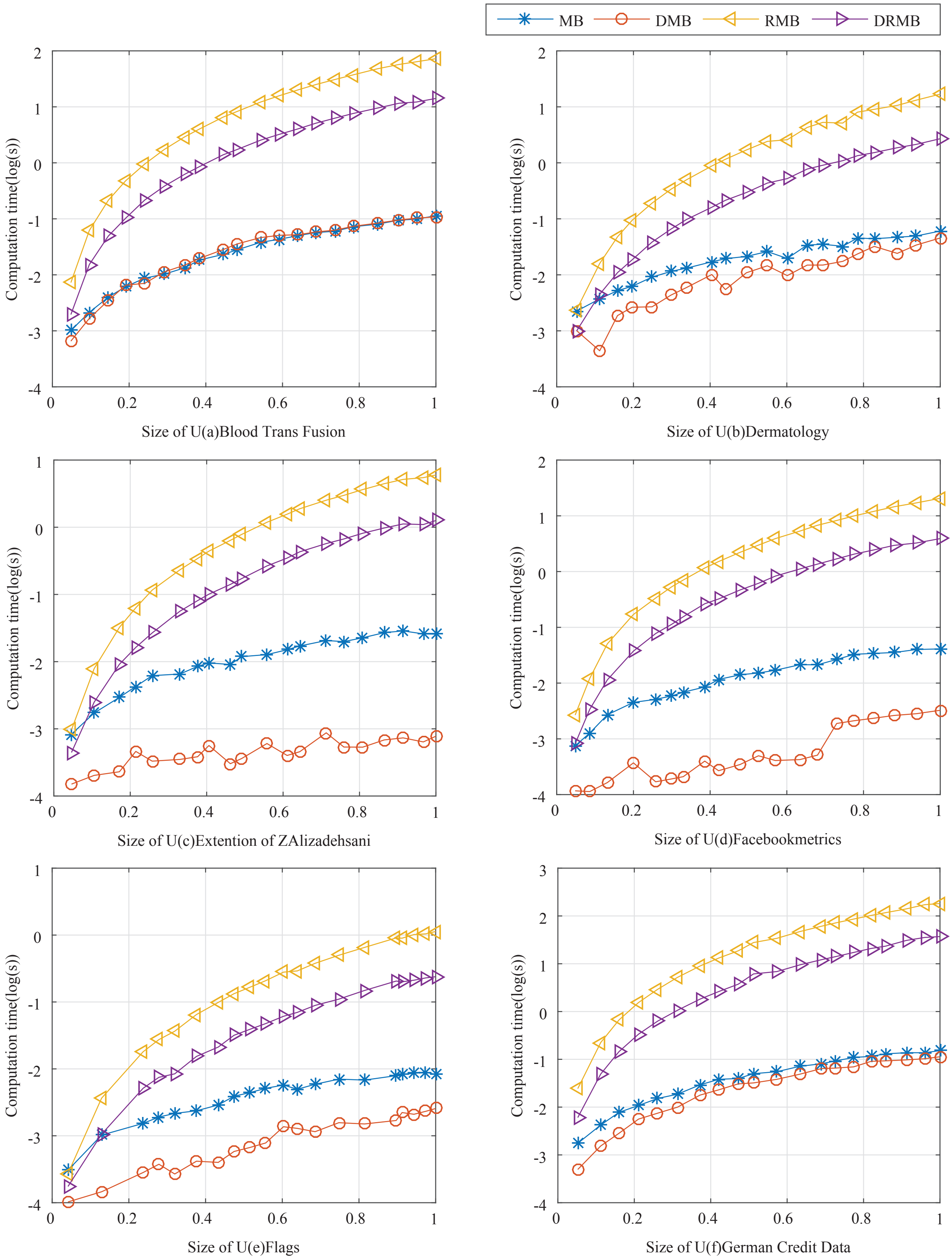

Figure 1 | Computational time of Algorithm 3 when the size of *U* increasing gradually (adding attributes).

