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Many-level fuzzy rough approximation spaces induced by many-level fuzzy preorders and the related ditopological structures

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Abstract

We present a many-level version for the Pawlak – Dubois&Prade theory of rough approximation of fuzzy sets. Basing on the many-level upper and lower fuzzy rough approximation operators, we define the measure of rough approximation that in a certain sense characterizes the quality of the obtained approximation. Further, the fuzzy rough approximation operators give rise to two alternative topological-type structures considered in the paper.

Keywords: Many-level fuzzy rough approximation operators, measure of fuzzy rough approximation, LM-fuzzy (di)topologies, M-level L-fuzzy (di)-topologies.

1 Introduction

A problem that became especially actual in the last quarter of the previous century was to deal with information systems using big volumes of data and other similar situations. Answering this challenge, Z. Pawlak in his celebrated paper [11] has introduced the concept of a rough set and developed the basics of the corresponding theory. In [4] D. Dubois and H. Prade have introduced a fuzzy version of a rough set; later the theory of fuzzy rough sets was developed in different directions. In this paper, we present a many-level version of rough approximation for L-fuzzy sets and develop a method that allows to estimate, in a certain sense, the quality of this approximation. Further, we present here a certain motivation for many-level approach in the theory of rough approximation of fuzzy sets.

One of possible interpretations of a rough set is as follows. Assume we are looking from some distance at a plane filled up with pixels, and D is a domain in this plane. Then we may be sure that a pixel, say p, is in the domain D. Let $l(D)$ be the set of all such pixels. Further, for some pixels, we may be hesitating whether they are inside D or not. Let $bd(D)$ be the set of all such pixels and let $u(D) = l(D) \cup bd(D)$. Obviously, $l(D)$ and $u(D)$ can be viewed as respectively the lower and the upper Pawlak's rough approximations of the domain D and $bd(D)$ as its boundary set. But now imagine that we change the distance from which the observation is made. Then we can expect that the lower and upper rough approximations of the domain D vary depending on the distance from which the observation is made. Thus lower and upper rough approximations of D become *functions* $l_d(D)$ and $u_d(D)$ of the parameter d (*the distance of observation*). To manage with this and other similar cases in crisp, as well as in fuzzy cases, one can use many-level lower and upper fuzzy rough approximation operators. Further, we develop a method allowing to estimate the quality of obtained approximation. It is based on the measures of approximation introduced here. Continuing the previous example, these measures characterize, respectively, "how precisely D is covered by $u_d(D)$" and "how precisely $l_d(D)$ is covered by D".

Another example. Assume we make an approximation of an object. It may happen that at some stages this approximation is not as precise as at the others. And the transfer from lq'less precise" to "more precise" should be done in a "smooth" way. Our approach presents a model how this transition can be done.

2 Many-level L-fuzzy relations

2.1 Basic definitions

Let $L = (L, \leq_L, \wedge_L, \vee_L, *)$ be an integral commutative complete lattice monoid (in particular, $L = [0, 1]$ and $*$ a lower semi-continuous t-norm), see, e.g. [8], and let $M = (M, \leq_M, \wedge_M, \vee_M)$ be a complete infinitely distributive lattice.

Definition 2.1 *An M-level L-fuzzy relation on a set*

X is a mapping $R : X \times X \times M \to L$. An M-level L-fuzzy relation on a set X is called:

1. reflexive, *if* $R(x,x,\alpha) = 1_L \ \forall\ x \in X, \alpha \in M$;
2. separated, *if* $R(x,y,\alpha) = 1_L\ \forall \alpha \in M \Longrightarrow\ x = y$;
3. symmetric, *if* $R(x,y,\alpha) = R(y,x,\alpha)\ \forall x,y \in X, \alpha \in M$;
4. transitive, *if* $R(x,y,\alpha) * R(y,z,\alpha) \leq R(x,z,\alpha)\ \forall x,y,z \in X, \alpha \in M$.
5. *A reflexive transitive M-level L-fuzzy relation is called an* M-level L-fuzzy preoder *or an* LM-fuzzy preoder *for short.*

A pair (X,R), where X is a set and $R : X \times X \times M \to L$ is an LM-fuzzy preoder, is called an LM-fuzzy preodered space.

Above we considered *level-wise* properties of an M-level L-fuzzy relation R. Now we collect properties showing the behavior of the relation R between different levels $\alpha \in M$.

Definition 2.2 *An M-level L-fuzzy relation R on a set X is called*

1. non-increasing, *if* $\alpha \leq \beta \Longrightarrow R(x,y,\alpha) \geq R(x,y,\beta)\ \forall x,y \in X,\ \forall \alpha,\beta \in M$;
2. upper semi-continuous, *if* $R\left(x,y,\bigvee_{i\in I}\alpha_i\right) = \bigwedge_{i\in I} R(x,y,\alpha_i)\ \forall x,y \in X,\ \forall \{\alpha_i \mid i \in I\} \subseteq M$.
3. lower semi-continuous, *if* $R\left(x,y,\bigwedge_{i\in I}\alpha_i\right) = \bigvee_{i\in I} R(x,y,\alpha_i)\ \forall x,y \in X,\ \forall \{\alpha_i \mid i \in I\} \subseteq M$;
4. global, *if it satisfies conditions* $(\perp)$ *and* $(\top)$:
   $(\perp)$ $R(x,y,0_M) = 1_L$ *for all* $x,y \in X$;
   $(\top)$ $R(x,y,1_M) = 1_L$ *if* $x = y$ *and* $R(x,y,1_M) = 0_L$ *otherwise* .

Given LM-fuzzy preodered spaces (X,R_X) and (Y,R_Y), a mapping $f : X \to Y$ is called monotone non-decreasing, or just monotone for short if $R_X(x,x',\alpha) \leq R_Y(f(x),f(x'),\alpha)$ for all $x,x' \in X$ and all $\alpha \in M$.

The category of LM-fuzzy preodered spaces and their monotone mappings is denoted by LM-**PREL**.

