

# Characterizing Positive Definite Matrices with t-norms

J.Recasens

Dept. Tecnologia de l'Arquitectura  
ETS Arquitectura del Vallès  
Universitat Politècnica de Catalunya  
C. Pere Serra 1-15  
j.recasens@upc.edu

## Abstract

In this work symmetric positive definite matrices with non-negative entries will be characterized by using the Yager's family of t-norms. It can always be assumed that such an  $n \times n$  matrix corresponds to a reflexive and symmetric fuzzy relation  $A$  on a set of cardinality  $n$  and then  $A$  is positive definite if and only if it is transitive with respect to a specific t-norm  $T_\lambda$  of Yager's family with  $\lambda$  depending on  $n$ . The result will be applied to give alternative proofs of the following two facts.

- Every min-indistinguishability operator separating points on a finite set has a positive definite matrix.
- Every ultrametric on a finite set is Euclidean.

**Keywords:** Positive definite matrix, Tolerance relation, Transitivity, Indistinguishability operator, Yager's family of t-norms, Euclidean metric space.

## 1 Introduction

Positive definite matrices appear in many branches of Mathematics such as in the study of quadratic forms, optimization problems and classification of quadric (hyper)-surfaces; but what is more important in Artificial Intelligence is that in linear algebra the matrix associated to an inner product of a vector space is positive definite. In this case, an Euclidean distance can be defined on this space. In many AI problems, in particular in machine learning, vision or cluster analysis, a distance on a universe  $X$  is needed [7]. In these cases, the use of kernels on  $X$  allows us to define an Euclidean-like distance  $d$  on  $X$  with all its good properties [6, 7].

There are many classical characterizations of positive definite matrices (we recall the main important two in Proposition 1.6).

Surprisingly, the use of t-norms has been proved a good tool for studying these matrices as it can be found in [5, 11, 9].

In [5, 11] it has been proved that if a reflexive and symmetric fuzzy relation is positive semi-definite, then it is transitive with respect to the continuous Archimedean t-norm  $T_{\arccos}$  having the function  $\arccos$  as additive generator; in [11], it is also proved that it is also transitive with respect to the continuous Archimedean t-norm  $T_{\sqrt{1-x}}$  having the function  $t(x) = \sqrt{1-x}$  as additive generator. The reciprocal of both assertions is not true but in [9] the following two characterizations in which an additional embedding property of metric spaces is involved can be found.

**Proposition 1.1.** *A tolerance relation  $E = (x_{ij})$  on a set  $X = \{x_1, x_2, \dots, x_n\}$  of finite cardinality  $n$  is positive definite if and only if it is a  $T_{\sqrt{1-x}}$ -indistinguishability operator and  $X$  with the distance  $d(x_i, x_j) = \sqrt{2}\sqrt{1-x_{ij}}$ ,  $1 \leq i, j \leq n$ , is embeddable into  $\mathbb{R}^n$  in such a way that the images of the points of  $X$  lie on the hypersphere  $\mathbb{S}^{n-1}$ .*

**Proposition 1.2.** *A tolerance relation  $E$  on a set  $X = \{x_1, x_2, \dots, x_n\}$  of finite cardinality is positive definite if and only if it is a  $T_{\arccos}$ -indistinguishability operator and  $X$  with the distance  $d(x_i, x_j) = \arccos x_{ij}$ ,  $1 \leq i, j \leq n$  can be mapped isometrically into  $\mathbb{S}^{n-1}$  with the spherical metric.*

In the current work symmetric positive definite matrices with non-negative entries will be characterized thanks to the Yager's family of t-norms. More precisely, after proving that the entries of a positive definite matrix  $A$  with non-negative values can be assumed to be in the unit interval, it will be characterized as being transitive with respect to a specific t-norm  $T_\lambda$  of the Yager's family where  $\lambda$  depends on the order of  $A$  (Proposition 3.5).

The rest of this section contains the basic definitions and properties of indistinguishability operator, tolerance relation and positive definite matrix.

**Definition 1.3.** [8] Let  $T$  be a  $t$ -norm and  $X$  a set. A fuzzy relation  $E$  on  $X$  is a  $T$ -indistinguishability operator if and only if for all  $x, y, z \in X$

- $E(x, x) = 1$  (Reflexivity)
- $E(x, y) = E(y, x)$  (Symmetry)
- $T(E(x, y), E(y, z)) \leq E(x, z)$  ( $T$ -transitivity).