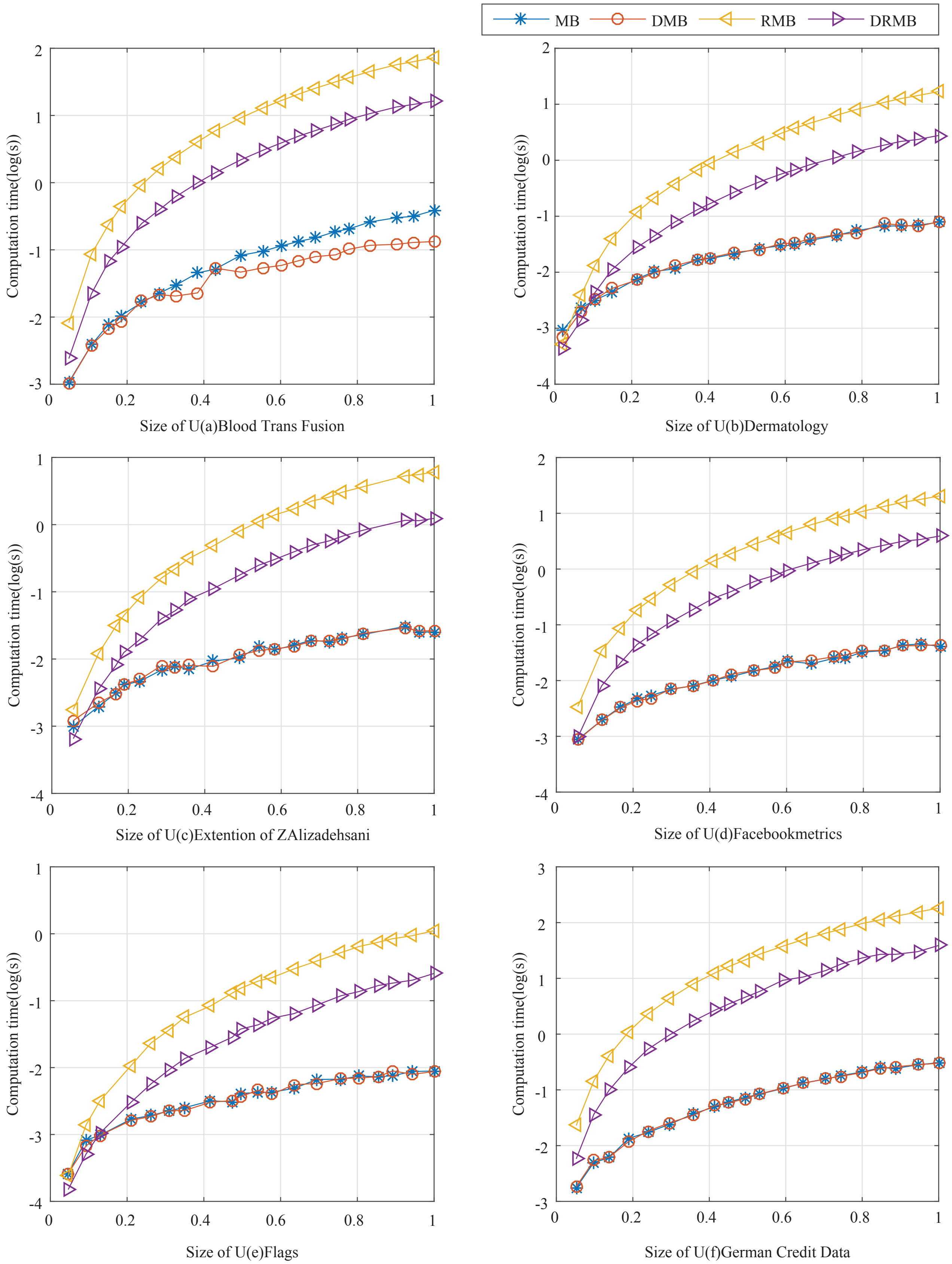

Figure 2 Computational time of Algorithm 5 when the size of *U* increasing gradually (deleting attributes).

