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Aggregation of Partial T–Indistinguishability Operators

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Abstract

We focus on the aggregation of partial T–indistinguishability operators. We provide a characterization of those functions that allow to merge a collection of partial T-indistinguishability operators into a new one in terms of $(+, \max)$-tuples. Moreover, we see that this is equivalent to the characterization by means of (T, T_M)-tuples given by Calvo et al. in [8]. Also, we analyze the aggregation of a collection $(E_i)_{i=1}^n$ of partial T_i-indistinguishability operators. Moreover, we show that a generalized inter-exchange composition function condition is a sufficient condition to guarantee that a function merges partial T_i-indistinguishability operators into a single one. Finally, we give different expressions of those aggregation functions defined by means of the additive generators of the corresponding t-norms and another specific function.

Keywords: Aggregation function, generalized inter-exchange composition function, partial indistinguishability operator, partial pseudo–metric, (+, max)-tuple, t-norm, t-conorm, (T, T_M)-tuple, weighted function.

1 Introduction

Aggregation functions constitute an important tool in the field of information fusion (see [4, 5]). The information can be given by means of fuzzy relations that depend on different applications (see [11, 18, 25, 26, 32]). The problem of aggregating fuzzy relations has received considerable attention from the fuzzy community researching (see, [12, 20, 21, 27, 28, 29]), for this reason, in this work, we pay special attention in the preservation of the properties of partial T–indistinguishability operators by means of aggregation functions.

The early work of partial T-indistinguishability operators were introduced by Trillas in [30], so that given a t–norm $T : [0, 1]^2 \to [0, 1]$, a T–indistinguishability operator on a (non–empty) set X is a fuzzy relation $E : X \times X \to [0, 1]$ verifying for all $x, y, z \in X$ the following properties: (i) $E(x, x) = 1$, (ii) $E(x, y) = E(y, x)$, (iii) $T(E(x, y), E(y, z)) \le E(x, z)$. A T–indistinguishability operator is a mathematical tool for classifying objects when a measure presents some kind of uncertainty. They are also known as measures of similarity, in fact, the greater $E(x, y)$ the more similar are x and y. Throughout this contribution we will assume that the reader is familiar with the basic notions of triangular norms, for more details see [17].

The T–indistinguishability operators are widely related to pseudo-metrics. In fact, one can find different methods, in the actual literature, to generate pseudo-metrics from indistinguishability operators, and viceversa (see [1, 2, 15, 19, 21, 24, 29, 31]). According to [10], a pseudo-metric on a (non-empty) set X is a function $d : X \times X \to [0, \infty]$ which satisfies the following axioms for all $x, y, z \in X$: (i) $d(x, x) = 0$, (ii) $d(x, y) = d(y, x)$, (iii) $d(x, z) \le d(x, y) + d(y, z)$. Pseudo-metrics are measures of dissimilarity, the smaller $d(x, y)$ the more similar are x and y, and these measures are the antecedent of partial-pseudo-metrics.

The notion of partial pseudo-metric was introduced by Matthews (see [6]), he wanted to develop a suitable mathematical tool for quantitative models in denotational semantics. A partial pseudo-metric on a non-empty set X was defined as a function $p : X \times X \to [0, \infty[$ which satisfies for all x, y, z the following: $(i)\, p(x, x) \le p(x, y)$, $(ii)\, p(x, y) = p(y, x)$, $(iii)\, p(x, y) + p(z, z) \le p(x, z) + p(z, y)$. In the partial framework, a closely related notion of indistinguishability operator is considered in the works by Demirci [9] and by Bukatin et al. [6]. This new type of indistinguishability operator was called partial T-indistinguishability operator and was introduced in [14]. So that, we recall that a partial T-indistinguishability operator E on a non-empty set X is a fuzzy relation $E : X \times X \to [0,1]$ satisfying the following properties for any $x, y, z \in X$: $(i)\, E(x,y) \leq E(x,x), (ii)\, E(x,y) = E(y,x)$, and $(iii)\, T(E(x,z), E(z,y)) \leq T(E(x,y), E(z,z))$. A partial T-indistinguishability operator E on a non-empty set X will be called a partial T-equality provided that it satisfies the following additional property for any $x, y, z \in X$: $(i')\, E(x,y) = E(x,x) = E(y,y)$ if and only if $x = y$, i.e., it is a partial T-indistinguishability operator on a set X that separates points. In the literature, there are partial indistinguishability operators that are not an indistinguishability one, as for instance: $E_k(x,y) = k$, where $k \in]0,1[$. Note that, E_k is not a indistinguishability operator, since $E_k(x,x) \neq 1$ for all x.

In the last decades, the problem of merging a collection of indistinguishability operators into a single one has been addressed by different researchers (see [20, 27, 28, 29]). Concretely, in [27] the functions that aggregate indistinguishability operators have been characterized by means of triangle triplets. Recently, other characterizations of those functions that aggregate indistinguishability operators have been given by Mayor and Recasens in [23], in terms of T-triangular triplets. Also, there are other characterizations, in terms of T-triangular triplets, of those functions that aggregate, the so-called relaxed indistinguishability operators in the sense of [14] (i.e., fuzzy relations verifying only properties (ii) and (iii) of a T–indistinguishability operator) have also been given in [7].