If  $E(x, y) = 1$  if and only if  $x = y$ , then  $E$  separates points.

**Definition 1.4.** A reflexive and symmetric fuzzy relation  $E$  on  $X$  is called a proximity or tolerance relation.

If  $X = \{x_1, x_2, \dots, x_n\}$  is a finite set, then a tolerance relation  $E$  can be identified with the matrix  $E = (x_{ij})_{i,j=1,2,\dots,n}$  where  $x_{ij} = E(x_i, x_j)$ .

**Definition 1.5.** A symmetric  $n \times n$  real matrix  $A$  is positive definite if  $\vec{u}^t A \vec{u} > 0$  for every non-zero vector  $\vec{u} \in \mathbb{R}^n$  and positive semi-definite if  $\vec{u}^t A \vec{u} \geq 0$ .

There are many characterizations of positive definiteness. The next proposition recalls a couple of them and can be found in any book of linear algebra.

**Proposition 1.6.** Let  $A$  be a symmetric  $n \times n$  real matrix. The following statements are equivalent.

- a)  $A$  is positive definite.
- b) All eigenvalues of  $A$  are positive.
- c) All principal minors of  $A$  are positive, where the  $k$ -th principal minor of a matrix  $A$  is the determinant of its upper-left  $k \times k$  sub-matrix.

In the proof of Proposition 3.2 we will need the following result.

**Proposition 1.7.** The product of positive definite matrices is a positive definite matrix.

## 2 Needed Results

In this section important known results will be presented that will be used in Section 3 to our characterization.

**Definition 2.1.** Let  $X$  be a set. A mapping  $d : X \times X \rightarrow [0, \infty]$  is a pseudodistance or pseudometric if and only if for all  $x, y, z \in X$

- $d(x, x) = 0$
- $d(x, y) = d(y, x)$

- $d(x, y) + d(y, z) \geq d(x, z)$

If  $d(x, y) = 0$  if and only if  $x = y$ , then  $d$  is a distance or metric on  $X$ .

The next proposition shows the way indistinguishability operators are related to distances for continuous Archimedean  $t$ -norms.

**Proposition 2.2.** [12] Let  $T$  be a continuous Archimedean  $t$ -norm,  $t$  an additive generator of  $T$  and  $X$  a set. A fuzzy relation  $E$  on  $X$  is a  $T$ -indistinguishability operator if and only if  $t \circ E$  is a pseudodistance on  $X$ .  $E$  separates points if and only if  $t(E)$  is a distance.

**Lemma 2.3.** Let  $T$  be a  $t$ -norm,  $X$  a set,  $x, y \in X$  and  $E$  a  $T$ -indistinguishability operator on  $X$ . If  $E(x, y) = 1$ , then for all  $z \in X$ ,  $E(x, z) = E(y, z)$  (i.e.: the columns of  $E$  corresponding to  $x$  and  $y$  coincide).

*Proof.* Since  $E$  is  $T$ -transitive,

$$E(y, z) = T(E(x, y), E(y, z)) \leq E(x, z).$$

Similarly,  $E(x, z) \leq E(y, z)$ .  $\square$

The next result states that a positive definite  $T$ -indistinguishability operator must separate points.

**Proposition 2.4.** Let  $T$  be a  $t$ -norm,  $X$  a set and  $E$  a  $T$ -indistinguishability operator on  $X$  which is positive definite. Then  $E$  separates points.

*Proof.* If  $E$  does not separate points, then there exists  $x, y \in X$  ( $x \neq y$ ) with  $E(x, y) = 1$  and thanks to the previous Lemma 2.3,  $E$  has two equal columns. Therefore, its determinant is 0 and cannot be positive definite.  $\square$

**Definition 2.5.** If a metric space  $(S, d)$  is isometrically embeddable in an Euclidean space, we will say that  $d$  is Euclidean.

Corollary 2.7 provides a characterization of metric spaces isometrically embeddable into an Euclidean space. This result together with the ones in [2] recalled below will be the cornerstone to prove the characterization of positive definite matrices with the use of the Yager's family of  $t$ -norms.