2.2 Construction of an M-level L-fuzzy relation from a quasi-pseudometric

We construct an M-level L-fuzzy relation R_ρ from an ordinary quasi-pseudometric ρ on a set X. A similar construction in case ρ is a metric was considered in [16] where it was used in the study of many-valued bornologies.

Let $L = M = [0,1]$ be the unit intervals viewed as lattices and let $* : L \times L \to L$ be a continuous t-norm. Further, let X be a set and $\rho : X \times X \to [0,1]$ be a quasi-pseudometric on this set. We define an M-level L-fuzzy relation $R_\rho : X \times X \times [0,1] \to [0,1]$ by setting

$$R_\rho(x,y,\alpha) = \begin{cases} \frac{\alpha}{\alpha + (1-\alpha)\rho(x,y)} & \text{if } \alpha \neq 0 \text{ or } \rho(x,y) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that $R_\rho(x,y,\cdot) : [0,1] \to [0,1]$ is continuous for all $x,y \in [0,1]$. Modifying the proof of Proposition 7.1 in [16] we can get the following result:

Theorem 2.3 *For any quasi-pseudometric $\rho : X \times X \to [0,1]$, many-level fuzzy relation $R_\rho : X \times X \times [0,1] \to [0,1]$ is reflexive, upper and lower semi-continuous and global. If ρ is a pseudometric, then R_ρ is symmetric. If ρ is a quasi-metric, then R_ρ is separated. The relation R_ρ is transitive in cases of the product t-norm $* = \cdot$ and hence for any weaker continuous t-norm. If ρ is an ultra pseudometric, then relation R_ρ is transitive for any continuous t-norm $*$.*

Corollary 2.4 *In cases $* = \cdot$ and $* = *_L$ the mapping $R_\rho : X \times X \times [0,1] \to [0,1]$ is a global continuous LM-fuzzy preoder for any quasi-pseudpmetric $\rho : X \times X \to [0,1]$. If ρ is an ultra pseudometric, then R_ρ is a global continuous LM-fuzzy preoder for any continuous t-norm.*

Remark 2.5 If we start with an arbitrary quasi-pseudometric $d : X \times X \to [0,\infty)$, then for the definition of the relation R_d we can take its equivalent quasi-pseudometric $\rho : X \times X \to [0,1]$ defined by $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$. In this case M-level L-fuzzy relation R_ρ can be rewritten as $R_d(x,y,\alpha) = \frac{\alpha(1+d(x,y))}{\alpha+d(x,y)}$ if $\alpha \neq 0$ or $d(x,y) \neq 0$ and $R_d(x,y,0) = 1$ if $d(x,y) = 0$.

3 M-level rough approximation of an L-fuzzy set

Given an LM-fuzzy preoder $R : X \times X \times M \to L$, we define *the upper M-level L-rough approximation operator* $u_R : L^X \times M \to L^X$ and *the lower M-level L-rough approximation operator* $l_R : L^X \times M \to L^X$ by $u_R(A,\alpha)(x) = \bigvee_y (R(y,x,\alpha) * A(y)$ and $l_R(A,\alpha)(x) = \bigwedge_y (R(x,y,\alpha) \mapsto A(y)\ \ \forall A \in L^X$, respectively.

Theorem 3.1 *M-level L-rough approximation operators satisfy the following properties:*

(1u) $u_R(a_X,\alpha) = a_X \forall \alpha \in M$ *where* $a_X : X \to L$ *is the constant function with value a;*

(2u) $A \leq u_R(A,\alpha)\ \forall A \in L^X,\ \forall \alpha \in M$;

(3u) $u_R(\bigvee_i A_i, \alpha) = \bigvee_i u_R(A_i, \alpha)$
$\forall \{A_i \mid i \in I\} \subseteq L^X \ \forall \alpha \in M$;

(4u) $u_R(u_R(A, \alpha), \alpha) = u_R(A, \alpha) \ \forall A \in L^X \ \forall \alpha \in M$;

(5u) $\alpha \leq \beta \Longrightarrow u_R(A, \alpha) \geq u_R(A, \beta)$

(1l) $l_R(a_X, \alpha) = a_X \ \forall \ \alpha \in M$;

(2l) $A \geq l_R(A, \alpha) \ \forall A \in L^X \ \forall \ \alpha \in M$;

(3l) $l_R(\bigwedge_i A_i, \alpha) = \bigwedge_i l_R(A_i, \alpha)$
$\forall \{A_i \mid i \in I\} \subseteq L^X \ \forall \alpha \in M$;

(4l) $l_R(l_R(A, \alpha), \alpha) = l_R(A, \alpha) \ \forall A \in L^X \ \forall \alpha \in M$.

(5l) $\alpha \leq \beta \Longrightarrow l_R(A, \alpha) \leq l_R(A, \beta)$

Besides, if R is lower semi-continuous, then

(6u) $u_R(A, \bigwedge_j \alpha_j) = \bigvee_j (u_R(A, \alpha_j) \ \forall \{\alpha_j \mid j \in J\} \subseteq A$;

(6l) $l_R(A, \bigwedge_j \alpha_j) = \bigwedge_j (l_R(A, \alpha_j) \ \forall \{\alpha_j \mid j \in J\} \subseteq A$.