The rest of this contribution is organized as follows. Section 2 is devoted to the characterization of functions that merge a collection of partial T_i–indistinguishability operators into another one. In the case that all T_i are equal to T, the main characterization is based on functions $F : [0,1]^n \to [0,1]$ that transform an n–dimensional $(T,\, T_M)$-tuple into a 1–dimensional $(T,\, T_M)$-tuple. The former characterization on F is equivalent to the characterization of functions $G : [0, +\infty]^n \to [0, +\infty]$ that transform an n–dimensional (+, max)-tuple into a 1–dimensional (+, max)-tuple. In Subsection 2.1, we provide that a generalized inter-exchange composition property is a sufficient condition to guarantee that a function aggregates partial T_i–indistinguishability operators. Some examples of this type of functions are also provided.

2 Aggregation of partial T-indistinguishability operators

The problem of how to combine a collection of partial T–indistinguishability operators into a single one has already been addressed by Calvo el al. in [8]. They characterized the functions $F : [0,1]^n \to [0,1]$ that merge partial T-indistinguishability operators into a new one by means of those functions that transform n-dimensional (T, T_M)-tuples into a 1-dimensional (T, T_M)-tuple, where T_M is the minimum t-norm. We recall the general idea of the above aggregation problem.

Definition 1. *A function $F : [0,1]^n \to [0,1]$ aggregates partial T–indistinguishability operators if $F(E_1, \ldots, E_n)$ is a partial T–indistinguishability operator on X for any set X and any collection $(E_1, \ldots, E_n)$ of partial T–indistinguishability operators on X, where $F(E_1, \ldots, E_n)$ is the fuzzy binary relation given by $F(E_1, \ldots, E_n)(x,y) = F(E_1(x,y), \ldots, E_n(x,y))$. The funtion F will be called partial indistinguishability aggregation function.*

According to the previous definition and the results of the aforesaid work [8], we recall that $F = T$ aggregates partial T-indistinguishability operators into a new one. In particular, given a collection $(E_i)_{i=1}^n$ of partial T_P-indistinguishability operators, where T_P is the product t-norm, we can ensure that the function $F(E_1, \ldots, E_n)(x,y) = \prod_{i=1}^{n} E_i(x,y)$ is also a partial T_P-indistinguishability operator.

Now, we recall the main characterization of the functions that merge a collection of partial T-indistinguishability into a single one (see [8]), since our new characterization is based on it. But, to achieve our purpose we need to recall also the next two definitions on (T, T_M)-tuples and to introduce the new type of $(+, \max)$-tuples.

Definition 2. *Let T be a t–norm. We say that $(a, b, c, d, d', d'') \in [0,1]^6$ is a $(T,\, T_M)$-tuple if and only if $T(a,b) \leq T(c,d)$, $T(a,c) \leq T(b,d')$, $T(c,b) \leq T(a,d'')$, $a \leq \min\{d, d'\}$, $b \leq \min\{d, d''\}$, and $c \leq min\{d'', d'\}$.*

The above definition can be extended to the n-dimentional case as follows:

Definition 3. *Let $n \in \mathbb{N}$ and let T be a t–norm and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{d}', \mathbf{d}'' \in [0,1]^n$, $n > 1$, we say that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{d}', \mathbf{d}'')$ is an (n–dimensional) $(T,\, T_M)$-tuple provided that $(a_i, b_i, c_i, d_i, d_i', d_i'')$ is a $(T,\, T_M)$-tuple for all $i = 1, \ldots, n$, where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{d} = (d_1, \ldots, d_n)$, $\mathbf{d}' = (d_1', \ldots, d_n')$, and $\mathbf{d}'' = (d_1'', \ldots, d_n'')$.*

We mention the main characterization of the functions that merge a collection of partial T-indistinguishability into a single one [8].

Theorem 1. *Let $n \in \mathbb{N}$ and let $F : [0,1]^n \to [0,1]$. The following assertions are equivalent:*

1) *F aggregates partial T–indistinguishability operators,*

2) *F transforms an n–dimensional (T, T_M)-tuple into a 1-dimensional (T, T_M)-tuple.*

Now, we introduce two new concepts that we will relate to the definitions of (T, T_M)-tuples and that allows to give another characterization of those functions that aggregate partial T-indistinguishability operators.

Definition 4. *We say that* $(a, b, c, d, d', d'') \in [0, +\infty)^6$ *is a* 1*-dimensional* $(+, \max)$*–tuple if and only if* $a + b \geq d + c$, $a + c \geq b + d'$, $b + c \geq a + d''$, $a \geq \max(d, d')$, $b \geq \max(d, d'')$, *and* $c \geq \max(d', d'')$.

The last definition can be extended to the n-dimensional case as follows:

Definition 5. *We say that* $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{d}', \mathbf{d}'') \in ([0, +\infty)^n)^6$ *, where* $n > 1$*, is an n–dimensional* $(+, \max)$*-tuple whenever* $(a_i, b_i, c_i, d_i, d_i', d_i'')$ *is a* 1*–dimensional* $(+, \max)$*-tuple, for all* $i = 1, \ldots, n$*, where* $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{d} = (d_1, \ldots, d_n)$, $\mathbf{d}' = (d_1', \ldots, d_n')$, *and* $\mathbf{d}'' = (d_1'', \ldots, d_n'')$.

In the following theorem we deal with the two kind of tuples, i.e., (T, T_M)-tuples and $(+, \max)$-tuples.

