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Quasi-i-Boolean algebras vs. quasi-m Boolean algebras

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Abstract

We introduce the quasi-i-Boolean algebras and the quasi-m Boolean algebras and we prove their categorically equivalence.

Keywords: Quasi-Wajsberg algebra, Quasi-MV algebra, Quasi-i-Boolean algebra, Quasi-m Boolean algebra, Wajsberg algebra, MV algebra, I-Boolean algebra, Boolean algebra

1 Introduction

The departure point for the research was the introducing of the *quasi-MV algebras* in 2006 [15], as generalizations of MV algebras, following an investigation into the foundations of quantum computing (see [4]); since then, many papers investigated them [17, 1, 6, 14]. Then, the introducing of the *quasi-Wajsberg algebras* in 2010 [2], as generalizations of Wajsberg algebras; they are categorically equivalent to quasi-MV algebras, just as Wajsberg algebras are categorically equivalent to MV algebras.

In paper [10], starting from the quasi-Wajsberg algebras, whose regular algebras are the Wajsberg algebras, we have introduced a theory of quasi-algebras (of logic) vs. a theory of regular algebras (of logic), in the commutative case. We then have developed the theory in the preprints [11, 12]. In [12], we have introduced the quasi-i-Boolean algebras, whose regular algebras are the i-Boolean algebras.

In [13], we have noted that, in fact, there are two kinds of "quasi"-generalizations, that we have called: "quasi-" and "quasi-m", corresponding to the two kinds of algebras: *M algebras* (inside the commutative algebras of logic) and *commutative unital magmas*, respectively ("m" comes from *magma*). Thus, (in [13] is the non-commutative case) we have generalized the *M algebras* to *quasi-M algebras*, as the most general quasi-algebras, by generalizing the principal, defining property (M) to (q-M), and we have generalized the *commutative unital magmas* to *commutative quasi-m unital magmas*, as the most general quasi-m algebras, by generalizing the principal, defining property (U) to (qm-U).

The *Wajsberg algebra* and the *i-Boolean algebra* belong to the "world of commutative algebras of logic", more precisely to the class of *M algebras*; the *MV algebra* and the *Boolean algebra* belong to the "world of commutative algebras", more precisely to the class of *commutative unital magmas*.

This small paper is organized as follows. In Section 2, we recall some basic things about regular algebras (structures), from [10], and about regular-m algebras (structures), from [13]; we recall the i-Boolean algebras and the Boolean algebras, and their categorically equivalence. In Section 3, we recall some basic things about quasi-algebras (quasi-structures), from [10], and about quasi-m algebras (structures), from [13]; we introduce the quasi-i-Boolean algebras and the quasi-m Boolean algebras and we prove their categorically equivalence (the main result); finally, we present some examples.

2 Regular algebras vs. regular-m algebras

2.1 Introduction to a theory of regular algebras (structures)

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an *algebra* of type $(2, 0)$ through this paper, where a binary relation $\leq$ can be *defined* by: $x \leq y \overset{def.}{\Longleftrightarrow} x \rightarrow y = 1$.
Equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a *structure*, where $\leq$ is a binary relation on A, $\rightarrow$ is a binary operation (an implication) on A and $1 \in A$, all *connected* by the equivalence: $x \leq y \Longleftrightarrow x \rightarrow y = 1$.

Consider the property: for all $x \in A$,
(M) $1 \rightarrow x = x$.

Definitions 2.1 [10]

(1) The algebra $\mathcal{A} = (A, \rightarrow, 1)$ (structure $\mathcal{A} = (A, \leq, \rightarrow, 1)$) is called *regular*, if it satisfies the property (M).

(1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\rightarrow$, 1 ($\leq$, $\rightarrow$, 1, respectively) is also called *regular*, if it satisfies (M).

(1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a regular algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *regular*.

An *M algebra* is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifying (M) [13]. Note that the class of M algebras is the largest class of regular algebras. Other examples of regular algebras are the BCI, BCK algebras, the Hilbert algebras, the commutative implicative-groups [13], the Wajsberg algebras etc.

Let us consider the following subsets of A [10]:
$U \stackrel{def.}{=} \{x \rightarrow y \mid x, y \in A\}$, $\quad V \stackrel{def.}{=} \{1 \rightarrow x \mid x \in A\}$,
$V_M \stackrel{def.}{=} \{x \in A \mid x \stackrel{(M)}{=} 1 \rightarrow x\}$.
By (M), we have that: $V_M = V = U = A$, and this is the basic, definable property of regular algebras (structures). In the theory of regular algebras (structures), we determine which properties depend on (M) (are proved by (M)) and which are independent of (M); this is the necessary step to develop further the theory of quasi-algebras (quasi-structures).