Proof We omit the proofs of properties (1u) - (4u) and (1l) - (4l) since they can be easily obtained by modifying the proofs of the corresponding properties of fuzzy rough approximation operators in the proof of Proposition 4 in [12] for the many-level case. Properties (5u) and (5l) are clear from the definitions. To prove properties (6u) and (6l) let $A \in L^X$, $x \in X$ and $\{\alpha_j, \mid j \in J\} \subset M$ be given. Then

$$u_R\left(A, \bigwedge_j a_j\right)(x) = \bigvee_y R\left(y, x, \bigwedge_j \alpha_j\right) * A(y) =$$

$$\bigvee_y \left(\bigvee_j R(y, x, \alpha_j) * A(y)\right) =$$

$$\bigvee_j \bigvee_y (R(y, x, \alpha_j) * A(y)) = \bigvee_j (u_R(A, \alpha_j)(x)$$

and

$$l_R(A, \bigwedge_j \alpha_j)(x) = \bigwedge_y \left(R(x, y, \bigwedge_j \alpha_j) \mapsto A(y)\right) =$$

$$\bigwedge_y \left(\bigvee_j R(x, y, \alpha_j) \mapsto A(y)\right) =$$

$$\bigwedge_y \bigwedge_j (R(x, y, \alpha_j) \mapsto A(y)) =$$

$$\bigwedge_j \left(\bigwedge_y R(x, y, \alpha_j) \mapsto A(y)\right) = \bigwedge_j l_R(A, \alpha_j)(x).$$

□

The proof of the following theorem can be done by level-wise modification in the proofs of Proposition 3 and Theorem 1 in [12].

Theorem 3.2 *The equalities $u_R(l_R(A, \alpha), \alpha) = l_R(A, \alpha)$ and $l_R(u_R(A, \alpha), \alpha) = u_R(A, \alpha)$ hold for every L-fuzzy set A and every $\alpha \in M$,.*

Definition 3.3 *The pair (u_R, l_R), where $R : X \times X \times M \to L$ is an LM-fuzzy preoder on X and $u_R, l_R : L^X \times M \to L^X$ are M-level L-rough approximation operators is called an M-level L-rough approximative pair, and the corresponding triple (X, u_R, l_R) is called an M-level L-rough approximation space.*

Definition 3.4 *Let (X, u_{R_X}, l_{R_X}) and (Y, u_{R_Y}, l_{R_Y}) be M-level L-rough approximation spaces. A mapping $f : X \to Y$ is called continuous[1] if it satisfies the following two conditions:*

(con 1) $f(u_{R_X}(A, \alpha)) \leq u_{R_Y}(f(A), \alpha) \ \forall A \in L^X \ \forall \alpha \in M$,

(con 2) $f^{-1}(l_{R_Y}(B, \beta)) \leq l_{R_X}(f^{-1}(B, \beta)) \ \forall B \in L^Y, \forall \beta \in M$.

*By LM-**RAS** we denote the category of M-level L-rough approximation spaces and their continuous mappings.*

Theorem 3.5 *If $f : (X, R_X) \to (Y, R_Y)$ is a monotone mapping then the mapping $f : (X, u_{R_X}, l_{R_X}) \to (Y, u_{R_Y}, l_{R_Y})$ is continuous.*

Proof To prove the first property, we fix $A \in L^X$, $\alpha \in M$, $y \in Y$ and reason as follows:

$$f(u_{R_X}(A, \alpha))(y) = \bigvee_{f(x)=y} u_{R_X}(A, \alpha)(x) =$$

$$\bigvee_{f(x)=y} \left(\bigvee_{x'} R(x', x, \alpha) * A(x')\right) \leq$$

$$\bigvee_{x'} R_Y(f(x'), f(x), \alpha) * A(x') =$$

$$\bigvee_{x'} R_Y(f(x'), y, \alpha) * A(x') \leq$$

$$\bigvee_{y'} R_Y(y', y, \alpha) * f(A)(y') = u_{R_Y}(f(A), \alpha)(y).$$

To prove the second property, we fix $B \in L^Y$, $\beta \in M$, $x \in X$ and reason as follows:

$$f^{-1}(l_{R_Y}(B, \beta)(x)) = l_{R_Y}(B, \alpha) f(x) =$$

$$\bigwedge_{y'} R_Y(f(x), y', \beta) \mapsto B(y') \leq$$

$$\bigwedge_{x'} R_Y(f(x), f(x'))(\beta) \mapsto B(f(x')) \leq$$

$$\bigwedge_{x'} R_X(x, x', \beta) \mapsto B(f(x')) =$$

$$\bigwedge_{x'} R_X(x, x', \beta) \mapsto f^{-1}(B)(x') = l_{R_X}(f^{-1}(B, \beta))(x).$$

□

Corollary 3.6 *Assigning the space (X, u_{R_X}, l_{R_X}) to an LM-fuzzy predoredered set (X, R) and interpreting a monotone mapping $f : (X, R_X) \to (Y, R_Y)$ as a mapping $f : (X, u_{R_X}, l_{R_X}) \to (Y, u_{R_Y}, l_{R_Y})$, we get an embedding functor from the category LM-**PREL** of LM-fuzzy preodered sets into the category LM-**RAS**.*

[1]the justification of this term will be clarified later

4 LM-fuzzy ditopology induced by rough approximation operators on an LM-fuzzy preodered set

Many-level upper and lower L-rough approximation operators give rise to two ditopological type structures on the underlying set. These ditopologies will be the subject of this and the next section. However, first we have to specify terminology concerning topological structures in fuzzy environment. Following the terminology initiated in [9] and now accepted by many authors, by an L-topology on a set X we call a family $T \subseteq L^X$ of its L-fuzzy subsets such that $0_L \in T$; $U, V \in T \Rightarrow U \wedge V \in T$ and $U_i \in T \ \forall i \in I \Rightarrow \bigvee_i U_i \in T$. On the other hand, by an LM-fuzzy topology we call a mapping $\mathcal{T} : L^X \to M$ such that $\mathcal{T}(0_X) = 1_M$; $U, V \in L^X \Rightarrow \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$; $\mathcal{T}(\bigvee_i U_i) \geq \bigwedge_i \mathcal{T}(U_i)$. In an analogous way, an L-co-topology and an LM-fuzzy co-topology on a set X are defined. In case $L = M$ we just speak about L-topologies and L-co-topologies. Since in the context of our research L-topologies (LM-fuzzy topologies) and L-co-topologies (resp. LM-fuzzy co-topologies) generally are unrelated, we stick to the terminology introduced in the works by M. L. Brown and his co-authors, see e.g. [2], and use the terms L-ditopology and LM-fuzzy ditopology respectively.