Theorem 2. *Let T be a strict and continuous Archimedean t-norm with additive generator t. A* $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{d}', \mathbf{d}'')$ *tuple of* $([0, 1]^n)^6$ *is an n–dimensional* (T, T_M)*-tuple if and only if* $(t(\mathbf{a}), t(\mathbf{b}), t(\mathbf{c}), t(\mathbf{d}), t(\mathbf{d}'), t(\mathbf{d}''))$ *is an n–dimensional* $(+, \max)$*-tuple of* $([0, +\infty)^n)^6$.

Proof. Calling $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{d} = (d_1, \ldots, d_n)$, $\mathbf{d}' = (d_1', \ldots, d_n')$, $\mathbf{d}'' = (d_1'', \ldots, d_n'')$. If $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{d}', \mathbf{d}'')$ is an n–dimensional (T, T_M)-tuple of $([0, 1]^n)^6$ we have that every $(a_i, b_i, c_i, d_i, d_i', d_i'')$, for all $i = 1, \ldots, n$ is a (T, T_M)-tuple, then

$$T(a_i, b_i) \leq T(c_i, d_i) \Leftrightarrow t(a_i) + t(b_i) \geq t(c_i) + t(d_i),$$

$$T(a_i, c_i) \leq T(b_i, d_i') \Leftrightarrow t(a_i) + t(c_i) \geq t(b_i) + t(d_i'),$$

$$T(c_i, b_i) \leq T(a_i, d_i'') \Leftrightarrow t(c_i) + t(b_i) \geq t(a_i) + t(d_i''),$$

and $a_i \leq \min(d_i, d_i')$, $b_i \leq \min(d_i, d_i'')$, $c_i \leq \min(d_i', d_i'')$ or equivalently $t(a_i) \geq \max(t(d_i), t(d_i'))$, $b_i \geq \max(t(d_i), t(d_i''))$, $c_i \geq \max(t(d_i'), t(d_i''))$.

Furthermore, $(t(a_i), t(b_i), t(c_i), t(d_i), t(d_i'), t(d_i''))$ is a 1-dimensional $(+, \max)$–tuple for all $i = 1, \ldots, n$ and therefore $(t(\mathbf{a}), t(\mathbf{b}), t(\mathbf{c}), t(\mathbf{d}), t(\mathbf{d}'), t(\mathbf{d}''))$ is an n-dimensional $(+, \max)$–tuple, where $t(\mathbf{x}) = (t(x_1), \ldots, t(x_n))$, for all $\mathbf{x} \in [0, 1]^n$. □

By means of the Theorems 1 and 2 we have the following characterization.

Theorem 3. *Let T be a strict and continuous Archimedean t-norm. A function* $F : [0, 1]^n \to [0, 1]$ *aggregates partial T-indistinguishability operators if and only if there is a function* $G : [0, +\infty)^n \to [0, +\infty)$ *defined by* $G = t \circ F \circ t^{-1}$ *that transforms an n–dimensional* $(+, \max)$*–tuple into a* 1*–dimensional* $(+, \max)$*–tuple, where t is the additive generator of the t–norm.*

Proof. (⇒) First, we assume that F aggregates partial T–indistinguishability operators which is equivalent to say that F transforms an n-dimensional (T, T_M)–tuple into a 1–dimensional (T, T_M)–tuple. Now, we will consider a set $X = \{x, y, z\}$ where x, y, z are different elements and a collection $(E_i)_{i=1}^n$ of partial T–indistinguishability operators on X. Then, the tuple $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{d}', \mathbf{d}'')$ is a (T, T_M) tuple, where $a_i = E_i(x, y) = E_i(y, x)$, $b_i = E_i(x, z) = E_i(z, x)$, $c_i = E_i(z, y) = E_i(y, z)$, $d_i = E_i(x, x)$, $d_i' = E_i(y, y)$, $d_i'' = E_i(z, z)$, for all $i = 1, \ldots, n$. Hence, taking into account the Theorem 1 we have that $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}), F(\mathbf{d}), F(\mathbf{d}'), F(\mathbf{d}''))$ is a 1–dimensional (T, T_M)–tuple, so that

$$T(F(\mathbf{a}), F(\mathbf{b})) \leq T(F(\mathbf{c}), F(\mathbf{d})),$$

$$T(F(\mathbf{a}), F(\mathbf{c})) \leq T(F(\mathbf{b}), F(\mathbf{d}')),$$

$$T(F(\mathbf{c}), F(\mathbf{b})) \leq T(F(\mathbf{a}), F(\mathbf{d}'')),$$

and

$$F(\mathbf{a}) \leq T_M(F(\mathbf{d}), F(\mathbf{d}')), F(\mathbf{b}) \leq T_M(F(\mathbf{d}), F(\mathbf{d}'')),$$

$$F(\mathbf{c}) \leq T_M(F(\mathbf{d}'), F(\mathbf{d}'')).$$

From the representation theorem of the t–norm T [22] the above inequalities are equivalent to the following ones

$$\begin{array}{rcl} t(F(\mathbf{a})) + t(F(\mathbf{b})) & \geq & t(F(\mathbf{c})) + t(F(\mathbf{d})), \\ t(F(\mathbf{a})) + t(F(\mathbf{c})) & \geq & t(F(\mathbf{b})) + t(F(\mathbf{d}')), \\ t(F(\mathbf{c})) + t(F(\mathbf{b})) & \geq & t(F(\mathbf{a})) + t(F(\mathbf{d}'')), \end{array}$$

and

$$\begin{array}{rcl} F(t^{-1}(t(\mathbf{a}))) & \leq & T_M(F(t^{-1}(t(\mathbf{d}))), F(t^{-1}(t(\mathbf{d}')))), \\ F(t^{-1}(t(\mathbf{b}))) & \leq & T_M(F(t^{-1}(t(\mathbf{d}))), F(t^{-1}(t(\mathbf{d}'')))), \\ F(t^{-1}(t(\mathbf{c}))) & \leq & T_M(F(t^{-1}(t(\mathbf{d}'))), F(t^{-1}(t(\mathbf{d}'')))). \end{array}$$

