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On L^M-valued F-transforms and L^M-valued fuzzy rough sets

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Abstract

The goal of this contribution is to introduce and study the theory of L^M-valued F-transforms, which is essentially the many-valued generalizations of the notion of lattice-valued F-transforms. Further, we associate the concepts of L^M-valued fuzzy rough sets with L^M-valued F-transforms and obtain some interesting results. Moreover, we show that the notion of F-transforms studied earlier can be considered as the particular case of L^M-valued F-transforms.
Keywords: L^M-valued fuzzy set, L^M-valued fuzzy partitions, L^M-valued F-transforms.

1 Introduction

Since the inception of notion of fuzzy transform (F-transform), by Perfilieva [7], the theory attracted interest of many of the researchers. It has now been significantly developed and opened a new page in the theory of semi-linear spaces. The main idea of the F-transform is to factorize (or fuzzify) the precise values of independent variables by a closeness relation, and precise values of dependent variables are averaged to an approximate value. The theory of F-transform has already been elaborated and extended from real valued to lattice-valued functions (cf., [7], [8]), from fuzzy sets to parametrized fuzzy sets (cf., [15]) and from the single variable to the two (or more variables) (cf., [1], [2], [16]). The various studies carried out in the line of applications of F-transform, e.g., denoising [11], scheduling [4], trading [18], time series [6], numerical solutions of partial differential equations [3], data analysis [9], and neural network approaches [17].

The theory of L-valued fuzzy transform was further studied by a number of researchers. Recently, the F-transforms for functions in two variables is studied in [12] and the relationship of fuzzy transforms with fuzzy rough sets and fuzzy topologies is studied in [10], while mathematical morphology is studied in [14]. In continuation to such studies, in this paper, we introduce and study the theory of L^M-valued F-transforms.

The structure of the paper is the following. In the Section 2, some basic definitions, L^M-valued fuzzy sets, L^M-valued fuzzy relations and L^M-valued fuzzy rough sets are recalled/introduced. L^M-valued fuzzy partitions and L^M-valued direct F-transforms are introduced in Section 3. Finally, in Section 4, we discuss the concept of L^M-valued inverse F-transforms.

2 Preliminaries

In this section, we recall the concepts of residuated lattices, L^M-valued fuzzy sets and L^M-valued rough sets, which will be used in the main text of the paper.

Definition 2.1 [5] *A* **residuated lattice** L *is an algebra* $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *such that*

1. $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice with the least element* 0 *and the greatest element* 1*;*
2. $(L, \otimes, 1)$ *is a commutative monoid and*
3. $\forall\, a, b, c \in L$*;*

$$a \otimes b \leq c \text{ iff } a \leq b \rightarrow c,$$

i.e. $(\rightarrow, \otimes)$ *is an adjoint pair on* L.

A residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is **complete** if it is complete as a lattice.

Proposition 2.1 [5] *Let* $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *be a residuated lattice. Then for all* $a, b, c \in L$,

1. $a \rightarrow 1 = 1, 1 \rightarrow a = a$*;*

2. $a \leq (b \to a \otimes b)$;

3. $a \otimes (a \to b) \leq b$;

4. $a \otimes (\bigvee_{j\in J} b_j) = \bigvee_{j\in J} (a \otimes b_j)$;

5. $a \to \bigwedge_{j\in J} b_j = \bigwedge_{j\in J} (a \to b_j)$;

6. $\bigvee_{j\in J} a_j \to b = \bigwedge_{j\in J} (a_j \to b)$.

In the following, by M, we denote a complete infinitely distributive lattice $(M, \leq_M, \wedge_M, \vee_M)$. The least and greatest elements of M are denoted by 0_M and 1_M respectively. As different from the lattice L, we do not exclude here the trivial case, when M is an one-element lattice $\bullet$ and hence in this case $0_M = 1_M$. For details on the theory associated with L^M-valued fuzzy set, we refer [13].

Now, we recall the definitions of L^M-valued fuzzy sets and L^M-valued fuzzy relations from [13].