2.1.1 Implicative-Boolean algebras

There are many equivalent definitions of Boolean algebras (see [16]). We recall here the definition introduced in 2009 [8] and presented also in [3], motivated by the axioms system of the classical propositional logic:

(G1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,

(G2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$,

(G3) $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$.

An *implicative-Boolean algebra*, or an *i-Boolean algebra* for short, is an algebra

$$\mathcal{A} = (A, \rightarrow, ^-, 1)$$

of type $(2, 1, 0)$ verifying: for all $x, y, z \in A$,

(K) $\quad x \rightarrow (y \rightarrow x) = 1$,

(pimpl-1) $\quad [x \rightarrow (y \rightarrow z)] \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,

(DN1) $\quad (y^- \rightarrow x^-) \rightarrow (x \rightarrow y) = 1$,

(An) $\quad x \rightarrow y = 1 = y \rightarrow x \implies x = y$ (Antisymmetry).

Let $\mathcal{A}$ be an i-Boolean algebra. The following properties hold, among many others [12]:

(M) $1 \rightarrow x = x$;

(Re) (Reflexivity) $x \rightarrow x = 1$,

(L) (Last element) $x \rightarrow 1 = 1$,

(F) (First element) $0 \rightarrow x = 1$,

(B) $(y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,

(*) $y \rightarrow z = 1 \implies (x \rightarrow y) \rightarrow (x \rightarrow z) = 1$,

(**) $y \rightarrow z = 1 \implies (z \rightarrow x) \rightarrow (y \rightarrow x) = 1$,

(D) $y \rightarrow [(y \rightarrow x) \rightarrow x] = 1$,

(Ex) (Exchange) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(Tr) (Trans.) $x \rightarrow y = 1 = y \rightarrow z \implies x \rightarrow z = 1$,

(impl) (implicative) $(x \rightarrow y) \rightarrow x = x$;

(Neg) $x^- = x \rightarrow 0$;

(DN) (Double negation) $(x^-)^- = x$;

(Neg1-0) $1^- = 0$,

(Neg0-1) $0^- = 1$;

(Neg2) $x \rightarrow y = 1 \Longrightarrow y^- \rightarrow x^- = 1$,

(Neg3) $y \rightarrow x^- = x \rightarrow y^-$;

(Neg6) $x \rightarrow x^- = x^-$,

(DN2) $x \rightarrow y = y^- \rightarrow x^-$,

(DN6) $x^- \rightarrow x = x$.

Note that an i-Boolean algebra is a regular algebra, since (M) holds.

2.2 Introduction to a theory of regular-m algebras (structures)

Let $\mathcal{A} = (A, \odot, 1)$ be a commutative algebra of type $(2, 0)$ or let $\mathcal{A} = (A, \leq, \odot, 1)$ be a commutative structure, through this paper, where commutative means $x \odot y = y \odot x$, for all $x, y \in A$.

Consider the property: for all $x \in A$,
(U) $1 \odot x = x$.

Definitions 2.2 [13]

(1m) The commutative algebra $\mathcal{A} = (A, \odot, 1)$ of type $(2, 0)$ (structure $\mathcal{A} = (A, \leq, \odot, 1)$) is called *regular-m*, if it satisfies the property (U).

(1'm) Any commutative algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\odot$, 1 ($\leq$, $\odot$, 1, respectively) is also called *regular-m*, if it satisfies (U).

(1"m) Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a regular-m algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *regular-m*.

A *commutative unital magma* is a commutative algebra $\mathcal{A} = (A, \odot, 1)$ verifying (U). Note that their class is the largest class of regular-m algebras ("m" comes from *magma*). Other examples of regular-m algebras are the commutative monoids, the commutative groups, the MV algebras etc.

Let us consider the following subset of A [13]:

$$V_U \stackrel{def.}{=} \{x \in A \mid x \stackrel{(U)}{=} 1 \odot x\} \subseteq A.$$

Note that, if property (U) holds, then $A \subseteq V_U$, hence $V_U = A$, and this is the basic, definable property of regular-m algebras (structures). In the theory of regular-m algebras (structures), we determine which properties depend on (U) (are proved by (U)) and which are independent of (U); this is the necessary step to develop further the theory of quasi-m algebras (structures).

2.2.1 Boolean algebras

Boolean algebras were introduced in 1854 by George Boole. The most used definition is as a *complemented, bounded, distributive lattice*, namely:

A *Boolean algebra* is an algebra

$$\mathcal{A} = (A, \wedge, \vee, ^-, 0, 1)$$

of type $(2, 2, 1, 0, 0)$ verifying: for all $x, y, z \in A$,

(mWid)	$x \wedge x = x$,
(mVid)	$x \vee x = x$;
(mWcomm)	$x \wedge y = y \wedge x$,
(mVcomm)	$x \vee y = y \vee x$;
(mWass)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(mVass)	$x \vee (y \vee z) = (x \vee y) \vee z$;
(mAbs1)	$x \wedge (x \vee y) = x$,
(mAbs2)	$x \vee (x \wedge y) = x$;
(mDis1)	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(mDis2)	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
(mUW)	$1 \wedge x = x$,
(mUV)	$0 \vee x = x$;
(mWx)	$x \wedge x^- = 0$,
(mVx)	$x \vee x^- = 1$.