**Proposition 2.6.** [10] Let  $(S, d)$ ,  $S = \{x_0, x_1, \dots, x_n\}$ , be a finite metric space of  $n + 1$  points. Then  $d$  is Euclidean if and only if the matrix  $A$  with entries  $x_{ij} = \frac{1}{2}(d_{0i}^2 + d_{0j}^2 - d_{ij}^2)$ ,  $i, j = 1, \dots, n$  where  $d_{ij}$  stands for  $d(x_i, x_j)$  is positive semi-definite.

If  $A = (x_{ij})$  is a reflexive and symmetric fuzzy relation, its associated  $n \times n$  matrix has ones in its diagonal, so that for all  $i = 1, 2, \dots, n$ , we have

$$1 = x_{ii} = \frac{d_{0i}^2 + d_{0i}^2 - d_{ii}^2}{2} = \frac{2d_{0i}^2}{2} = d_{0i}^2$$

and hence

$$d_{0i} = 1.$$

Moreover, from

$$x_{ij} = \frac{d_{0i}^2 + d_{0j}^2 - d_{ij}^2}{2} = \frac{2 - d_{ij}^2}{2}$$

we get

$$d_{ij} = \sqrt{2}\sqrt{1 - x_{ij}} \text{ for all } 1 \leq i, j \leq n.$$

**Corollary 2.7.** *Let  $E = (x_{ij})_{i,j=1,2,\dots,n}$  be a tolerance relation on a finite set  $X = \{x_1, \dots, x_n\}$ . Then  $E$  is a positive definite matrix if and only if  $d : X \cup \{x_0\} \times X \cup \{x_0\} \rightarrow [0, 1]$  defined for all  $x_i, x_j \in X$  by  $d(x_i, x_j) = \sqrt{2}\sqrt{1 - x_{ij}}$  if  $1 \leq i, j \leq n$ ,  $d(x_0, x_i) = 1$  if  $i \neq 0$  and  $d(x_0, x_0) = 0$  is a metric and  $X \cup \{x_0\}$  is isometrically embeddable in an Euclidean space.*

If we restrict the distance defined in the last corollary to  $X$ , we obtain the following result.

**Corollary 2.8.** *If  $E = (x_{ij})_{i,j=1,2,\dots,n}$  is a positive definite tolerance relation on a finite set  $X = \{x_1, \dots, x_n\}$ , then  $d : X \times X \rightarrow [0, 1]$  defined for all  $x_i, x_j \in X$  by  $d(x_i, x_j) = \sqrt{1 - x_{ij}}$  is a metric and  $X$  is isometrically embeddable in an Euclidean space.*

The reciprocal of the previous corollary is not true as is shown in the next counterexample.

**Counterexample 2.9.** *The determinant of the matrix*

$$A = \begin{pmatrix} 1 & 0.1 & 0.8 \\ 0.1 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{pmatrix}$$

*is  $\det(A) = -0.028 < 0$  and hence  $A$  is not positive definite but  $\sqrt{1 - 0.1} = 0.949$ ,  $\sqrt{1 - 0.8} = 0.447$  and  $\sqrt{1 - 0.7} = 0.548$  are the sides of a triangle and hence determine an Euclidean distance.*

The next proposition provides a relationship between distances and L-indistinguishability operators where L stands for the Lukasiewicz t-norm.

**Proposition 2.10.** *[3, 8] Let  $E : X \times X \rightarrow [0, 1]$  be a tolerance relation on a set  $X$ .  $E$  is an L-indistinguishability operator if and only if  $1 - E$  is a pseudometric on  $X$ .*

Hence, every distance can be written in the form  $1 - E$  where  $E$  is an L-indistinguishability operator (separating points).

The last result that will be needed in Section 3 is due to M. Deza and H. Maehara in [2] where the authors study the values  $c$  for which, given a set  $X$  of finite cardinality  $n$  and a distance  $d$  on  $X$ , the power of  $d$  to  $c$  ( $d^c$ ) is Euclidean. In particular they prove that

for a set of cardinality 6, the greatest value  $c_6$  of  $c$  is  $\frac{1}{2} \log_2 \frac{3}{2} \sim 0.2924$ . Of course, as the cardinality  $n$  of the set increases, the corresponding greatest value  $c_n$  decreases.

Thanks to a result by Blumenthal [1],  $c_4 = \frac{1}{2}$ .