Definition 2.2 *Let $L = (L, \leq, \wedge, \vee, \otimes, \to)$ be a residuated lattice and $M = (M, \leq, \wedge, \vee)$ be a complete infinitely distributive lattice. Then an* L^M**-valued fuzzy set** *λ of a nonempty set X is a map $\lambda : X \to L^M$.*

Remark 2.1 *It is to note here that an L^M-valued set λ can also be represented as the family $\{\lambda^\alpha \in L^X : \alpha \in M\}$ of L-valued sets of X, which can be ordered by the elements of M, where $\lambda^\alpha(x) = \lambda(x)(\alpha)$. Further, for fixed $M = \bullet$ and $L = [0, 1]$, the L^M-valued set λ turns out to be a fuzzy set.*

Definition 2.3 *Let X be a nonempty set and $\lambda^\alpha : X \to L, \alpha \in M$ be an L-valued fuzzy subset of X. The* **core** *of the λ^α is defined as a classical set*

$$core(\lambda^\alpha) = \{x \in X : \lambda^\alpha(x) = 1, \alpha \in M\}.$$

If $core(\lambda^\alpha) \neq \phi$ then λ^α is called a L-valued normal fuzzy set.

Also, for all $a \in L$, $\underline{a} \in L^X$ such that $\underline{a}(x) = a,\ x \in X$ denotes L- valued constant fuzzy set.

Proposition 2.2 *Let $\nu, \mu \in L^M$ and $\alpha \in M$. The following are the induced operations of multiplication $\otimes$, implication $\to$, intersection $\wedge$, union $\vee$ and negation $\neg$ from L to L^M.*

1. $(\nu \otimes \mu)(\alpha) = \nu(\alpha) \otimes \mu(\alpha)$;

2. $(\nu \to \mu)(\alpha) = \nu(\alpha) \to \mu(\alpha)$;

3. $(\nu \vee \mu)(\alpha) = \nu(\alpha) \vee \mu(\alpha)$;

4. $(\nu \wedge \mu)(\alpha) = \nu(\alpha) \wedge \mu(\alpha)$;

5. $\neg\nu(\alpha) = \nu(\alpha) \to 0$.

Now, we define the concept of L^M-valued fuzzy relation by extending the notion of L-valued fuzzy relation to its many-valued setting.

Definition 2.4 *Let X be a nonempty set. An* L^M**-fuzzy relation** *R is a map $R : X \times X \to L^M$. Further, R is said to be*

1. **reflexive** *if $R(x, x)(\alpha) = 1_L$, for all $x \in X, \alpha \in M$;*

2. **transitive** *if $R(x, y)(\alpha) \otimes R(y, z)(\alpha) \leq R(x, z)(\alpha)$, for all $x, y, z \in X, \alpha \in M$.*
   A reflexive and transitive L^M-valued fuzzy relation R is called an L^M**-valued fuzzy preorder**.

Next, we introduce the concept of L^M-valued fuzzy approximation operators by considering the L^M-valued fuzzy sets and L^M-valued fuzzy preorder relations. Such concepts can be considered as a many-valued version of the notion of L-valued fuzzy approximation operators and differs from [13] in the sense of nature of L^M-valued fuzzy relations.

Definition 2.5 *Let X be a nonempty set and R be an L^M-valued fuzzy relation on X. A pair (X, R) is called an* L^M**-valued fuzzy approximation space**.

Definition 2.6 *Let λ be an L-valued fuzzy set and $R : X \times X \to L^M$ be an L^M-valued fuzzy relation on a set X. The* L^M**-valued upper(lower) fuzzy approximation operator** *is a map $\overline{R}(\underline{R}) : L^X \to (L^M)^X$ and defined as follows:*

$$\overline{R}(\lambda)(y)(\alpha) = \bigvee_{x\in X} (R(x, y)(\alpha) \otimes \lambda(x)),$$

$$\underline{R}(\lambda)(x)(\alpha) = \bigwedge_{y\in X} (R(x, y)(\alpha) \to \lambda(y))$$