Note that a Boolean algebra (that can be defined equivalently as an algebra $(A, \wedge, ^-, 1)$ of type $(2, 1, 0)$) is a regular-m algebra, since (mUW), and the dual (mUV), hold.

2.3 Implicative-Boolean algebras vs. Boolean algebras

The i-Boolean algebras are categorically equivalent to the Boolean algebras just as the Wajsberg algebras are categorically equivalent to the MV algebras, namely we have:

Theorem 2.3 *[3]*
(1) Let $\mathcal{A} = (A, \rightarrow, ^-, 1)$ *be an i-Boolean algebra.*
Define $\Phi(\mathcal{A}) \overset{def.}{=} (A, \wedge, \vee, ^-, 0, 1)$ *as follows: for every* $x, y \in A$, $x \wedge y \overset{def.}{=} (x \rightarrow y^-)^-$,
$x \vee y \overset{def.}{=} (x^- \wedge y^-)^- = x^- \rightarrow y$, $\quad 0 \overset{def.}{=} 1^-$.
Then, $\Phi(\mathcal{A})$ *is a Boolean algebra.*
(1') Conversely, let $\mathcal{A} = (A, \wedge, \vee, ^-, 0, 1)$ *be a Boolean algebra.*
Define $\Psi(\mathcal{A}) \overset{def.}{=} (A, \rightarrow, ^-, 1)$ *as follows: for every* $x, y \in A$, $x \rightarrow y \overset{def.}{=} (x \wedge y^-)^- = x^- \vee y$.
Then, $\Psi(\mathcal{A})$ *is an i-Boolean algebra.*
(2) The mappings Φ *and* Ψ *are mutually inverse.*

3 Quasi-algebras vs. quasi-m algebras

3.1 Introduction to a theory of quasi-algebras (quasi-structures)

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an *algebra* of type $(2, 0)$ or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a *structure*. Consider the properties: for all $x, y \in A$,

(q-M) $1 \rightarrow (x \rightarrow y) = x \rightarrow y$,
(M1) $1 \rightarrow 1 = 1$.

Definitions 3.1 [10]
(q1) The algebra $\mathcal{A} = (A, \rightarrow, 1)$ (structure $\mathcal{A} = (A, \leq, \rightarrow, 1)$) is called *quasi-algebra* (*quasi-structure*, respectively), if it satisfies the properties (q-M) and (M1).
(q1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\rightarrow$, 1 ($\leq$, $\rightarrow$, 1, respectively) is also called *quasi-algebra* (*quasi-structure*), if it satisfies (q-M) and (M1).
(q1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a quasi-algebra (quasi-structure) $\mathcal{A}' = (A, \sigma)$, is also called *quasi-algebra* (*quasi-structure*).

A *quasi-M algebra* is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifying (q-M) and (M1) [13]. Note that their class is the largest class of quasi-algebras. Other examples of quasi-algebras are the quasi-BCI, the quasi-BCK algebras [10], the quasi-Hilbert algebras [11], the commutative quasi-implicative-groups [13], the quasi-Wajsberg algebras [2].

Remark 3.2 [10] (q-M) is different of (M) if and only if $V_M = V = U \subset A$, and this is the basic, definable property of quasi-algebras (quasi-structures).

For every quasi-algebra (quasi-structure) $\mathcal{A}$, the subset $V_M = V = U$ of A will be called the *regular* set of $\mathcal{A}$ and will be denoted by $R(A)$:

$$R(A) \overset{def.}{=} V_M = V = U \subseteq A.$$

The quasi-algebra (quasi-structure) $\mathcal{A}$ is called *q-proper*, if $R(A) \neq A$ (i.e. (M) $\not\Longleftrightarrow$ (q-M)); otherwise, $\mathcal{A}$ is a regular algebra (structure). The elements of $R(A)$ are called the *regular elements* of A, the elements of $A \setminus R(A)$ are called the *quasi-elements* of A.

Theorem 3.3 [10] Let $\mathcal{A} = (A, \rightarrow, 1)$ be a q-proper quasi-algebra. Then, $\mathcal{R}(\mathcal{A}) = (R(A), \rightarrow, 1)$ is a regular algebra.

In a q-proper quasi-algebra, we shall call *quasi-property*, a property that is dependent of (q-M), and we shall call *regular property*, a property that is independent of (q-M).