4.1 LM-fuzzy topology on an LM-preordered set

Let (X, R) be an LM-fuzzy preordered set and let $l_R : L^X \times M \to L^X$ be the lower LM-rough approximation operator induced on this set. Then the properties $(1l) - (4l)$ of l_R collected in Theorem 3.1 mean that for every level α the restriction of the mapping l_R to the set $L^X \times \{\alpha\}$, that is the mapping $l_R : L^X \times \{\alpha\} \to L^X$ can be interpreted as the interior operator on L^X see, e.g. [10], [13], [14]). In particular, if R is global, then $l_R(A, 0_M)(x) = \inf_{y \in X} A(y)$ and $l_R(A, 1_M)(x) = A(x)$

Hence, by setting $T_\alpha = \{A \in L^X : l_R(A, \alpha) = A\}$, we obtain the L-topology corresponding to this L-fuzzy interior operator. Moreover, the property $(3l)$ allows to conclude that it is actually an Alexandroff L-topology (see e.g. [1], [3]), that is the intersection axiom holds also for infinite families of L-fuzzy sets. Thus for each α the family T_α satisfies the axioms of an Alexandroff L-topology:

1. $0_X, 1_X \in T_\alpha$;
2. $\{A_i : i \in I\} \subseteq T_\alpha \Longrightarrow \bigwedge_i A_i \in T_\alpha$;
3. $\{A_i : i \in I\} \subseteq T_\alpha \Longrightarrow \bigvee_i A_i \in T_\alpha$

Taking such L-topologies for all $\alpha \in M$, we obtain the family $\{T_\alpha : \alpha \in M\}$. Besides, since $l_R(A, \beta) \leq l_R(A, \alpha)$ whenever $\beta \leq \alpha$, we conclude that

$$\beta \leq \alpha \implies T_\beta \subseteq T_\alpha,$$

that is the family $\{T_\alpha : \alpha \in M\}$ is non-decreasing. In particular, $T_0 = \{a_X \mid a : X \to L^X\}$. This means that T_0 consists of all constants and hence is the indiscrete stratified L-topology. On the other hand $T_1 = L^X$, that is T_1 is the discrete L-topology.

We use this family in order to construct an LM-fuzzy topology from this indexed set of L-topologies. To do this in a coordinated way, in addition, we assume that M is a De Morgan algebras, that is a completely distributive lattice endowed with an order reversing involution ${}^c : M \to M$. We define

$$\mathcal{T}(A) = \bigvee \{\alpha \in M : A \in T_{\alpha^c}.\}$$

Theorem 4.1 *If the lattice M is completely distributive, then $\mathcal{T}$ is an LM-fuzzy topology on the LM-preordered set (X, R), that is*

1. $\mathcal{T}(0_X) = 0_M$;
2. $\mathcal{T}(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ *for all* $\{A_i : i \in I\} \subseteq L^X$;
3. $\mathcal{T}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ *for all* $\{A_i : i \in I\} \subseteq L^X$.

Proof The first property is obvious, since $0_X \in T_\alpha$ for all $\alpha \in M$.

To prove the second property, take any family $\{A_i : i \in I\} \subseteq L^X$ and assume that $\bigwedge_i \mathcal{T}(A_i) = \alpha$. In case $\alpha = 0_M$ the inequality is obvious, therefore we assume that $\alpha > 0_M$. Take any $\beta \lhd \alpha$ where $\lhd$ is the wedge below relation on the completely distributive lattice M. From the definition of $\mathcal{T}$ it is clear that $A_i \in T_{\beta^c}$ for every $i \in I$ and hence, recalling that T_{β^c} is an Alexandroff L-topology, we conclude that also $\bigwedge_i A_i \in T_{\beta^c}$. Therefore $\mathcal{T}(\bigwedge_i A_i) \geq \beta^c$. Since this is true for any $\beta \lhd \alpha$ and lattice M is completely distributive, we conclude that $\mathcal{T}(\bigwedge_i A_i) \geq \alpha^c = \bigwedge_i \mathcal{T}(A_i)$.

To prove the third property, take any family $\{A_i : i \in I\} \subseteq L^X$ and assume that $\bigvee_i \mathcal{T}(A_i) = \alpha$. In case $\alpha = 0_M$ the inequality is obvious, therefore we assume that $\alpha > 0_M$. Take any $\beta \lhd \alpha$. From the definition of $\mathcal{T}$ it is clear that $A_i \in T_{\beta^c}$ for every $i \in I$ and hence, recalling that T_{β^c} is an L-topology, we conclude that also $\bigvee_i A_i \in T_{\beta^c}$. Therefore $\mathcal{T}(\bigwedge_i A_i) \geq \beta$. Since this is true for any $\beta \lhd \alpha$ and the lattice M is completely distributive, we conclude that $\mathcal{T}(\bigvee_i A_i) \geq \alpha = \bigwedge_i \mathcal{T}(A_i)$. □

4.2 LM-fuzzy co-topology on an LM-preorderd set

Let (X, R) be an LM-fuzzy preordered set and let $u_R : L^X \times M \to L^X$ be the upper L-rough approximation operator induced by the LM-relation R on the set X. Then properties $(1u)-(4u)$ of the upper LM-rough approximation operator u_R mean that the restriction of u_R to $L^X \times \{\alpha\}$, can be interpreted as an L-fuzzy closure operator on the set L^X (This fact is well-known, see, e.g. [10], [13], [14]). Besides,

$$\alpha \leq \beta \Longrightarrow u_R(A,\alpha) \geq u_R(A,\beta);$$

$$u_R(A,0_M)(x) = \sup_{y\in X} A(y), \quad u_R(A,1_M)(x) = A(x).$$

Now, given $\alpha \in M$ the family $K_\alpha = \{A \in L^X : u_R(A,\alpha) = A\}$, is the Alexandroff L-co-topology corresponding to this L-fuzzy closure operator. This means that