Remark 2.2 *We can interpret the L^M-valued upper(lower) fuzzy approximation operator as a map $\overline{R}(\underline{R}) : L^X \to L^{M\times X}$. It can also be expressed as a family of L-valued upper(lower) fuzzy approximation operators $\{\overline{R}^\alpha(\underline{R}^\alpha) : L^X \to L^X : \alpha \in M\}$ defined as:*

$$\overline{R}^\alpha(\lambda)(x) = \overline{R}(\lambda)(x)(\alpha),$$

$$\underline{R}^\alpha(\lambda)(x) = \underline{R}(\lambda)(x)(\alpha), \forall \lambda \in L^X, \forall x \in X.$$

Proposition 2.3 *Let (X, R) be an L^M-valued fuzzy approximation space and $\overline{R}, \underline{R} : L^X \to (L^M)^X$ be the induced upper (lower) L^M-valued fuzzy approximation operator. Then for $\lambda \in L^X, \alpha, \beta \in M, x \in X$ and $a \in L$;*

1. $\overline{R}(0_X)(x, 0_M) = 0_L, \underline{R}(1_X)(x, \alpha) = 1_L$;
2. $\overline{R}(\bigvee_{j\in J} \lambda_j) = \bigvee_{j\in J} \overline{R}(\lambda_j), \underline{R}(\bigwedge_{j\in J} \lambda_j) = \bigwedge_{j\in J} \underline{R}(\lambda_j)$;
3. $\overline{R}(\underline{a} \otimes \lambda)(x, \alpha) = \underline{a} \otimes \overline{R}(\lambda)(x, \alpha)$,
   $\underline{R}(\underline{a} \to \lambda)(x, \alpha) = \underline{a} \to \underline{R}(\lambda)(x, \alpha)$;
4. R is reflexive $\Longleftrightarrow \underline{R}^\alpha(\lambda) \leq \lambda \Longleftrightarrow \lambda \leq \overline{R}^\alpha(\lambda)$.

Now, we show the relationship between the classes of L^M-valued fuzzy relations and properties of L^M-valued fuzzy approximation operators.

Proposition 2.4 *Let (X, R) be an L^M-valued fuzzy approximation space and R be reflexive. Then $\underline{R}(0_X)(x, 0_M) = 0_L$ and $\overline{R}(1_X)(x, \alpha) = 1_L$.*

3 L^M-valued fuzzy partitions and L^M-valued F-transforms

In this section, we introduce and study the notion of L^M-valued fuzzy partitions and L^M-valued F-transforms. Moreover, we discuss some properties of L^M-valued F-transforms.

Definition 3.1 *A collection $\mathit{\Pi}_X$ of L^M-valued normal fuzzy sets $\{A_\xi : \xi \in \Lambda\}$ in X is an L^M-**valued fuzzy partition** of X, if the corresponding collection of family of ordinary sets $\{core(A^\alpha_\xi) : \xi \in \mathit{\Lambda}, \alpha \in M\}$ is a partition of X. A pair $(X, \mathit{\Pi}_X)$, where $\mathit{\Pi}_X$ is an L^M-valued fuzzy partition of X, is called a **space with an L^M-valued fuzzy partition**.*

Remark 3.1 *We can interpret an L^M-valued fuzzy set $A_\xi : X \to L^M$ as a map $A_\xi : X \times M \to L$ defined by $A^\alpha_\xi(x) = A_\xi(x)(\alpha)$.*

Let $\mathit{\Pi}_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X. Then it can be represented by the following L^M-valued reflexive fuzzy relation $R_{\mathit{\Pi}_X}$, where

$$R^\alpha_{\mathit{\Pi}_X}(x, y) = R_{\mathit{\Pi}_X}(x, y)(\alpha) = A_\xi(x)(\alpha), \text{if } y \in core(A^\alpha_\xi). \tag{3.1}$$

The above L^M-valued fuzzy relation $R_{\mathit{\Pi}_X}$ can be decomposed into constituent L^M-valued fuzzy relations R_ξ, where $R^\alpha_\xi(x, y) = R_\xi(x, y)(\alpha)$. Now, for all $\xi \in \Lambda$,

$$R_\xi(x, y)(\alpha) = \begin{cases} A_\xi(x)(\alpha), & \text{if } y \in core(A^\alpha_\xi), \\ 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

From [10], it can be prove that the L^M-valued fuzzy relation R_ξ is an L^M-valued fuzzy preorder.