Now, considering in the first group of inequalities the following identity $t(F(\mathbf{x})) = (t \circ F \circ t^{-1})t(\mathbf{x})$ for all $\mathbf{x} \in [0, 1]^n$ and calling $G = t \circ F \circ t^{-1}$ we have that

$$G(t(\mathbf{a})) + G(t(\mathbf{b})) \geq G(t(\mathbf{c})) + G(t(\mathbf{d})),$$

$$G(t(\mathbf{a})) + G(t(\mathbf{c})) \geq G(t(\mathbf{b})) + G(t(\mathbf{d}')),$$

$$G(t(\mathbf{c})) + G(t(\mathbf{b})) \geq G(t(\mathbf{a})) + G(t(\mathbf{d}'')),$$

and since t is a non-increasing function the second group of inequalities is equivalent to the following

$$G(t(\mathbf{a})) \geq \max(G(t(\mathbf{d})), G(t(\mathbf{d}'))),$$

$$G(t(\mathbf{b})) \geq \max(G(t(\mathbf{d})), G(t(\mathbf{d}''))),$$

$$G(t(\mathbf{c})) \geq \max((G(t(\mathbf{d}')), G(t(\mathbf{d}''))),$$

with $(t(\mathbf{a}), t(\mathbf{b}), t(\mathbf{c}), t(\mathbf{d}), t(\mathbf{d}'), t(\mathbf{d}'')) \in [0,+\infty)^n$ and where $t(\mathbf{x}) = (t(x_1), \ldots, t(x_n)) \in [0,+\infty)^n$ for any $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$, which shows us that G transforms an n–dimensional $(+, \max)$–tuple into a 1–dimensional $(+, \max)$–tuple.

$(\Leftarrow)$ We have to see that F transforms an n-dimensional (T, T_M)–tuple into a 1-dimensional (T, T_M)–tuple, since this is equivalent to say that F aggregates partial T-indistinguishability operators into a new one. So that, we will consider an n–dimensional (T, T_M)–tuple, i.e., $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{d}', \mathbf{d}'')$, and applying the Theorem 2 we obtain that $(t(\mathbf{a}), t(\mathbf{b}), t(\mathbf{c}), t(\mathbf{d}), t(\mathbf{d}'), t(\mathbf{d}''))$ is an n–dimensional $(+, \max)$–tuple. Therefore, as G transforms an n–dimensional $(+, \max)$–tuple into a 1–dimensional $(+, \max)$–tuple, we have

$$G(t(\mathbf{a})) + G(t(\mathbf{b})) \geq G(t(\mathbf{c})) + G(t(\mathbf{d})),$$

$$G(t(\mathbf{a})) + G(t(\mathbf{c})) \geq G(t(\mathbf{b})) + G(t(\mathbf{d}')),$$

$$G(t(\mathbf{c})) + G(t(\mathbf{b})) \geq G(t(\mathbf{a})) + G(t(\mathbf{d}'')),$$

and

$$G(t(\mathbf{a})) \geq \max(G(t(\mathbf{d})), G(t(\mathbf{d}'))),$$

$$G(t(\mathbf{b})) \geq \max(G(t(\mathbf{d})), G(t(\mathbf{d}'))),$$

$$G(t(\mathbf{c})) \geq \max((G(t(\mathbf{d})), G(t(\mathbf{d}'))),$$

and since $G = t \circ F \circ t^{-1}$, we get easily

$$T(F(\mathbf{a}), F(\mathbf{b})) \leq T(F(\mathbf{c}), F(\mathbf{d})),$$

$$T(F(\mathbf{a}), F(\mathbf{c})) \leq T(F(\mathbf{b}), F(\mathbf{d}')),$$

$$T(F(\mathbf{c}), F(\mathbf{b})) \leq T(F(\mathbf{a}), F(\mathbf{d}'')),$$

$$F(\mathbf{a}) \leq T_M(F(\mathbf{d}), F(\mathbf{d}')), F(\mathbf{b}) \leq T_M(F(\mathbf{d}), F(\mathbf{d}'')),$$

$$F(\mathbf{c}) \leq T_M(F(\mathbf{d}'), F(\mathbf{d}'')).$$

Therefore, F transforms an n–dimensional (T, T_M)–tuple into a 1-dimensional (T, T_M)–tuple. □

The next result provides a necessary condition of the functions that transforms an n–dimensional $(+, \max)$–tuple into a 1–dimensional $(+, \max)$–tuple.