1. $1_X \in K_\alpha$;
2. $\{A_i : i \in I\} \subseteq K_\alpha \Longrightarrow \bigvee_i A_i \in K_\alpha$;
3. $\{A_i : i \in I\} \subseteq K_\alpha \Longrightarrow \bigwedge_i A_i \in K_\alpha$

Taking such L- co-topologies for all $\alpha \in M$, we obtain the family $\{K_\alpha : \alpha \in M\}$. Besides, since

$$\beta \leq \alpha \Longrightarrow u_R(\cdot,\beta) \geq u_R(\cdot,\alpha)$$

we conclude that

$$\beta \leq \alpha \implies K_\beta \subset K_\alpha,$$

that is the family $\{K_\alpha : \alpha \in M\}$ is non-decreasing. To use this family of L-co-topologies in order to define an (Alexandroff) LM-fuzzy co-topology $\mathcal{K}$ on the set X, as in the previous subsection, we assume that lattice M is completely distributive and is endowed with an order reversing involution $^c : M \to M$. Now, by setting

$$\mathcal{K}(A) = \bigvee \{\alpha \in M : A \in K_{\alpha^c}.\}$$

we obtain a mapping $\mathcal{K} : L^X \times M \to L^X$.

Theorem 4.2 *$\mathcal{K}$ is an LM-fuzzy co-topology on the set X, that is*

1. $\mathcal{K}(1_X) = 1_M$;
2. $\mathcal{K}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{K}(A_i)$ *for all* $\{A_i : i \in I\} \subseteq L^X$;
3. $\mathcal{K}(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{K}(A_i)$ *for all* $\{A_i : i \in I\} \subseteq L^X$.

Proof The first property is obvious, since $1_X \in K_\alpha$ for all $\alpha \in M$.

To prove the second property, take any family $\{A_i : i \in I\} \subseteq L^X$ and assume that $\bigwedge_i \mathcal{K}(A_i) = \alpha$. In case $\alpha = 0_M$ the inequality is obvious, therefore we assume that $\alpha > 0_M$. Take any $\beta \lhd \alpha$ where $\lhd$ is the wedge-below relation in the completely distributive lattice M. Then from the definition of $\mathcal{K}$ it is clear that $A_i \in K_{\beta^c}$ for every $i \in I$, and hence, recalling that K_{β^c} is an Alexandroff L-co-topology, we conclude that also $\bigvee_i A_i \in K_{\beta^c}$. Therefore $\mathcal{K}(\bigvee_i A_i) \geq \beta$. Since this is true for any $\beta \lhd \alpha$ and lattice M is completely distributive, we conclude that $\mathcal{K}(\bigvee_i A_i) \geq \alpha = \bigwedge_i \mathcal{K}(A_i)$.

To prove the third property, take any family $\{A_i : i \in I\} \subseteq L^X$ and assume that $\bigvee_i \mathcal{K}(A_i) = \alpha$. In case $\alpha = 0_M$ the inequality is obvious, therefore we assume that $\alpha > 0_M$. Take any $\beta \lhd \alpha$. Then from the definition of $\mathcal{K}$ it is clear that $A_i \in K_{\beta^c}$ for every $i \in I$, and hence, recalling that K_{β^c} is an Alexandroff L-co-topology, we conclude that also $\bigvee_i A_i \in K^c_\beta$. Therefore $\mathcal{K}(\bigvee_i A_i) \geq \beta$. Since this is true for any $\beta \lhd \alpha$ and lattice M is completely distributive, we conclude that $\mathcal{K}(\bigwedge_i A_i) \geq \alpha = \bigwedge_i \mathcal{K}(A_i)$. □

4.3 The case of an MV-algebra

In this section we assume that L is an MV-algebra, see e.g. [6]. This means that $(\alpha \mapsto 0_L) \mapsto 0_L = \alpha$, and hence by setting $\alpha' = \alpha \mapsto 0_L$, we obtain an order reversing involution $' : L \to L$ on the lattice L. This involution is extended point-wise to the L-powerset L^X as $A'(x) = A(x) \mapsto 0_L$. Besides, we continue to assume that M is a De Morgan algebra.

Generally LM-fuzzy topology $\mathcal{T}$ and LM-fuzzy co-topology constructed above are unrelated and hence we cannot view the obtained structures as an LM-fuzzy topology. As we will see here, in case L is an MV-algebra and the LM-fuzzy preorder relation R on a set X is symmetric, then the pair $(\mathcal{T},\mathcal{K})$ is an L-fuzzy topology where the mapping $\mathcal{T} : L^X \to M$ determines the degree of openness of L-fuzzy subsets of X while the mapping $\mathcal{K} : L^X \to M$ determines the degree of closeness of L-fuzzy subsets of X and $\mathcal{T}(A',\alpha) = \mathcal{K}(\alpha, A)'$ for every $A \in L^X$.

Indeed, it is easy to notice that in case of an MV-algebra L and the symmetric L-fuzzy preoder R, we have $u_R(A,\alpha)' = l_R(A',\alpha)$ for every $A \in L^X$ and every $\alpha \in M$.