Remark 3.2 *It is easy to see that for any L^M-valued fuzzy partition $\mathit{\Pi}_X = \{A_\xi : \xi \in \Lambda\}$ of X, the representing L^M-valued fuzzy relation $R_{\mathit{\Pi}_X}$ can be decomposed into the union of L^M-valued fuzzy preorder relations R_ξ, i.e.,*

$$R_{\mathit{\Pi}_X} = \bigcup_{\xi\in\mathit{\Lambda}} R_\xi. \tag{3.3}$$

Now, we introduce the concept of L^M-valued F-transform based on the notion of L^M-valued fuzzy partition. The newly introduced F-transforms are essentially the many-valued generalizations of the notion of F-transforms studied earlier. We begin with the following notion of L^M-valued direct F-transform.

Definition 3.2 *Let $\lambda \in L^X$ and $\mathit{\Pi}_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X. The L^M-**valued direct $F^\uparrow$-transform** of λ with respect to L^M-valued fuzzy partition $\mathit{\Pi}_X = \{A_\xi : \xi \in \Lambda\}$ is an L^M-valued fuzzy set $F^\uparrow_\xi[\lambda]$ defined as*

$$F^\uparrow_\xi[\lambda](\alpha) = \bigvee_{x\in X} (\lambda(x) \otimes A_\xi(x)(\alpha)).$$

Remark 3.3 *We can represent the above expression as a collection of family of lattice elements $\{F^\uparrow_{\xi,\alpha} : \xi \in \Lambda, \alpha \in M\}$, and defined as*

$$F^\uparrow_{\xi,\alpha}[\lambda] = F^\uparrow_\xi[\lambda](\alpha), \forall\lambda \in L^X.$$

In Proposition 3.1, we show that the L^M-valued direct $F^\uparrow$-transform of λ with respect to L^M-valued fuzzy partition $\mathit{\Pi}_X$ is included into the range of the L^M-valued upper fuzzy approximation of λ in the space $(X, R_{\mathit{\Pi}})$.

Proposition 3.1 *Let $\mathit{\Pi}_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X, represented by L^M-valued fuzzy relation $R_{\mathit{\Pi}_X}$ such that $R_{\mathit{\Pi}_X} = \bigcup_{\xi\in\mathit{\Lambda}} R_\xi$. Then*

1. *in the L^M-valued fuzzy approximation space $(X, R_{\mathit{\Pi}_X})$, the L^M-valued upper fuzzy approximation $\overline{R}_{\mathit{\Pi}_X}(\lambda)$ of $\lambda \in L^X$ is determined by the L^M-valued direct $F^\uparrow$-transform of λ with respect to $\mathit{\Pi}_X$, i.e., for all $y \in X$, there exists $\xi_y \in \Lambda$ such that*
$$\overline{R}^\alpha_{\mathit{\Pi}_X}(\lambda)(y) = F^\uparrow_{\xi_y,\alpha}[\lambda], \tag{3.4}$$
2. *in the L^M-valued fuzzy approximation space (X, R_ξ), the L^M-valued upper fuzzy approximation $\overline{R}_\xi(\lambda)$ of $\lambda \in L^X$ is given by*

$$\overline{R}^\alpha_\xi(\lambda)(y) = \begin{cases} F^\uparrow_{\xi,\alpha}[\lambda], & y \in core(A^\alpha_\xi) \\ \lambda(y), & otherwise, \end{cases} \tag{3.5}$$

3. *for all* $\xi \in \Lambda$,

$$\overline{R}_\xi(\lambda) \leq \overline{R}_{\Pi_X}(\lambda). \quad (3.6)$$

Proof:

1. Suppose for any $y \in X$, there exists $\xi_y \in \Lambda$ such that $y \in core(A^\alpha_{\xi_y})$. By using (3.1), we have

$$R_{\Pi_X}(x,y)(\alpha) = A_{\xi_y}(x)(\alpha).$$

Thus

$$\begin{aligned}\overline{R}^\alpha_{\Pi_X}(\lambda)(y) &= \bigvee_{x\in X} (\lambda(x) \otimes R_{\Pi_X}(x,y)(\alpha)) \\ &= \bigvee_{x\in X} (\lambda(x) \otimes A_{\xi_y}(x)(\alpha)) \\ &= F^\uparrow_{\xi_y,\alpha}[\lambda].\end{aligned}$$

2. Let us fix some $\xi \in \Lambda$ and consider (X, R_ξ). Suppose that $y \in core(A^\alpha_\xi)$. By (3.2), we get

$$\begin{aligned}\overline{R}^\alpha_\xi(\lambda)(y) &= \bigvee_{x\in X} (\lambda(x) \otimes R_\xi(x,y)(\alpha)) \\ &= \bigvee_{x\in X} (\lambda(x) \otimes A_\xi(x)(\alpha)) \\ &= F^\uparrow_{\xi,\alpha}[\lambda].\end{aligned}$$

On the other hand, if $y \notin core(A^\alpha_\xi)$, then

$$\begin{aligned}\overline{R}^\alpha_\xi(\lambda)(y) &= \bigvee_{x\in X} (\lambda(x) \otimes R_\xi(x,y)(\alpha) \\ &\geq \lambda(y) \otimes R_\xi(y,y)(\alpha) \\ &= \lambda(y).\end{aligned}$$

3. Let $\xi \in \Lambda$. If $y \in core(A^\alpha_\xi)$, $\alpha \in M$, then by (3.4) and (3.5), $\overline{R}^\alpha_\xi(\lambda)(y) = \overline{R}^\alpha_{\Pi_X}(\lambda)(y)$. If $y \notin core(A^\alpha_\xi)$, then (3.6) follows from Proposition 2.3.

From the Proposition 3.1, it is clear that an L^M-valued upper fuzzy approximation $\overline{R}_\xi(\lambda)$ of λ closely corresponds to the ξ-th L^M-valued direct $F^\uparrow$-transform approximation of λ. Therefore the ξ-th components of $F^\uparrow[\lambda]$ satisfy all properties of L^M-valued upper fuzzy approximation operators discussed above. Now, we have the following.

Proposition 3.2 *Let $\lambda, \rho \in L^X$ and $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X. Then for all $\xi \in \Lambda, a \in L$ and $\alpha \in M$.*

1. $F^\uparrow_{\xi,\alpha}[\underline{a}] = a$,

2. $F^\uparrow_{\xi,\alpha}[\lambda] \leq F^\uparrow_{\xi,\alpha}[\rho]$, *if* $\lambda \leq \rho$,

3. $F^\uparrow_{\xi,\alpha}[\lambda \vee \rho] = F^\uparrow_{\xi,\alpha}[\lambda] \vee F^\uparrow_{\xi,\alpha}[\rho]$,

4. $F^\uparrow_{\xi,\alpha}[\underline{a} \otimes \lambda] = a \otimes F^\uparrow_{\xi,\alpha}[\lambda]$,

5. $\lambda(x_\xi) \leq F^\uparrow_{\xi,\alpha}[\lambda]$, *if* $x_\xi \in core(A^\alpha_\xi)$.

Next, we introduce the following concept of an L^M-valued direct $F^\downarrow$-transform.

Definition 3.3 *Let $\lambda \in L^X$ and $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X. The* **L^M-valued direct $F^\downarrow$-transform** *of λ with respect to L^M-valued fuzzy partition $\Pi_X = \{A_\xi : \xi \in \Lambda\}$, is an L^M-valued fuzzy set $F^\downarrow_\xi[\lambda]$ and defined as*

$$F^\downarrow_\xi[\lambda](\alpha) = \bigwedge_{y\in X} (A_\xi(y)(\alpha) \rightarrow \lambda(y)).$$

Remark 3.4 *We can represent the above expression as a collection of family of lattice elements $\{F^\downarrow_{\xi,\alpha} : \xi \in \Lambda, \alpha \in M\}$, and defined as*

$$F^\downarrow_{\xi,\alpha}[\lambda] = F^\downarrow_\xi[\lambda](\alpha), \forall \lambda \in L^X.$$

In Proposition 3.3, we show that the L^M-valued direct $F^\downarrow$-transform of λ w.r.t. L^M-valued fuzzy partition Π_X is included into the range of the L^M-valued lower fuzzy approximation of λ in the space $(X, R^T_{\Pi_X})$, where $R^T_{\Pi_X}$ is the transpose of R_{Π_X}.