Proposition 1. *Let $G : [0,\infty)^n \to [0,\infty)$ be a function. 1) If G is an additive function, then G transforms an n–dimensional $(+, \max)$–tuple into a 1–dimensional $(+, \max)$–tuple. 2) If G transforms an n–dimensional $(+, \max)$–tuple into a 1–dimensional $(+, \max)$–tuple then G is a non-decreasing function and $G(\mathbf{e} + \mathbf{e}') \geq \frac{G(\mathbf{e})+G(\mathbf{e}')}{2}$, (inequality of Jensen).*

Proof. Let $(\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{e}, \mathbf{e}', \mathbf{e}'',)$ be an n–dimensional $(+, \max)$-tuple, then $(a'_i, b'_i, c'_i, e_i, e'_i, e''_i)$ is a 1–dimensional $(+, \max)$-tuple for all $i = 1, \ldots, n$,

$$a'_i + b'_i \geq c'_i + e_i,$$

$$a'_i + c'_i \geq b'_i + e'_i,$$

$$b'_i + c'_i \geq a'_i + e''_i,$$

and $a'_i \geq \max(e_i, e'_i)$, $b'_i \geq \max(e_i, e''_i)$, $c'_i \geq \max(e'_i, e''_i)$.

Furthermore,

$$\mathbf{a}' + \mathbf{b}' \geq \mathbf{c}' + \mathbf{e},$$

$$\mathbf{a}' + \mathbf{c}' \geq \mathbf{b}' + \mathbf{e}',$$

$$\mathbf{c}' + \mathbf{b}' \geq \mathbf{a}' + \mathbf{e}'',$$

and $\mathbf{a}' \geq \max(\mathbf{e}, \mathbf{e}')$, $\mathbf{b}' \geq \max(\mathbf{e}, \mathbf{e}'')$, $\mathbf{c}' \geq \max(\mathbf{e}', \mathbf{e}'')$. Now, considering that G is an additive function and as a consequence it is a non-decreasing function, we have

$$G(\mathbf{a}') + G(\mathbf{b}') \geq G(\mathbf{c}') + G(\mathbf{e}),$$

$$G(\mathbf{a}') + G(\mathbf{c}') \geq G(\mathbf{b}') + G(\mathbf{e}'),$$

$$G(\mathbf{c}') + G(\mathbf{b}') \geq G(\mathbf{a}') + G(\mathbf{e}''),$$

and $G(\mathbf{a}') \geq \max(G(\mathbf{e}), G(\mathbf{e}'))$, $G(\mathbf{b}') \geq \max(G(\mathbf{e}), G(\mathbf{e}''))$, $G(\mathbf{c}') \geq \max(G(\mathbf{e}'), G(\mathbf{e}''))$. Therefore, $(G(\mathbf{a}'), G(\mathbf{b}'), G(\mathbf{c}'), G(\mathbf{e}), G(\mathbf{e}'), G(\mathbf{e}''))$ is a 1-dimensional $(+, \max)$–tuple.

In the second case, we consider the following n-dimensional $(+, \max)$-tuple $(\mathbf{e} + \mathbf{e}', \mathbf{e} + \mathbf{e}'', \mathbf{e}' + \mathbf{e}'', \mathbf{e}, \mathbf{e}', \mathbf{e}'')$ for any $\mathbf{e}, \mathbf{e}', \mathbf{e}'' \in [0,+\infty]^n$. As G transforms an n-dimensional $(+, \max)$-tuple into a 1-dimensional $(+, \max)$-tuple, we have

$$\begin{array}{lll} 1) & G(\mathbf{e}+\mathbf{e}') + G(\mathbf{e}+\mathbf{e}'') & \geq G(\mathbf{e}) + G(\mathbf{e}'+\mathbf{e}''), \\ 2) & G(\mathbf{e}+\mathbf{e}') + G(\mathbf{e}'+\mathbf{e}'') & \geq G(\mathbf{e}') + G(\mathbf{e}+\mathbf{e}''), \\ 3) & G(\mathbf{e}+\mathbf{e}'') + G(\mathbf{e}'+\mathbf{e}'') & \geq G(\mathbf{e}'') + G(\mathbf{e}+\mathbf{e}'), \end{array}$$

and

$$\begin{array}{llll} 4) & G(\mathbf{e}+\mathbf{e}') & \geq \max(G(\mathbf{e}), G(\mathbf{e}')) & \geq G(\mathbf{e}), \\ 5) & G(\mathbf{e}+\mathbf{e}'') & \geq \max(G(\mathbf{e}), G(\mathbf{e}'')) & \geq G(\mathbf{e}), \\ 6) & G(\mathbf{e}'+\mathbf{e}'') & \geq \max(G(\mathbf{e}'), G(\mathbf{e}'')) & \geq G(\mathbf{e}'). \end{array}$$

From 4) or 5) or 6) we have that G is a non–decreasing function.

From 1) and 2) we get $2G(\mathbf{e} + \mathbf{e}') \geq G(\mathbf{e}) + G(\mathbf{e}')$,

From 1) and 3) we get $2G(\mathbf{e} + \mathbf{e}'') \geq G(\mathbf{e}) + G(\mathbf{e}'')$,

From 2) and 3) we get $2G(\mathbf{e}' + \mathbf{e}'') \geq G(\mathbf{e}') + G(\mathbf{e}'')$. Therefore, $G(\mathbf{e} + \mathbf{e}') \geq \frac{G(\mathbf{e})+G(\mathbf{e}')}{2}$. □

2.1 Aggregation of partial indistinguishability operators defined by different t-norms

Here, the problem is how to combine a collection $(E_i)_{i=1}^n$ of partial T_i–indistinguishability operators into a single one. To achieve this aim, we need to introduce the notion of partial T_i–indistinguishability operators aggregation function.