Recalling the definition of L-topology T_α and L-co-topology K_α, for every $A \in L^X$ we have:

$$A \in K_\alpha \Longleftrightarrow u_R(A,\alpha) = A \Longleftrightarrow$$

$$u_R(A,\alpha)' = A' \Longleftrightarrow l_R(A',\alpha) = A' \Longleftrightarrow A \in T_\alpha.$$

Now we have

$$\mathcal{K}(A) = \bigvee \{\alpha \in M \mid A \in K_{\alpha^c}\} =$$

$$\bigvee \{\alpha \in M \mid u_R(A,\alpha^c) = A\} =$$

$$\bigvee \{\alpha \in M \mid u_R(A,\alpha^c)' = A'\} =$$

$$\bigvee \{\alpha \in M \mid l_R(A',\alpha^c) = A'\} = \mathcal{T}(A').$$

Thus we have the following theorem:

Theorem 4.3 *If L is an MV-algebra and the LM-fuzzy relation R on a set X, then $\mathcal{T}(A) = \mathcal{K}(A')$ for every $A \in L^X$ and hence the mapping $\mathcal{T} : L^X \to M$ is an LM-fuzzy topology on X.*

5 Measure of M-level L-fuzzy rough approximation and induced M-level L-fuzzy ditopologies

5.1 Measure of inclusion of L-fuzzy sets

Definition 5.1 *By setting $A \hookrightarrow B = \bigwedge_{x\in X}(A(x) \mapsto B(x))$ where $A, B \in L^X$ and $\mapsto: L \times L \to L$ is the residuum corresponding to the operation $*$ by Galois connection, we obtain a mapping $\hookrightarrow: L^X \times L^X \to L$. We call $A \hookrightarrow B$ by the measure of inclusion of the L-fuzzy set A into the L-fuzzy set B.*

Proposition 5.2 *Relation $\hookrightarrow: L^X \times L^X \to L$ satisfies the following properties for all $\{A_i \mid i \in I\} \subseteq L^X$, $A \in L^X, \{B_i \mid i \in I\} \subseteq L^X$, $B \in L^Y$:*

(1) $(\bigvee_i A_i) \hookrightarrow B = \bigwedge_i (A_i \hookrightarrow B)$;

(2) $A \hookrightarrow (\bigwedge_i B_i) = \bigwedge_i (A \hookrightarrow B_i)$;

(3) $A \hookrightarrow B = 1_L$ *whenever* $A \leq B$;

(4) $1_X \hookrightarrow A = \bigwedge_x A(x)$;

(5) $(A \hookrightarrow B) \leq (A * C \hookrightarrow B * C)\ \forall A, B, C \in L^X$;

(6) $(A \hookrightarrow B) * (B \hookrightarrow C) \leq (A \hookrightarrow C)\ \forall A, B, C \in L^X$;

(7) $(\bigwedge_i A_i) \hookrightarrow (\bigwedge_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$;

(8) $(\bigvee_i A_i) \hookrightarrow (\bigvee_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$.

The proof of these properties is easy and can be found in the recent works of different authors.

5.2 Measure of M-level L-fuzzy rough approximation

Let (X, R) be an LM-fuzzy preodered set. Given an L-fuzzy set $A \in L^X$, we define the measure $\mathcal{U}(A, \cdot) : M \to L$ of its upper M-level L-fuzzy rough approximation by $\mathcal{U}_R(A, \alpha) = u_R(A, \alpha) \hookrightarrow A$. Respectively, the measure $\mathcal{L}(A, \cdot) : M \to L$ of its lower M-level L-fuzzy rough approximation is defined by $\mathcal{L}_R(A, \alpha) = A \hookrightarrow l_R(A, \alpha)$. If R is symmetric then it is easy to see that $\mathcal{U}_R(A, \alpha) = \mathcal{L}_R(A, \alpha)$ for every L-fuzzy set A. In this case we call it by the measure of M-level L-fuzzy rough approximation of an L-fuzzy set A.

The above defined measures of lower and upper M-level rough approximation of L-fuzzy sets give rise to the M-level operators of upper and lower L-fuzzy rough approximation $\mathcal{U}_R : L^X \times M \to L$ and $\mathcal{L}_R : L^X \times M \to L$ and the operator of M-level L-fuzzy rough approximation $\mathfrak{R}_R : L^X \times M \to L$ in case R is symmetric. In the next theorem we collect the main properties of these operators.

Theorem 5.3 *1. $\mathcal{U}_R(a_X, \alpha) = 1_L\ \forall \alpha \in M$;*

2. $\mathcal{L}_R(a_X, \alpha) = 1_L\ \forall \alpha \in M$;

3. $\mathcal{U}_R(u_R(A, \alpha), \alpha) = 1_L\ \forall A \in L^X,\ \forall \alpha \in M$;

4. $\mathcal{L}_R(l_R(A), \alpha), \alpha) = 1_L\ \forall A \in L^X\ \forall \alpha \in M$;

5. $\mathcal{U}_R(\bigvee_i A_i, \alpha) \geq \bigwedge_i \mathcal{U}_R(A_i, \alpha)$ $\forall \{A_i \mid i \in I\} \subseteq L^X\ \forall \alpha \in M$;

6. $\mathcal{U}_R(\bigwedge_i A_i, \alpha) \geq \bigwedge_i \mathcal{U}_R(A_i, \alpha)$ $\forall \{A_i \mid i \in I\} \subseteq L^X,\ \forall \alpha \in M$;

7. $\mathcal{L}_R(\bigwedge_i A_i, \alpha) \geq \bigwedge_i \mathcal{L}_R(A_i, \alpha)$ $\forall \{A_i \mid i \in I\} \subseteq L^X,\ \forall \alpha \in M$;

8. $\mathcal{L}_R(\bigvee_i A_i, \alpha) \geq \bigwedge_i \mathcal{L}_R(A_i, \alpha)$ $\forall \{A_i \mid i \in I\} \subseteq L^X,\ \forall \alpha \in M$;

9. $\mathcal{U}_R(a_X * A, \alpha) \geq \mathcal{U}_R(A, \alpha)$ for all $A \in L^X$ and all constants a_X;

10. $\mathcal{L}_R(a_X \mapsto A, \alpha) \geq \mathcal{L}_R(A, \alpha)$ for all $A \in L^X$ and all constants a_X.