### 3 The Characterization

In this section we will characterize positive definite matrices with non-negative entries with the use of the Yager's family of t-norms.

First, let us prove (Proposition 3.2) that we can restrict the characterization to tolerance relations (i.e.: the matrix  $A = (a_{ij})$  satisfies  $a_{ii} = 1$  and  $0 \leq a_{ij} = a_{ji} \leq 1$ ).

**Lemma 3.1.** *Let  $A = (a_{ij})$   $1 \leq i, j \leq n$  be a symmetric positive definite matrix. Then for all  $1 \leq i, j \leq n$*

- a)  $a_{ii} > 0$ .
- b)  $a_{ii} \cdot a_{jj} \geq a_{ij}^2$ .

*Proof.* a) Trivial thanks to Proposition 1.6c).

b)

$$\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = a_{ii} \cdot a_{jj} - a_{ij} \cdot a_{ji} = a_{ii} \cdot a_{jj} - a_{ij}^2 \geq 0.$$

□

**Proposition 3.2.** *Let  $A = (a_{ij})$   $1 \leq i, j \leq n$  be a symmetric matrix with non-negative entries and  $a_{ii} > 0$  for all  $1 \leq i \leq n$  and  $D = \text{diag}(\frac{1}{\sqrt{a_{11}}}, \frac{1}{\sqrt{a_{22}}}, \dots, \frac{1}{\sqrt{a_{nn}}})$  be the  $n \times n$  matrix with diagonal  $\frac{1}{\sqrt{a_{11}}}, \frac{1}{\sqrt{a_{22}}}, \dots, \frac{1}{\sqrt{a_{nn}}}$  and zeros otherwise. Then  $E = D \cdot A \cdot D$  is a positive definite tolerance relation if and only if  $A$  is a positive definite matrix.*

*Proof.*  $E = (e_{ij})$ ,  $1 \leq i, j \leq n$  is the matrix with entries  $e_{ij} = \frac{a_{ij}}{\sqrt{a_{ii}} \cdot \sqrt{a_{jj}}}$ .

- a) For  $1 \leq i \leq n$ ,  $e_{ii} = \frac{a_{ii}}{\sqrt{a_{ii}} \cdot \sqrt{a_{ii}}} = 1$  and  $E$  is reflexive.
- b) Thanks to the previous lemma,

$$1 \geq e_{ij} = \frac{a_{ij}}{\sqrt{a_{ii}} \cdot \sqrt{a_{jj}}} \geq 0.$$

If  $A$  is positive definite, then  $E$ , being the product of positive definite matrices also is positive definite (Proposition 1.7) and thanks to Lemma 3.1b),  $0 \leq e_{ij} \leq 1$ .

Reciprocally,  $A = D^{-1} \cdot E \cdot D^{-1}$  which is positive definite if  $E$  is.

□

Yager's family of t-norms is defined as follows.

**Example 3.3.** [4] The Yager's family of continuous Archimedean t-norms  $(T_\lambda)_{\lambda \in (0, \infty)}$  is defined for all  $x, y \in [0, 1]$  by

$$T_\lambda(x, y) = \max((1 - (1 - x)^\lambda + (1 - y)^\lambda)^{\frac{1}{\lambda}}, 0).$$

$t_\lambda$  defined by  $t_\lambda(x) = (1 - x)^\lambda$  for all  $x \in [0, 1]$  is an additive generator of  $T_\lambda$ .

#### Properties 3.4.

- All t-norms of Yager's family are non-strict.
- If  $\lambda > \mu$ , then  $T_\lambda(x, y) \geq T_\mu(x, y)$  for all  $x, y \in [0, 1]$ .
- If  $\lambda = 1$ , then we recover the Łukasiewicz t-norm  $L$  and  $t_1(x) = 1 - x$  is an additive generator of  $L$ .
- $\lim_{\lambda \rightarrow \infty} T_\lambda(x, y) = \min(x, y)$  for all  $x, y \in [0, 1]$ .

Now applying the results of the previous section we get the following characterization of positive definite matrices.

**Proposition 3.5. Characterization of Positive Definite Matrices.** For a given natural number  $n$ , let  $c_n$  be the greatest value satisfying that for every distance  $d$  on any finite set of cardinality  $n$ ,  $d^{c_n}$  is an Euclidean distance. Then a fuzzy relation  $E : X \times X \rightarrow [0, 1]$  on a set  $X = \{x_1, x_2, \dots, x_n\}$  of cardinality  $n$  is a  $T_{\frac{1}{2^{c_n+1}}}$ -indistinguishability operator separating points if and only if it is positive definite.