Definition 6. *A function $F : [0,1]^n \to [0,1]$ aggregates a collection $(E_i)_{i=1}^n$ of partial T_i–indistinguishability operators on any non-empty set X into a new partial T–indistinguishability operator if $F(E_1,\ldots,E_n)$ is partial T–indistinguishability operator on X, where $F(E_1,\ldots,E_n)$ is the fuzzy binary relation given by $F(E_1,\ldots,E_n)(x,y) = F(E_1(x,y),\ldots,E_n(x,y))$. The function F should be called partial T_i–indistinguishability operators aggregation function.*

Next, we give sufficient condition to guarantee that a function aggregates partial T_i-indistinguishability operators. In order to achieve it, we need to introduce the generalized version of the inter-exchange composition functions condition (ICFE) introduced in [8].

Definition 7. *Let T_i, $i = 1,\ldots,n$ be a family of t-norms. We will say that a t-norm T and a function $F : [0,1]^n \to [0,1]$ satisfy the generalized inter–exchange composition functions (GICFE, for short) provided that*

$$T(F(\mathbf{a}),F(\mathbf{b})) = F(T_1(a_1,b_1),\ldots,T_n(a_n,b_n)),$$

for all $\mathbf{a},\mathbf{b} \in [0,1]^n$.

Note that, if all T_i are equal to T the above condition coincides with the inter-exchange composition functions equality (ICFE) whose condition holds by every t-norm T and every i-th projection. We also know that there are functions that aggregate partial T-indistinguishability operators but they do not satisfy the ICFE (see [8]). By means of Definition 7 we can show the following.

Proposition 2. *Let E_i, $i = 1,\ldots,n$ be a family of partial T_i–indistinguishability operators, and let T be another t-norm. If $F : [0,1]^n \to [0,1]$, $n \geq 1$, is a non–decreasing function and T satisfy the generalized inter–exchange composition functions, then F aggregates the collection $(E_i)_{i=1}^n$ of partial T_i–indistinguishability operators into a partial T–indistinguishability operator.*

Proof. We need to show that $F(E_1,\ldots,E_n)$ is a partial T-indistinguishability operator. The symmetry follows directly from the symmetry of each E_i. The next inequality $F(E_1,\ldots,E_n)(x,y) \leq F(E_1,\ldots,E_n)(x,x)$ follows from the increasingness of F and from the inequality $E_i(x,y) \leq E_i(x,x)$, for all $i = 1,\ldots,n$.

Now we must to show that,

$$T(F(E_1(x,z),\ldots,E_n(x,z)),F(E_1(z,y),\ldots,E_n(z,y)))$$
$$\leq$$
$$T(F(E_1(x,y),\ldots,E_n(x,y)),F(E_1(z,z),\ldots,E_n(z,z))).$$

As each E_i is a partial T_i–indistinguishability operator we have that

$$T_i(E_i(x,z),E_i(z,y)) \leq T_i(E_i(x,y),E_i(z,z))$$

and since F is non-decreasing we obtain

$$F(T_1(E_1(x,z),E_1(z,y)),\ldots,T_n(E_n(x,z),E_n(z,y)))$$
$$\leq$$
$$F(T_1(E_1(x,y),E_1(z,z)),\ldots,T_n(E_n(x,y),E_n(z,z))).$$

Applying the given condition we have

$$T(F(E_1(x,z),\ldots,E_n(x,z)),F(E_1(z,y),\ldots,E_n(z,y))) =$$
$$F(T_1(E_1(x,z),E_1(z,y)),\ldots,T_n(E_n(x,z),E_n(z,y))) \leq$$
$$F(T_1(E_1(x,y),E_1(z,z)),\ldots,T_n(E_n(x,y),E_n(z,z))) =$$
$$T(F(E_1(x,y),\ldots,E_n(x,y)),F(E_1(z,z),\ldots,E_n(z,z))).$$

To sum up, $F(E_1,\ldots,E_n)$ is a partial T–indistinguishability operator, this means that F aggregates a collection of partial T_i–indistinguishability operators into a partial T–indistinguishability operator. □

The next example shows that there are functions that aggregate partial T_i-indistinguishability operators but they do not satisfy the GICFE.

Example 1. *Fix $k \in]0,1[$. Consider the function $F_k : [0,1]^n \to [0,1]$ defined by*

$$F_k(\mathbf{a}) = k,$$

for all $\mathbf{a} \in [0,1]^n$. It is not hard to verify that F_k aggregates a collection $(E_i)_{i=1}^n$ of partial T_i-indistinguishability operators into a partial T_P-indistinguishability operator. However, F_k, T_P and any $T_i, i = 1,\ldots,n$ do not satisfy the GICFE. Indeed,

$$k^2 = T_P(k,k) = T_P(F_k(\mathbf{a}),F_k(\mathbf{b})) <$$
$$F_k(T_1(a_1,b_1),\ldots,T_n(a_n,b_n)) = k,$$

for all $\mathbf{a},\mathbf{b} \in [0,1]^n$.