Proof (1) Referring to Theorem 3.1 and applying Proposition 5.2 we have $\mathcal{U}_R(a_X, \alpha) = u_R(a_X, \alpha) \hookrightarrow a_X = a_X \hookrightarrow a_X = 1_M$.
(2) Referring to Theorem 3.1 and applying Proposition 5.2, we have
$\mathcal{L}_R(a_X, \alpha) = a_X \hookrightarrow l_R(a_X, \alpha) = a_X \hookrightarrow a_X = 1_M$.
(3) Referring to Theorem 3.1 and applying Proposition 5.2, we have $\mathcal{U}_R(u_R(A, \alpha)) = u_R(u_R(A, \alpha), \alpha) \hookrightarrow u_R(A, \alpha) = u_R(A, \alpha) \hookrightarrow u_R(A, \alpha) = 1_M$.
(4) Referring to Theorem 3.1 and applying Proposition 5.2, we have $\mathcal{L}_R(l_R(A, \alpha), \alpha) = l_R(A, \alpha) \hookrightarrow l_R(l_R(A, \alpha), \alpha) = l_R(A, \alpha) \hookrightarrow l_R(A, \alpha) = 1_M$
(5) Referring to Theorem 3.1 and applying Proposition 5.2, we have $\mathcal{U}_R(\bigvee_i A_i, \alpha) = u_R(\bigvee_i A_i, \alpha) \hookrightarrow \bigvee_i A_i = \bigvee_i u_R(A_i, \alpha) \hookrightarrow \bigvee_i A_i$
(6) Referring to Theorem 3.1 and applying Proposition 5.2 we have $\mathcal{U}_R(\bigwedge_i A_i, \alpha) = u_R(\bigwedge_i A_i, \alpha) \hookrightarrow \bigwedge_i A_i \geq \bigwedge_i u_R(A, \alpha) \hookrightarrow \bigwedge_i A_i \geq \bigwedge_i (u_R(A_i, \alpha) \hookrightarrow A_i) = \bigwedge_i \mathcal{U}_R(A_i, \alpha)$.
(7) Referring to Theorem 3.1 and applying Proposition 5.2, we have $\mathcal{L}_R(\bigwedge_i A_i, \alpha) = (\bigwedge_i A_i) \hookrightarrow l_R(\bigwedge_i A_i, \alpha) = \bigwedge_i A_i \hookrightarrow \bigwedge_i l_R(A_i, \alpha) \geq \bigwedge_i A_i \hookrightarrow l_R(\bigwedge_i A_i, \alpha)$.
(8) Referring to Theorem 3.1 and applying and applying Proposition 5.2, we have $\mathcal{L}_R(\bigvee_i A_i, \alpha) = (\bigvee_i A_i) \hookrightarrow l_R(\bigvee_i A_i, \alpha) \geq \bigwedge_i A_i \hookrightarrow \bigwedge_i l_R(A_i, \alpha) \geq$

$\bigvee_i A_i \hookrightarrow l_R(\bigvee_i A_i, \alpha) \geq \bigwedge_i (l_R(A_i, \alpha) \hookrightarrow A_i) = \bigwedge_i \mathcal{L}(A_i, \alpha)$.
(9) $\mathcal{U}_R(\alpha * A, \alpha) = u(\alpha * A, \alpha) \hookrightarrow \alpha * A) = \bigwedge_{x\in X} \bigvee_{x'\in X}(R(x', x, \alpha) * A(x') * \alpha \mapsto \alpha * A(x)) \geq \bigwedge_{x\in X} \bigvee_{x'}(R(x', x, \alpha) * A(x') \mapsto A(x)) = \bigwedge_{x\in X}(u_R(A, \alpha)(x') \mapsto A(x)) = \mathcal{U}_R(A, \alpha) \hookrightarrow A$;
(10) Recall first that for every x we have $\alpha \mapsto l_R(A, \alpha)(x) = l_R(A \mapsto \alpha)(x)$. Indeed, $\alpha \mapsto l_R(A, \alpha)(x) = \alpha \mapsto \bigwedge_{x'}(R(x', x, \alpha) \mapsto A(x')) = \bigwedge_{x'}(\alpha \mapsto (R(x', x, \alpha) \mapsto A(x')) = \bigwedge_{x'}(R(x', x, \alpha) \mapsto (\alpha \mapsto A(x')) = l_R(\alpha \mapsto A, \alpha)(x)$.
Now we get the requested inequality as follows:
$\mathcal{L}_R(A, \alpha) = A \hookrightarrow l_R(A, \alpha) = \bigwedge_x (A(x) \mapsto l_R(A, \alpha)(x)) \leq \bigwedge_x (\alpha \mapsto A)(x) \mapsto (\alpha \mapsto l_R(A, \alpha)(x) = \bigwedge_x (\alpha \mapsto A)(x) \mapsto (\alpha \mapsto l_R(A, \alpha)(x)) =$
$\mathcal{L}_R(\alpha \mapsto A, \alpha)$. □

5.3 Examples of measures for many-level fuzzy rough approximation of L-fuzzy sets

The case of Łukasiewicz t-norm Let $*_L$ be the Łukasiewicz t-norm on the interval $L = [0, 1]$, and $\mapsto_L$: $L \times L \to L$ be the corresponding residuum. Then, given an M-level L-fuzzy relation R on a set X, $A \in L^X$ and $\alpha \in M$, we have:
$\mathcal{U}_R(A, \alpha) = \bigwedge_x \bigwedge_{x'}(2 - A(x) + A(x') - R(x, x', \alpha))$
$\mathcal{L}_R(A, \alpha) = \bigwedge_x \bigwedge_{x'}(2 - A(x) + A(x') - R(x', x, \alpha))$.
In particular, if $R : X \times X \times M \to [0, 1]$ is global, then $\mathcal{R}_R(A, 0_M) = 1_L$ and $\mathcal{R}_R(A, 1_M) = 1 - \inf_{x,x'} \mid A(x) - A(x') \mid$ for all $A \in L^X$.

The case of the minimum t-norm Let $* = \wedge$ be the minimum t-norm on the unit interval $L = [0, 1]$, and $\mapsto$: $L \times L$ be the corresponding residuum, Then
$\mathcal{U}_R(A, \alpha) = \inf_{x,x'}(A(x') \wedge R(x, x', \alpha) \mapsto A(x))$,
$\mathcal{L}_R(A, \alpha) = \inf_{x,x'}(A(x') \wedge R(x', x, \alpha) \mapsto A(x))$.
In particular, if R is global, then $\mathcal{R}_R(A, \alpha) = 1_L$ $\forall A \in L^X, \alpha \in M$.