3.1.1 Quasi-implicative-Boolean algebras

We have introduced the notion of quasi-implicative-Boolean algebra in the preprint [12], as follows:
A *quasi-implicative-Boolean algebra*, or a *quasi-i-Boolean algebra* for short, is an algebra

$$\mathcal{A} = (A, \rightarrow, ^-, 1)$$

of type $(2,1,0)$ verifying: for all $x,y,z \in A$,
(K) $x \to (y \to x) = 1$,
(pimpl-1) $[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$,
(DN1) $(y^- \to x^-) \to (x \to y) = 1$,
(q-An) $x \to y = 1 = y \to x \implies 1 \to x = 1 \to y$ (quasi-Antisymmetry),
(q-M) $1 \to (x \to y) = x \to y$,
(DN) $(x^-)^- = x$,
$(q-\overline{M(1 \to x)})$ $1 \to (1 \to x)^- = (1 \to x)^-$.

Note that a quasi-i-Boolean algebra is a quasi-algebra, since (q-M) and (M1) hold. Indeed, by [10], $x \to 1 \overset{(q-M)}{=} 1 \to (x \to 1) \overset{(K)}{=} 1$, hence (L) holds and (L) implies (M1). Note also that, if (M) holds, then any quasi-i-Boolean algebra is an i-Boolean algebra.

Theorem 3.4 *[12] Let $\mathcal{A} = (A, \to, ^-, 1)$ be a q-proper quasi-i-Boolean algebra. Then,*

$$\mathcal{R}(\mathcal{A}) = (R(A), \to, ^-, 1)$$

is an i-Boolean algebra.

The quasi-i-Boolean algebras verify the regular properties: (Re), (L), (F), (B), (*), (**), (D), (Ex), (Tr), (Neg1-0), (Neg0-1), (Neg2), (Neg3), (DN2), among others [12], and the following quasi-properties, among others [12]:
(q-I1) $x \to y = (1 \to x) \to y$,
(q-I2) $x \to y = x \to (1 \to y)$;
(q-impl) (quasi-implicative) $(x \to y) \to x = 1 \to x$;
(q-Neg) $x \to 0 = 1 \to x^- = (1 \to x)^-$,
(α) $x \to 0 = 1 \to x^-$, (β) $1 \to x^- = (1 \to x)^-$;
(q-Neg6) $x \to (1 \to x)^- = (1 \to x)^-$,
(q-DN6) $(1 \to x)^- \to x = 1 \to x$.

Proposition 3.5 *Let $(A, \to, ^-, 1)$ be a quasi-i-Boolean algebra. Define, for all $x, y \in A$,*
$x \wedge y \overset{def.}{=} (x \to y^-)^-$, $x \vee y \overset{def.}{=} (x^- \wedge y^-)^- = x^- \to y$,
$x \leq y \overset{def.}{\Longleftrightarrow} x \to y = 1$.
Then, we have: for all $x,y,z \in A$,
(WEx) $(x \wedge y) \to z = x \to (y \to z)$,
(j) $x \wedge y \leq x \leq x \vee y$,
(jj) $x \leq y$ *and* $a \leq b$ *imply* $x \wedge a \leq y \wedge b$, $x \vee a \leq y \vee b$.

3.2 Introduction to a theory of quasi-m algebras (structures)

Let $\mathcal{A} = (A, \odot, 1)$ be a commutative algebra or let $\mathcal{A} = (A, \leq, \odot, 1)$ be a commutative structure. Let us consider the following properties: for all $x, y \in A$,
(qm-U) $1 \odot (x \odot y) = x \odot y$,
(U1) $1 \odot 1 = 1$.

Definitions 3.6 [13]
(qm1) The commutative algebra $\mathcal{A} = (A, \odot, 1)$ (structure $\mathcal{A} = (A, \leq, \odot, 1)$) is called *quasi-m algebra (structure)*, if it satisfies the properties (qm-U), (U1).

(qm1') Any commutative algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\odot$, 1 ($\leq$, $\odot$, 1, respectively) is also called *quasi-m algebra (structure)*, if it satisfies (qm-U) and (U1).

(qm1") Any commutative algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a quasi-m algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *quasi-m algebra (structure)*.

A *commutative quasi-m unital magma* is a commutative algebra $\mathcal{A} = (A, \odot, 1)$ verifying (qm-U) and (U1) [13]. Note that their class is the largest class of quasi-m algebras. Other examples of quasi-m algebras are the commutative quasi-m monoids, quasi-m groups [13], the quasi-MV algebras [15] (in fact quasi-m MV algebras, by [13]).

Remark 3.7 [13] (qm-U) is different of (U) if and only if $V_U \subset A$, and this is the basic, definable property of quasi-m algebras (quasi-m structures).