Taking into account the given results in [20], we can show the following

Proposition 3. *Let $F : [0,1]^n \to [0,1]$ be a function and $T, T_1, T_2, \ldots, T_n$ continuous Archimedean t–norms with additive generators, $t, t_1, \ldots, t_n$ respectively. If there exists an additive function $s : (\mathbb{R}^+)^n \to \mathbb{R}^+$ such that $F = t^{(-1)} \circ s \circ (t_1 \times \ldots \times t_n)$, then F aggregates a collection $(E_i)_{i=1}^n$ of partial T_i–indistinguishability operators into a partial T–indistinguishability operator.*

Proof. We need to show the three conditions of a partial T–indistinguishability operator.

To see the first condition of a partial T-indistinguishability operator, i.e., $F(E_1, \ldots, E_n)(x,y) \le F(E_1, \ldots, E_n)(x,x)$, we need to distinguish several cases, in all of them we know that each E_i satisfies $E_i(x,y) \le E_i(x,x)$ and $s \circ (t_1 \times \ldots \times t_n)$ is a non-decreasing function.

1. If $s((t_1(E_1(x,y)), \ldots, t_n(E_n(x,y)))) \le t(0)$ and $s((t_1(E_1(x,x)), \ldots, t_n(E_n(x,x)))) \le t(0)$, due to the above considerations and from the fact that $t^{(-1)} = t^{-1}$ we get

   $$\begin{aligned} &F(E_1, \ldots, E_n)(x,y) = \\ &t^{-1} \circ s \circ (t_1 \times \ldots \times t_n)(E_1, \ldots, E_n)(x,y) \le \\ &t^{-1} \circ s \circ (t_1 \times \ldots \times t_n)(E_1, \ldots, E_n)(x,x) = \\ &F(E_1, \ldots, E_n)(x,x) \end{aligned}$$

2. If $s((t_1(E_1(x,y)), \ldots, t_n(E_n(x,y)))) > t(0)$ the inequality is also true.
3. If $s((t_1(E_1(x,x)), \ldots, t_n(E_n(x,x)))) > t(0)$, as we know that

   $$\begin{aligned} &t(0) < s((t_1(E_1(x,x)), \ldots, t_n(E_n((x,x)))) \le \\ &s((t_1(E_1(x,y), \ldots, t_n(E_n(x,y)))) \end{aligned}$$

   then the inequality follows directly.

Now, we will show the third condition of a partial T-indistinguishability operator to achieve that $F(E_1, \ldots, E_n)$ is a partial T–indistinguishability operator, i.e.,

$$T(F(E_1, \ldots, E_n)(x,z), F(E_1, \ldots, E_n)(z,y)) \le T(F(E_1, \ldots, E_n)(x,y), F(E_1, \ldots, E_n)(z,z)),$$

for this we will consider the representation theorem of the t-norm T [22]

To guarantee the previous inequality we need to split the proof in two cases:

1. If T is a strict t-norm and E_i, $i = 1, \ldots, n$ are partial T_i–indistinguishability operators we know that

   $$T_i(E_i(x,z), E_i(z,y)) \le T_i(E_i(x,y), E_i(z,z))$$

   or equivalently

   $$t_i(E_i(x,z)) + t_i(E_i(z,y)) \ge t_i(E_i(x,y)) + E_i(z,z)).$$

   Considering this inequality, the following notation $\mathbf{a} = (t_1(E_1(x,z)), \ldots, t_n(E_n(x,z)))$, $\mathbf{b} = (t_1(E_1(z,y)), \ldots, t_n(E_n(z,y)))$, $\mathbf{c} = (t_1(E_1(x,y)), \ldots, t_n(E_n(x,y)))$ and $\mathbf{d} = (t_1(E_1(z,z)), \ldots, t_n(E_n(z,z)))$, and the properties of s and t^{-1} functions, we obtain

   $$t^{-1}(s(\mathbf{a}) + s(\mathbf{b})) \le t^{-1}(s(\mathbf{c}) + s(\mathbf{d})).$$

   Now, replacing the function s by $s = id \circ s = t \circ t^{-1} \circ s$, and considering the representation theorem of a t–norm [22], we get the following equivalent inequality

   $$T\big((t^{-1} \circ s)(\mathbf{a}), (t^{-1} \circ s)(\mathbf{b})\big) \le T\big((t^{-1} \circ s)(\mathbf{c}), (t^{-1} \circ s)(\mathbf{d})\big).$$

2. If T is a non-strict t-norm we split the proof as follows
   (a) If $s(t_1(E_1(x,z)), \ldots, t_n(E_n(x,z))) > t(0)$ or $s(t_1(E_1(z,y)), \ldots, t_n(E_n(z,y))) > t(0)$, then the condition (iii) is true.
   (b) If $s(t_1(E_1(x,z)), \ldots, t_n(E_n(x,z))) \le t(0)$ and $s(t_1(E_1(z,y)), \ldots, t_n(E_n(z,y))) \le t(0)$, as $t^{(-1)} = t^{-1}$ the proof follows in the same way of the strict t-norm.

Furthermore, $F(E_1, \ldots, E_n)$ also satisfies the condition (iii) of a partial T–indistinguishability operator and therefore $F(E_1, \ldots, E_n)$ is a partial T–indistinguishability operator. □

The function F of the Proposition 3 satisfies the condition of the Definition 7 as we see in the next result.