Proposition 3.3 *Let $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X, represented by L^M-valued relation R_{Π_X}, such that $R_{\Pi_X} = \bigcup_{\xi\in\Lambda} R_\xi$. Then,*

1. *in the L^M-valued fuzzy approximation space $(X, R^T_{\Pi_X})$, the L^M-valued lower fuzzy approximation $\underline{R}^T_{\Pi_X}(\lambda)$ of $\lambda \in L^X$ is determined by the L^M-valued direct $F^\downarrow$-transform of λ with respect to Π_X, i.e., for any $x \in X$, there exists $\xi_x \in \Lambda$ such that*

$$\underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x) = F^\downarrow_{\xi_x,\alpha}[\lambda], \quad (3.7)$$

2. *in the L^M-valued fuzzy approximation space (X, R^T_ξ), the L^M-valued upper fuzzy approximation $\underline{R}^T_\xi(\lambda)$ of $\lambda \in L^X$ is given by*

$$\underline{R}^{T,\alpha}_\xi(\lambda)(x) = \begin{cases} F^\downarrow_{\xi,\alpha}[\lambda], & x \in core(A^\alpha_\xi) \\ \lambda(x), & otherwise, \end{cases} \quad (3.8)$$

3. *for all* $\xi \in \Lambda$,

$$\underline{R}^T_\xi(\lambda) \geq \underline{R}^T_{\Pi_X}(\lambda). \quad (3.9)$$

Proof:

1. Let us assume for $x \in X$ there exists $\xi_x \in \Lambda$ such that $x \in core(A^\alpha_{\xi_x})$. From the (3.1),
$$R^T_{\Pi_X}(x,y)(\alpha) = A_{\xi_x}(y)(\alpha).$$
Thus
$$\begin{aligned}\underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x) &= \bigwedge_{y\in X}(R^T_{\Pi_X}(x,y)(\alpha) \to \lambda(y))\\ &= \bigwedge_{y\in X}(A_{\xi_x}(y)(\alpha) \to \lambda(y))\\ &= F^{\downarrow}_{\xi_x,\alpha}[\lambda].\end{aligned}$$

2. Let $\xi \in \Lambda$ and consider (X, R^T_ξ). Suppose that $x \in core(A^\alpha_\xi)$. By (3.2),
$$\begin{aligned}\underline{R}^{T,\alpha}_{\xi}(\lambda)(x) &= \bigwedge_{y\in X}(R^T_{\xi}(x,y)(\alpha) \to \lambda(y))\\ &= \bigwedge_{y\in X}(A_{\xi}(y)(\alpha) \to \lambda(y))\\ &= F^{\downarrow}_{\xi,\alpha}[\lambda].\end{aligned}$$
On the other hand, if $x \notin core(A^\alpha_\xi)$, then
$$\begin{aligned}\underline{R}^{T,\alpha}_{\xi}(\lambda)(x) &= \bigwedge_{y\in X}(R^T_{\xi}(x,y)(\alpha) \to \lambda(y))\\ &\leq R^T_\xi(x,x)(\alpha) \to \lambda(x)\\ &= \lambda(x).\end{aligned}$$
Therefore (3.8) is proved.

3. Let us assume $x \in core(A^\alpha_\xi)$, $\xi \in \Lambda$, $\alpha \in M$, then by (3.7) and (3.8), $\underline{R}^{T,\alpha}_{\xi}(\lambda)(x) = \underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x)$. If $x \notin core(A^\alpha_\xi)$, then (3.9) follows from Proposition 2.3.

On the basis of Proposition 3.3, the L^M-valued lower fuzzy approximation $\underline{R}^\alpha_\xi(\lambda)$ is called the ξ-th L^M-valued direct $F^\downarrow$-transform approximation of λ.