For every quasi-m algebra (structure) $\mathcal{A}$, the subset V_U of A will be called the *regular-m* set of $\mathcal{A}$ and will be denoted by $Rm(A)$:

$$Rm(A) \overset{def.}{=} V_U \subseteq A.$$

The quasi-m algebra (structure) $\mathcal{A}$ is called *qm-proper*, if $Rm(A) \neq A$ (i.e. (U) $\not\Longleftrightarrow$ (qm-U)); otherwise, $\mathcal{A}$ is a regular-m algebra (structure). The elements of $Rm(A)$ are called the *regular-m elements* of A, the elements of $A \setminus Rm(A)$ are called the *quasi-m elements* of A.

Theorem 3.8 [13] Let $\mathcal{A} = (A, \odot, 1)$ be a qm-proper quasi-m algebra. Then, $\mathcal{R}m(\mathcal{A}) = (Rm(A), \odot, 1)$ is a regular-m algebra.

In a qm-proper quasi-m algebra, we shall call *quasi-m property*, a property that is dependent of (qm-U), and we shall call *regular-m property*, a property that is independent of (qm-U).

3.2.1 Quasi-m Boolean algebras

A *quasi-m Boolean algebra* is an algebra

$$(A, \wedge, \vee, ^-, 0, 1)$$

of type $(2,2,1,0,0)$ verifying: for all $x, y, z \in A$,
(UW1) $1 \wedge 1 = 1$,
(UV0) $0 \vee 0 = 0$;
(qm-Wid) $x \wedge x = 1 \wedge x$,
(qm-Vid) $x \vee x = 0 \vee x$;
(mWcomm) $x \wedge y = y \wedge x$,
(mVcomm) $x \vee y = y \vee x$;
(mWass) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(mVass) $x \vee (y \vee z) = (x \vee y) \vee z$;
(qm-Abs1) $x \vee (x \wedge y) = 1 \wedge x$,
(qm-Abs2) $x \wedge (x \vee y) = 0 \vee x$;

(IdU) $1 \wedge x = 0 \vee x$;
(qm-UW) $1 \wedge (x \wedge y) = x \wedge y$,
(qm-UV) $0 \vee (x \vee y) = x \vee y$;
(mDis1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(mDis2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
(mWx) $x \wedge x^- = 0$,
(mVx) $x \vee x^- = 1$;
(DN) $(x^-)^- = x$;
(dfV) $x \vee y = (x^- \wedge y^-)^-$;
(qm-$\overline{UW(1 \wedge x)}$) $1 \wedge (1 \wedge x)^- = (1 \wedge x)^-$,
(qm-$\overline{UV(0 \vee x)}$) $0 \vee (0 \vee x)^- = (0 \vee x)^-$.

Note that a quasi-m Boolean algebra (that can be defined equivalently as an algebra $(A, \wedge, ^-, 1)$ of type $(2, 1, 0)$) is a quasi-m algebra, since (qm-UW), (UW1), and the dual (qm-UV), (UV1), hold. Note also that, if (U) (here (UW) and (UV)) holds, then any quasi-m Boolean algebra is a Boolean algebra.

Theorem 3.9 Let $\mathcal{A} = (A, \wedge, \vee, ^-, 0, 1)$ be a qm-proper quasi-m Boolean algebra. Then,

$$\mathcal{R}m(\mathcal{A}) = (Rm(A), \wedge, \vee, ^-, 0, 1)$$

is a Boolean algebra.

Proposition 3.10 *Let $(A, \wedge, \vee, ^-, 0, 1)$ be a quasi-m Boolean algebra. We have:*
(i) $(x \wedge y^-)^- = x^- \vee y$,
(ii) (De Morgan laws) $(x \wedge y)^- = x^- \vee y^-$ and $(x \vee y)^- = x^- \wedge y^-$,
(iii) $0^- = 1$ and $1^- = 0$,
(iv) $x \vee 1 = 1$ and $x \wedge 0 = 0$,
(v)=(qm-Neg) $(x \wedge 1)^- = x^- \wedge 1$ and $(x \vee 0)^- = x^- \vee 0$.

3.3 Quasi-i-Boolean algebras vs. quasi-m Boolean algebras

The quasi-i-Boolean algebras are categorically equivalent to the quasi-m Boolean algebras just as the quasi-Wajsberg algebras are categorically equivalent to the quasi-m MV algebras, namely we have:

Theorem 3.11 *(See Theorem 2.3)*

(1) Let $\mathcal{A} = (A, \to, ^-, 1)$ be a quasi-i-Boolean algebra.
Define $\Phi(\mathcal{A}) \overset{def.}{=} (A, \wedge, \vee, ^-, 0, 1)$ as follows: for every $x, y \in A$, $x \wedge y \overset{def.}{=} (x \to y^-)^-$,
$x \vee y \overset{def.}{=} (x^- \wedge y^-)^- = x^- \to y$, $0 \overset{def.}{=} 1^-$.
Then, $\Phi(\mathcal{A})$ is a quasi-m Boolean algebra.