Proposition 4. *The function $F = t^{-1} \circ s \circ (t_1 \times \ldots \times t_n)$, where $t, t_1, \ldots, t_n$ are additive generators of a collection of strict continuous Archimedean t–norms $T, T_1, T_2, \ldots, T_n$, and $s : (\mathbb{R}^+)^n \to \mathbb{R}^+$ is an additive function, the following condition holds*

$$T(F(\mathbf{a})), F(\mathbf{b})) = F(T_1(a_1, b_1), \ldots, T_n(a_n, b_n)).$$

Proof. This is a matter of a simple computation. □

Observe that, in the Proposition 3 we can consider that all operators E_i, $i = 1, \ldots, n$, are partial indistinguishability operators w.r.t the same t–norm T, with additive generator t. Then, the function F should be $F = t^{(-1)} \circ s \circ (t \times \ldots \times t)$.

Remark 1. *Note that, any function* $F = t^{-1} \circ s \circ (t \times \ldots \times t)$, *where* t *is the additive generator of any strict t-norm* T *and* $s : (\mathbb{R}^+)^n \to \mathbb{R}^+$ *is an additive function, satisfies the condition ICFE given in [8] which guarantees that the function* F *aggregates a collection of partial* T*–indistinguishability operators.*

Example 2. *Considering in the above expression of* $F = t^{-1} \circ s \circ (t \times \ldots \times t)$ *the function* $s(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$, *with* $\sum_{i=1}^{n} w_i = 1$, *and taking the well-known additive generators* t *of different t-norms, we have*

1. *If* $t(x) = -\ln x$, *generator of the product t-norm, then* $F(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{w_i}$. *This function* F *is a weighted geometric mean.*
2. *If* $t(x) = \frac{1-x^\lambda}{\lambda}$ *with* $\lambda \in]-\infty, 0[$, *the generator of a Schweizer-Sklar t-norm, then* $F(x_1, \ldots, x_n) = \left(1 - \sum_{i=1}^{n} w_i(1 - x_i^\lambda)\right)^{1/\lambda}$. *This is a weighted power mean whose generator is* $g(x) = 1 - x^\lambda$.
3. *If* $t(x) = (-\ln x)^\lambda$ *with* $\lambda \in]0, +\infty[$, *generator of a Aczél–Alsina t-norm, then* $F(x_1, \ldots, x_n) = e^{-\left(\sum_{i=1}^{n} w_i(-\ln x_i)^\lambda\right)^{1/\lambda}}$. *This is a weighted quasi–arithmetic mean whose generator is* $g(x) = (-\ln x)^\lambda$.

Now, we mention a specific class of non–decreasing weighted functions that fit with the aggregation process of this work.

Proposition 5. *The function*

$$F(a_1, \ldots, a_n) = t^{-1}(\sum_{i=1}^{n} w_i t_i(a_i)) = t^{-1} \circ M_{\mathbf{w}}(t_i(a_i))$$

with $M_{\mathbf{w}}$ *denoting the weighted mean, where* $w_i > 0$, *for all* $i = 1, \ldots, n$ *and* $\sum_{i=1}^{n} w_i = 1$, *and* t, t_i *are additive generators of the given srict continuous Archimedean t-norms* T, T_i, *respectively, aggregates a collection* $(E_i)_{i=1}^n$ *of partial* T_i*-indistinguishability operators into another partial* T*-indistinguishability operator.*

Proof. The (i) and (ii) conditions of any partial T–indistinguishability operator follow directly from the same conditions of the E_i, $i = 1, \ldots, n$ partial T_i–indistinguishability operators and from the increasingness of F. From now on, we will show the condition (iii) of a partial T–indistinguishability operator. Because of any E_i, $i = 1, \ldots, n$, is a partial T_i–indistinguishability operator, we know that

$$T_i(E_i(x, z), E_i(z, y)) \le T_i(E_i(x, y), E_i(z, z))$$

and taking into account the representation theorem of the t–norm [22], we have that

$$t_i(E_i(x, z)) + t_i(E_i(z, y)) \ge t_i(E_i(x, y)) + t_i(E_i(z, z))$$

and from this

$$t^{-1}\left(\sum_{i=1}^{n} w_i t_i(E_i(x, z)) + \sum_{i=1}^{n} w_i t_i(E_i(z, y))\right) \le$$

$$t^{-1}\left(\sum_{i=1}^{n} w_i t_i(E_i(x, y)) + \sum_{i=1}^{n} w_i t_i(E_i(z, z))\right),$$

and considering that $I_d = t \circ t^{-1}$ in the above sumation we get the following equivalent expression

$$\begin{aligned} & T\left(F(E_1, \ldots, E_n)(x, z), F(E_1, \ldots, E_n)(z, y)\right) \\ &= t^{-1}\left(t(F(E_1, \ldots, E_n)(x, z)), t(F(E_1, \ldots, E_n)(z, y))\right) \\ &\le t^{-1}\left(t(F(E_1, \ldots, E_n)(x, y)), t(F(E_1, \ldots, E_n)(z, z))\right) \\ &= T\left(F(E_1, \ldots, E_n)(x, y), F(E_1, \ldots, E_n)(z, z)\right), \end{aligned}$$

furthermore $F(E_1, \ldots, E_n)$ satisfies the condition (iii) of a partial T–indistinguishability operator. Therefore, $F(E_1, \ldots, E_n)$ is a partial T–indistinguishability operator. □

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