*Proof.* If  $E$  is a  $T_{\frac{1}{2^{c_n+1}}}$ -indistinguishability operator separating points, then, thanks to Proposition 2.2,  $(1 - E)^{\frac{1}{2^{c_n+1}}}$  (and also  $d = (2(1 - E))^{\frac{1}{2^{c_n+1}}}$ ) is a distance on  $X$  that can be extended to a distance  $d'$  on  $X \cup \{x_0\}$  by  $d'(x_0, x_i) = 1$  for all  $1 \leq i \leq n$  and  $d'(x_0, x_0) = 0$ . Then  $D = d'^{c_n+1}$  is:  $D(x_i, x_j) = \sqrt{2} \sqrt{1 - x_{ij}}$  if  $1 \leq i, j \leq n$ ,  $D(x_0, x_i) = 1$  if  $i \neq 0$  and  $D(x_0, x_0) = 0$ .  $D$  is Euclidean and by Corollary 2.7  $E$  is positive definite. □

$c_n$  is not known except for very few values (for  $n = 2, 3, 4, 6$ , the corresponding  $c_n$  are  $c_2 = \infty$ ,  $c_3 = 1$ ,  $c_4 = \frac{1}{2}$ ,  $c_6 = \frac{1}{2} \log_2 \frac{3}{2} \sim 0.2924$  [2]) but in [2] a lower bound  $k_n$  for  $c_n$  is given. Namely,  $k_n = \frac{1}{2n} \log_2 e \sim \frac{0.7213}{n}$ . Therefore we have the following result.

**Proposition 3.6.** If a tolerance relation  $E : X \times X \rightarrow [0, 1]$  on a set  $X$  of cardinality  $n$  is  $T_{\frac{n+1}{\log_2 e}}$ -transitive, then it is positive semi-definite.

Also in [2] it is conjectured that the value of  $c_n$  is

$$c_n = \begin{cases} \frac{1}{2} \log_2 \left( \frac{n}{n-2} \right) & \text{if } n \text{ is even} \\ \frac{1}{2} \log_2 \left( \frac{n^2-1}{n^2-2n-1} \right) & \text{if } n \text{ is odd.} \end{cases}$$

From this we can conjecture the following.

#### Conjecture 3.7.

- A fuzzy relation  $E : X \times X \rightarrow [0, 1]$  on a set  $X$  of even cardinality  $n$  is a  $T_{\frac{1}{\log_2 \left( \frac{n+1}{n-1} \right)}}$ -indistinguishability operator if and only if it is positive definite.
- A fuzzy relation  $E : X \times X \rightarrow [0, 1]$  on a set  $X$  of odd cardinality  $n$  is a  $T_{\frac{1}{\log_2 \left( \frac{n^2+2n}{n^2-2} \right)}}$ -indistinguishability operator if and only if it is positive definite.

## 4 Concluding Remarks

In this work we have obtained a characterization of positive definite matrices with non-negative entries thanks to the Yager's family of t-norms that gives a surprising application of t-norms to a classical linear-algebraic problem.

We end this work by showing that Propositions 3.5 and 3.6 provide alternative proofs of two important well known facts.

- Since  $\min(x, y) \geq T_\lambda(x, y)$  for all  $\lambda \in (0, \infty)$  and  $x, y \in [0, 1]$ , every min-indistinguishability operator on a finite set is also a  $T_\lambda$ -indistinguishability operator for all  $\lambda \in (0, \infty)$ . From Proposition 3.5 it follows the next result (see [6] for an alternative proof).

**Proposition 4.1.** Every min-indistinguishability operator separating points on a finite set is positive definite.

- It is well known that  $E$  is a min-indistinguishability operator on a set  $X$  if and only if  $1 - E$  is a pseudoultrametric [8]. By the last proposition, if  $E$  separates points, then it is positive definite and therefore  $\sqrt{1 - E}$  is Euclidean by Corollary 2.8. Since powers of ultrametrics are also ultrametrics, we obtain a new proof of this well-known result ([13]).

**Proposition 4.2.** Every ultrametric on a finite set is Euclidean.

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