Proposition 3.4 *Let $\lambda, \rho \in L^X$ and $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X. Then for all $\xi \in \Lambda, a \in L$ and $\alpha \in M$.*

(1) $F^\downarrow_{\xi,\alpha}[\underline{a}] = a$,

(2) $F^\downarrow_{\xi,\alpha}[\lambda] \leq F^\downarrow_{\xi,\alpha}[\rho]$ *if* $\lambda \leq \rho$,

(3) $F^\downarrow_{\xi,\alpha}[\lambda \wedge \rho] = F^\downarrow_{\xi,\alpha}[\lambda] \wedge F^\uparrow_{\xi,\alpha}[\rho]$,

(4) $\lambda(x_\xi) \geq F^\downarrow_{\xi,\alpha}[\lambda]$ *if* $x_\xi \in core(A^\alpha_\xi)$,

(5) $F^\downarrow_{\xi,\alpha}[\underline{a} \to \lambda] = a \to F^\downarrow_{\xi,\alpha}[\lambda]$.

4 L^M-valued inverse $F^\uparrow(F^\downarrow)$-transform

This section introduces the concept of L^M-valued inverse $F^\uparrow(F^\downarrow)$-transform. Further, we show that L^M-valued inverse $F^\uparrow(F^\downarrow)$-transform can be expressed in terms of a combination of L^M-valued lower and upper (upper and lower) fuzzy approximation operators.

Definition 4.1 *Let $\lambda \in L^X$, $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X and $F^\uparrow[\lambda]$ be the family of L^M-valued direct $F^\uparrow$-transforms of λ with respect to Π_X. Then the L^M-**valued inverse** $F^\uparrow$**-transform** is a map $\hat{\lambda}^\uparrow : (L^M)^X \to L^X$ such that*
$$\hat{\lambda}^\uparrow_\alpha(x) = \bigwedge_{\xi\in\Lambda,\alpha\in M}(A_\xi(x)(\alpha) \to F^\uparrow_{\xi,\alpha}[\lambda]), \forall x \in X. \tag{4.1}$$

Proposition 4.1 *Let L^M-valued fuzzy partition $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ of X be represented by L^M-valued fuzzy relation R_{Π_X} and (X, R_{Π_X}) be the corresponding L^M-valued fuzzy approximation space. Then the L^M-valued inverse $F^\uparrow$-transform of $\lambda \in L^X$ is given by*
$$\hat{\lambda}^\uparrow_\alpha(x) = \bigwedge_{y\in X}(R_{\Pi_X}(x,y)(\alpha) \to \overline{R}^\alpha_{\Pi_X}(\lambda)(y)), \tag{4.2}$$
and therefore
$$\hat{\lambda}^\uparrow_\alpha = \underline{R}^\alpha_{\Pi_X}(\overline{R}^\alpha_{\Pi_X}(\lambda)). \tag{4.3}$$

Proof: Let the given assumptions hold and $\xi \in \Lambda$. Then as $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ is an L^M-valued fuzzy partition of X and $\{core(A^\alpha_\xi) : \xi \in \Lambda\}$ is a classical partition of X, for all $\xi \in \Lambda, \alpha \in M$, there exists $y_\xi \in core(A^\alpha_\xi)$. Now, from Equations (3.1) and (3.4), $R_{\Pi_X}(x, y_\xi)(\alpha) = A_\xi(x)(\alpha)$ and $\overline{R}^\alpha_{\Pi_X}(\lambda)(y_\xi) = F^\uparrow_{\xi,\alpha}[\lambda]$. Therefore
$$\begin{aligned}\hat{\lambda}^\uparrow_\alpha(x) &= \bigwedge_{\xi\in\Lambda,\alpha\in M}(A_\xi(x)(\alpha) \to F^\uparrow_{\xi,\alpha}[\lambda])\\ &= \bigwedge_{y_\xi\in X}(R_{\Pi_X}(x,y_\xi)(\alpha) \to \overline{R}^\alpha_{\Pi_X}(\lambda)(y_\xi)).\end{aligned}$$

It is clear that for any $y_\xi \in core(A^\alpha_\xi)$ the value of $(R_{\Pi_X}(x,y_\xi)(\alpha) \to \overline{R}^\alpha_{\Pi_X}(\lambda)(y_\xi))$ is same. Therefore
$$\hat{\lambda}^\uparrow_\alpha(x) = \bigwedge_{y\in X}(R_{\Pi_X}(x,y)(\alpha) \to \overline{R}^\alpha_{\Pi_X}(\lambda)(y)).$$

Equation (4.3) is the direct consequence of the representation (4.2) and Definition 2.6.