The case of the product t-norm Let $* = \cdot$ be the product t-norm on the unit interval [0,1] and $\mapsto$: $L \times L$ be the corresponding residuum, Then
$\mathcal{U}(A, \alpha) = \inf_{x,x'}(A(x') \cdot R(x, x', \alpha) \mapsto A(x))$,
$\mathcal{L}(A, \alpha) = \inf_{x,x'}(A(x') \cdot R(x', x, \alpha) \mapsto A(x))$. In particular in case R is global, $\mathcal{R}_R(A, 1_M) = 1_L$ and $\mathcal{R}_R(A, 0_M) = \inf_{x,x'\in X} \frac{A(x)}{A(x')}$.

5.4 On the category of M-level L-fuzzy rough approximation spaces

Given an LM-fuzzy preodered set (X, R), the quadruple $(X, R, \mathcal{U}_R, \mathcal{L}_R)$ is called by an M-level L-fuzzy rough approximation space.

Definition 5.4 *We call a mapping of M-level L-fuzzy rough approximation spaces $f : (X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X}) \to (Y, \mathcal{U}_{R_Y}, \mathcal{L}_{R_Y})$ continuous if*

(1con) $\mathcal{U}_{R_X}(f^{-1}(B), \alpha) \geq \mathcal{U}_Y(B, \alpha)$ $\forall B \in L^Y, \forall \alpha \in M$;

(2con) $\mathcal{L}_{R_X}(f^{-1}(B), \alpha) \geq \mathcal{L}_Y(B, \alpha)$ $\forall B \in L^Y$.

*Let ML-**FRAS** be the category whose objects are M-level L-fuzzy rough approximation spaces and whose morphisms are continuous mappings.*

By straightforward verification one can easily prove the following statement:

Theorem 5.5 *Let $R_X : X \times X \times M \to L$ and $R_Y : Y \times Y \times M \to L$ be LM-fuzzy preoders on sets X and Y respectively and let $f : (X, R_X) \to (Y, R_Y)$ be a monotone mapping. Then the mapping $f : (X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X}) \to (Y, \mathcal{U}_{R_Y}, \mathcal{L}_{R_Y})$ is continuous.*

Thus assigning the M-level L-fuzzy rough approximation space $(X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X})$ to an M-level L-fuzzy preoder space (X, R) and interpreting monotone mappings $f : (X, R_X) \to (Y, R_Y)$ as mappings $f : (X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X}) \to (Y, \mathcal{U}_{R_Y}, \mathcal{L}_{R_Y})$, we obtain an embedding functor from the category LM-**PREL** of LM-fuzzy preodered sets into the category LM-**RAS**.

5.5 Ditopological interpretation of M-level L-fuzzy rough approximation spaces

Let $(X, \mathcal{L}_R, \mathcal{U}_R)$ be an M-level L-fuzzy L-rough approximation space and let $\alpha \in M$ be fixed. Properties (1), (5) and (6) of Theorem 5.3 characterize the relation's $\mathcal{L}_R : L^X \times M \to L$ restriction to the set $L^X \times \{\alpha\}$ as a stratified L-fuzzy topology on the set X [10], [9]. In its turn, properties (2), (7) and (8) characterize the relation's $\mathcal{U}_R : L^X \times M \to L$ restriction to the set $L^X \times \{\alpha\}$ as a stratified L-fuzzy co-topology on a set X. This observation justifies the following definition:

Definition 5.6 *An M-level L-fuzzy topology on a set X is a mapping $\mathcal{L}_R : L^X \times M \to L$ satisfying properties (1), (5) and (6) of Theorem 5.3. Respectively, an M-level L-fuzzy co-topology on a set X is a mapping $\mathcal{U}_R : L^X \times M \to L$ satisfying properties (2), (7) and (8) of Theorem 5.3.*

Now from Theorem 5.5 we get the following:

Theorem 5.7 *Let (X, R) be an LM-fuzzy preodered set. Then the triple $(X, \mathcal{L}_X, \mathcal{U}_X)$ is a stratified M-level L-fuzzy ditopology.*

Theorem 5.8 *By assigning the M-level L-fuzzy ditopological space $(X, \mathcal{L}_{R_X}, \mathcal{U}_{R_X})$ to an LM-fuzzy-preodered set (X, R) and interpreting monotone mappings $f : (X, R_X) \to (Y, R_Y)$ as mappings of the corresponding M-level L-fuzzy ditopological spaces f :*

$(X, \mathcal{L}_{R_X}, \mathcal{U}_{R_X}) \to (Y, \mathcal{L}_{R_Y}, \mathcal{U}_{R_Y})$, *we get an embedding functor from the category of LM-fuzzy preordered sets into the category of M-level stratified L-fuzzy ditopological spaces.*

6 Conclusions

Basing on the research done in our papers [5], [15], [16], [7], we initiate here the *many level* approach to rough approximation for L-fuzzy sets, introduce the measure of the quality of this approximation and illustrate it with examples. Our special attention in this paper is made to two alternative topological interpretations of this approximation. As the main perspectives for the further work, we see both developing theoretical aspects of many-level rough fuzzy approximation of fuzzy sets, and applications to problems of practical nature. Concerning the theoretical issues, as first, we plan to develop further the qualitative approach to the theory of many-level fuzzy rough approximation for L-fuzzy sets in the framework of category theory. An investigation of the relations between the many-level approach to rough approximation and the theory of multigranual rough sets [17] is also one of the prospectives for the future work. As one of possible applications of our approach to practical problems, we see image processing. The idea of this application was sketched by an example in the Introduction. Besides, we guess that our approach could be helpful when studying some problems of decision making in fuzzy environment.

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