(1') Conversely, let $\mathcal{A} = (A, \wedge, \vee, ^-, 0, 1)$ be a quasi-m Boolean algebra.
Define $\Psi(\mathcal{A}) \overset{def.}{=} (A, \to, ^-, 1)$ as follows: for every $x, y \in A$, $x \to y \overset{def.}{=} (x \wedge y^-)^- \overset{(i)}{=} x^- \vee y$.
Then, $\Psi(\mathcal{A})$ is a quasi-i-Boolean algebra.

(2) The mappings Φ and Ψ are mutually inverse.

Proof. (1): First note that we have:
(a) $1 \wedge x = 1 \to x$ and (b) $0 \vee x = 1 \to x$.
Indeed, $1 \wedge x = (1 \to x^-)^- \overset{(\beta)}{=} ((1 \to x)^-)^- \overset{(DN)}{=} 1 \to x$, thus (a) holds.
$0 \vee x = 0^- \to x \overset{(Neg0-1)}{=} 1 \to x$, thus (b) holds too.

(UW1): $1 \wedge 1 = (1 \to 1^-)^- \overset{(Neg1-0)}{=} (1 \to 0)^- \overset{(q-Neg)}{=} 0 \to 0 \overset{(Re)}{=} 1$ and

(UV0): $0 \vee 0 = 0^- \to 0 \overset{(Neg0-1)}{=} 1 \to 0 \overset{(q-Neg)}{=} (1 \to 1)^- \overset{(Re)}{=} 1^- \overset{(Neg1-0)}{=} 0$.

(qm-Wid): $x \wedge x = (x \to x^-)^- \overset{(q-I2)}{=} (x \to (1 \to x^-))^- \overset{(\beta)}{=} [x \to (1 \to x)^-]^- \overset{(q-Neg6)}{=} ((1 \to x)^-)^- \overset{(DN)}{=} 1 \to x \overset{(a)}{=} 1 \wedge x$ and

(qm-Vid): $x \vee x = (x^- \wedge x^-)^- \overset{(qm-Wid)}{=} (1 \wedge x^-)^- \overset{(Neg0-1)}{=} (0^- \wedge x^-)^- = 0 \vee x$.

(mWcomm): $x \wedge y = (x \to y^-)^- \overset{(Neg3)}{=} (y \to x^-)^- = y \wedge x$ and

(mVcomm): $x \vee y = (x^- \wedge y^-)^- = (y^- \wedge x^-)^- = y \vee x$.

(mWass): $x \wedge (y \wedge z) = x \wedge (y \to z^-)^- = [x \to ((y \to z^-)^-)^-]^- \overset{(DN)}{=} [x \to (y \to z^-)]^-$ and $(x \wedge y) \wedge z \overset{(mWcomm)}{=} z \wedge (x \to y^-)^- = [z \to ((x \to y^-)^-)^-]^- \overset{(DN)}{=} [z \to (x \to y^-)]^- \overset{(Ex)}{=} [x \to (z \to y^-)]^- \overset{(Neg3)}{=} [x \to (y \to z^-)]^-$, hence $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

(mVass): $x \vee (y \vee z) = x \vee (y^- \wedge z^-)^- = [x^- \wedge ((y^- \wedge z^-)^-)^-]^- \overset{(DN)}{=} [x^- \wedge (y^- \wedge z^-)]^- \overset{(mWass)}{=} [(x^- \wedge y^-) \wedge z^-]^- \overset{(DN)}{=} [((x^- \wedge y^-)^-)^- \wedge z^-]^- = [(x \vee y)^- \wedge z^-]^- = (x \vee y) \vee z$.

(qm-Abs1): $x \vee (x \wedge y) \overset{(mVcomm)}{=} (x \wedge y) \vee x = (x \wedge y)^- \to x = ((x \to y^-)^-)^- \to x \overset{(DN)}{=} (x \to y^-) \to x \overset{(q-impl)}{=} 1 \to x \overset{(a)}{=} 1 \wedge x$ and

(qm-Abs2): $x \wedge (x \vee y) \overset{(mWcomm)}{=} (x \vee y) \wedge x = [(x^- \to y) \to x^-]^- \overset{(q-impl)}{=} (1 \to x^-)^- = 1 \wedge x \overset{(a)}{=} 1 \to x \overset{(b)}{=} 0 \vee x$.

(IdU): $1 \wedge x \overset{(a)}{=} 1 \to x \overset{(b)}{=} 0 \vee x$.