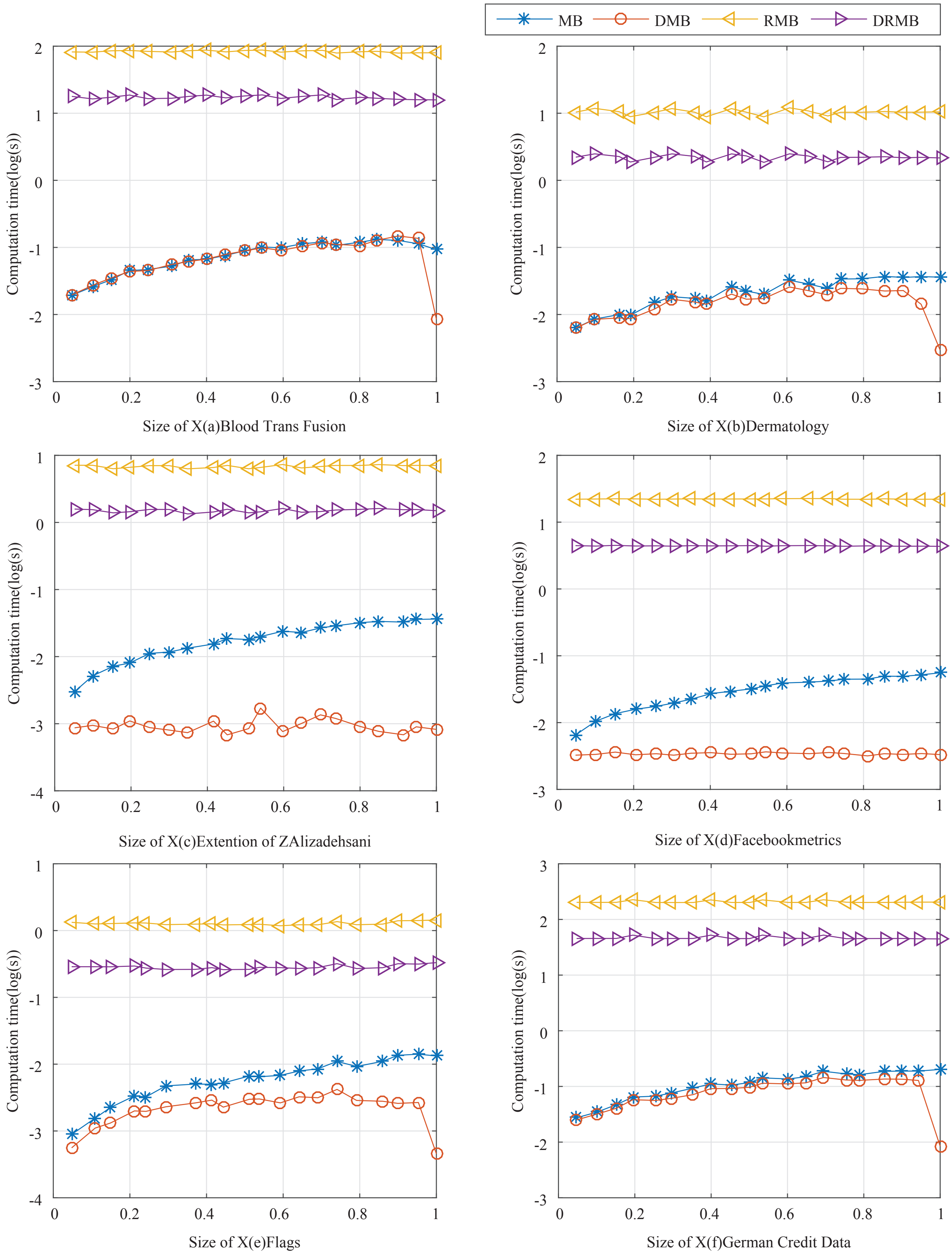

Figure 3 Computational time of Algorithm 3 when the size of X increasing gradually (adding attributes).