Definition 4.2 *Let $\lambda \in L^X$, $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ be an L^M-valued fuzzy partition of X and $F^\downarrow[\lambda]$ be the family of L^M-valued direct $F^\downarrow$-transforms of λ with respect to*

Π_X. *Then the* L^M**-valued inverse** $F^{\downarrow}$**-transform** *is a map* $\hat{\lambda}^{\downarrow} : (L^M)^X \to L^X$ *such that*

$$\hat{\lambda}^{\downarrow}_{\alpha}(y) = \bigwedge_{\xi \in \Lambda, \alpha \in M} (A_{\xi}(y)(\alpha) \otimes F^{\downarrow}_{\xi,\alpha}[\lambda]), \forall y \in X. \quad (4.4)$$

Proposition 4.2 *Let* L^M*-valued fuzzy partition* $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ *of* X *be represented by* L^M*-valued fuzzy relation* R_{Π_X} *and* (X, R_{Π_X}) *be the corresponding* L^M*-valued fuzzy approximation space. Then the* L^M*-valued inverse* $F^{\downarrow}$*-transform of* $\lambda \in L^X$ *is given by*

$$\hat{\lambda}^{\downarrow}_{\alpha}(y) = \bigvee_{x \in X} (R^T_{\Pi_X}(x, y)(\alpha) \otimes \underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x)), \quad (4.5)$$

and therefore

$$\hat{\lambda}^{\downarrow}_{\alpha} = \overline{R^{T,\alpha}_{\Pi_X}}(\underline{R}^{T,\alpha}_{\Pi_X}(\lambda)). \quad (4.6)$$

Proof: Let the given assumptions hold and $\xi \in \Lambda$. Then as $\Pi_X = \{A_\xi : \xi \in \Lambda\}$ is an L^M-valued fuzzy partition of X and $\{core(A^{\alpha}_{\xi}) : \xi \in \Lambda\}$ is a classical partition of X, for all $\xi \in \Lambda, \alpha \in M$, there exists $x_\xi \in core(A^{\alpha}_{\xi})$. Now, from Equations (3.1) and (3.7), $R^T_{\Pi_X}(x_\xi, y)(\alpha) = A_\xi(y)(\alpha)$ and $\underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x_\xi) = F^{\downarrow}_{\xi,\alpha}[\lambda]$. Therefore

$$\begin{aligned} \hat{\lambda}^{\downarrow}_{\alpha}(y) &= \bigvee_{\xi \in \Lambda, \alpha \in M} (A_\xi(y)(\alpha) \otimes F^{\downarrow}_{\xi,\alpha}[\lambda]) \\ &= \bigvee_{x_\xi \in X} (R^T_{\Pi_X}(x_\xi, y)(\alpha) \otimes \underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x_\xi)). \end{aligned}$$

For any choice of $x_\xi \in core(A^{\alpha}_{\xi})$ the value $(R^T_{\Pi_X}(x_\xi, y)(\alpha) \otimes \underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x_\xi))$ is same. Therefore

$$\hat{\lambda}^{\downarrow}_{\alpha}(y) = \bigvee_{x \in X} (R^T_{\Pi_X}(x, y)(\alpha) \otimes \underline{R}^{T,\alpha}_{\Pi_X}(\lambda)(x)).$$

Equation (4.6) is the direct consequence of the representation (4.5) and Definition 2.6.

5 Conclusion

In this contribution, we have introduced the notion of L^M-valued fuzzy partition and L^M-valued F-transforms. Further, we have established a relationship between L^M-valued fuzzy rough sets and L^M-valued F-transforms.

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