(qm-UW): $1 \wedge (x \wedge y) = 1 \wedge (x \to y^-)^- = [1 \to ((x \to y^-)^-)^-]^- \overset{(DN)}{=} [1 \to (x \to y^-)]^- \overset{(q-M)}{=} (x \to y^-)^- = x \wedge y$ and

(qm-UV): $0 \vee (x \vee y) = 0 \vee (x^- \to y) = 0^- \to (x^- \to y) \overset{(Neg0-1)}{=} 1 \to (x^- \to y) \overset{(q-M)}{=} x^- \to y = x \vee y$.

(mDis1): If $x \leq y \overset{def.}{\Longleftrightarrow} x \to y = 1$, then we shall prove:
(c) $(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$ and
(d) $(x \vee y) \wedge z \leq (x \wedge z) \vee (y \wedge z)$.

To prove (c): since $x \wedge z \leq z$ and $y \wedge z \leq z$, then

$(x\wedge z)\vee(y\wedge z) \le z\vee z \overset{(qm-Vid)}{=} 0\vee z$ and since $x\wedge z\le x$ and $y\wedge z \le y$, then $(x\wedge z)\vee(y\wedge z)\le x\vee y$, by (j) and (jj); then,
$(x\wedge z)\vee(y\wedge z) \overset{(qm-UV)}{=} 0\vee[(x\wedge z)\vee(y\wedge z)] \overset{(IdU)}{=}$
$1\wedge[(x\wedge z)\vee(y\wedge z)] \overset{(qm-Wid)}{=}$
$[(x\wedge z)\vee(y\wedge z)]\wedge[(x\wedge z)\vee(y\wedge z)] \overset{(jj)}{\le} (0\vee z)\wedge(x\vee y) \overset{(IdU)}{=}$
$(1\wedge z)\wedge(x\vee y) \overset{(mWass)}{=} 1\wedge[z\wedge(x\vee y)] \overset{(qm-UW)}{=}$
$z\wedge(x\vee y) \overset{(mWcomm)}{=} (x\vee y)\wedge z$.

To prove (d), first we prove

$$x\vee y\le z\to[(x\wedge z)\vee(y\wedge z)]. \qquad (1)$$

Indeed, since $z\to[(x\wedge z)\vee(y\wedge z)] \overset{(DN),(mWcomm)}{=}$
$z\to[(z\to x^-)\to(y\to z^-)^-] \overset{(Ex)}{=}$
$(z\to x^-)\to[z\to(y\to z^-)^-]$,
then (1) is equivalent to

$$x\vee y\le(z\to x^-)\to[z\to(y\to z^-)^-]. \qquad (2)$$

Now we prove

$$y\le z\to(y\to z^-)^-. \qquad (3)$$

Indeed, $y \overset{(D')}{\le} (y\to z^-)\to z^- \overset{(DN2),(DN)}{=}$
$z\to(y\to z^-)^-$, hence (3) holds.
From (3), by (*'), we obtain:
$(z\to x^-)\to y\le(z\to x^-)\to[z\to(y\to z^-)^-]$
and, by (K'), we obtain: $y\le(z\to x^-)\to y$; it follows by (Tr') that

$$y\le(z\to x^-)\to[z\to(y\to z^-)^-]. \qquad (4)$$

By (B'), we obtain:
$x^-\to(y\to z^-)^-\le(z\to x^-)\to[z\to(y\to z^-)^-]$
and by (K'), (DN2), we obtain:
$x\le(y\to z^-)\to x=x^-\to(y\to z^-)^-$.
It follows, by (Tr'), that

$$x\le(z\to x^-)\to[z\to(y\to z^-)^-]. \qquad (5)$$

From (4) and (5), we obtain, by (jj):
$x\vee y\le$
$((z\to x^-)\to[z\to(y\to z^-)^-])\vee$
$((z\to x^-)\to[z\to(y\to z^-)^-]) \overset{(qm-Vid)}{=}$
$0\vee((z\to x^-)\to[z\to(y\to z^-)^-]) \overset{(b)}{=}$
$1\to((z\to x^-)\to[z\to(y\to z^-)^-]) \overset{(q-M)}{=}$
$(z\to x^-)\to[z\to(y\to z^-)^-]$,
hence (2) holds, hence (1) holds.

Now, since (1) means
$(x\vee y)\to(z\to[(x\wedge z)\vee(y\wedge z)])=1$,
it follows, by (WEx), that
$[(x\vee y)\wedge z]\to[(x\wedge z)\vee(y\wedge z)]=1$, i.e. (d) holds.

By (c), (d) and (q-An), (β), (q-M), we obtain:
$(x\vee y)\wedge z=((x\vee y)\to z^-)^- \overset{(q-M)}{=}$
$[1\to((x\vee y)\to z^-)]^- \overset{(\beta)}{=} 1\to[(x\vee y)\to z^-]^- =$
$1\to[(x\vee y)\wedge z] \overset{(q-An)}{=} 1\to[(x\wedge z)\vee(y\wedge z)] =$
$1\to[(x\wedge z)^-\to(y\wedge z)] \overset{(q-M)}{=} (x\wedge z)^-\to(y\wedge z) =$
$(x\wedge z)\vee(y\wedge z)$, i.e. (mDis1) holds. The second part, (mDis2), has a similar proof.