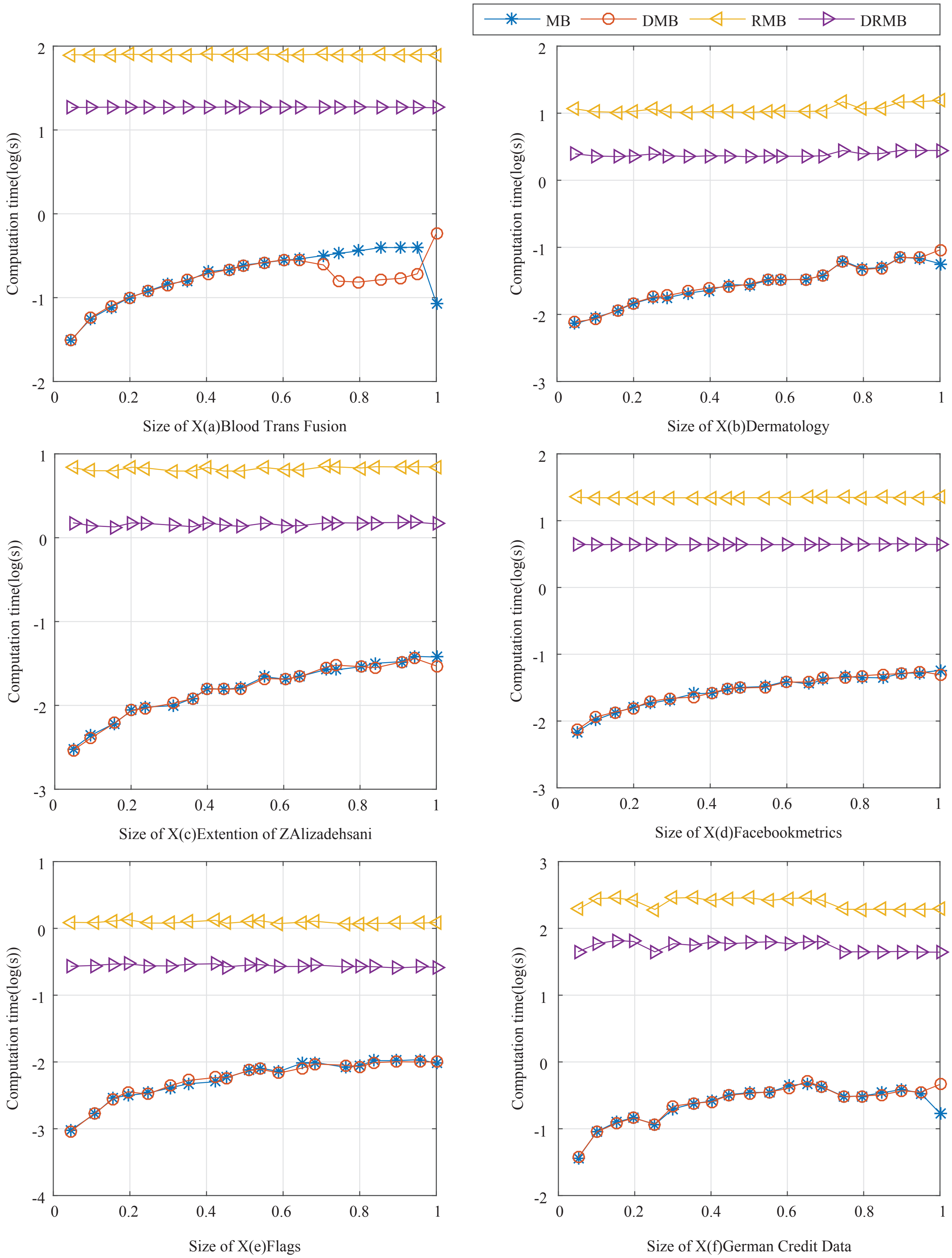

Figure 4 Computational time of Algorithm 5 when the size of *X* increasing gradually (deleting attributes).

AUTHORS' CONTRIBUTIONS

Jinjin Li and Peqiu Yu conceived and designed the study. Peiqiu Yu performed the experiments. Peiqiu Yu wrote the paper. Peiqiu Yu and Hongkun Wang reviewed and edited the manuscript. All authors read and approved the manuscript.

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