(mWx): $x\wedge x^-=(x\to(x^-)^-)^- \overset{(DN)}{=}$
$(x\to x)^- \overset{(Re)}{=} 1^- \overset{(Neg1-0)}{=} 0$ and

(mVx): $x\vee x^-=x^-\to x^- \overset{(Re)}{=} 1$.

(DN): By hypothesis. (dfV): By definition.

(qm-$\overline{UW(1\wedge x)}$) : $1\wedge(1\wedge x)^- \overset{(a)}{=}$
$1\to(1\to x)^- \overset{(q-\overline{M(1\to x)})}{=} (1\to x)^- \overset{(a)}{=} (1\wedge x)^-$ and

(qm-$\overline{UV(0\vee x)}$): $0\vee(0\vee x)^- \overset{(b)}{=}$
$1\to(1\to x)^- \overset{(q-\overline{M(1\to x)})}{=} (1\to x)^- \overset{(b)}{=} (0\vee x)^-$.

Thus, $\Phi(\mathcal{A})$ is a quasi-m Boolean algebra.

(1'): First note that we have:
(e) $1\to x=0\vee x \overset{(IdU)}{=} 1\wedge x$.
Indeed, $1\to x=(1\wedge x^-)^- \overset{(iii)}{=} (0^-\wedge x^-)^- \overset{(dfV)}{=} 0\vee x$.

(K): $x\to(y\to x)=x^-\vee(y\to x)=$
$x^-\vee(y^-\vee x) \overset{(mVcomm),(mVass)}{=} (x\vee x^-)\vee y^- \overset{(mVx)}{=}$
$1\vee y^- \overset{(iv)}{=} 1$.

(pimpl-1): $[x\to(y\to z)]\to[(x\to y)\to(x\to z)]$
$=[x^-\vee(y^-\vee z)]^-\vee[(x^-\vee y)^-\vee(x^-\vee z)] \overset{(DN),(ii)}{=}$
$[x\wedge(y\wedge z^-)]\vee[(x\wedge y^-)\vee x^-\vee z] \overset{(mDis2)}{=}$
$[x\wedge y\wedge z^-]\vee[(x\vee x^-\vee z)\wedge(y^-\vee x^-\vee z)] \overset{(mVx),(iv)}{=}$
$[x\wedge y\wedge z^-]\vee[1\wedge(y^-\vee x^-\vee z)] \overset{(IdU)}{=}$
$[x\wedge y\wedge z^-]\vee[0\vee(y^-\vee x^-\vee z)] \overset{(qm-UV)}{=}$
$[x\wedge y\wedge z^-]\vee[y^-\vee x^-\vee z] \overset{(ii)}{=}$
$[x\wedge y\wedge z^-]\vee[y\wedge x\wedge z^-]^- \overset{(mVx)}{=} 1$.

(DN1): $(y^-\to x^-)\to(x\to y)=$
$((y^-)^-\vee x^-)^-\vee(x^-\vee y) \overset{(DN)}{=}$
$(y\vee x^-)^-\vee(x^-\vee y) \overset{(mVx)}{=} 1$.

(q-An): $x\to y=1=y\to x$ means $x^-\vee y=1=y^-\vee x$, hence $0=y\wedge x^-$, by (DN), (ii), (iii). Then,
$1\to x=1^-\vee x \overset{(iii)}{=} 0\vee x=$
$(y\wedge x^-)\vee x \overset{(mVcomm),(mDis2)}{=} (y\vee x)\wedge(x^-\vee x) \overset{(mVx)}{=}$
$(y\vee x)\wedge 1=(y\vee x)\wedge(x^-\vee y)=(y\vee x)\wedge(y\vee x^-) \overset{(mDis2)}{=}$
$y\vee(x\wedge x^-) \overset{(mWx)}{=} y\vee 0 \overset{(e)}{=} 1\to y$.

(q-M): $1\to(x\to y)=1^-\vee(x^-\vee y) \overset{(iii)}{=}$
$0\vee(x^-\vee y) \overset{qm-UV)}{=} x^-\vee y=x\to y$.

(DN): By hypothesis.

(q-$\overline{M(1\to x)}$): $1\to(1\to x)^-=1^-\vee(1^-\vee x)^- \overset{(iii)}{=}$
$0\vee(0\vee x)^- \overset{(qm-\overline{UV(0\vee x)})}{=} (0\vee x)^-=(1^-\vee x)^-=$
$(1\to x)^-$.

Thus, $\Psi(\mathcal{A})$ is a quasi-i-Boolean algebra.

(2): Let
$(A, \to, ^-, 1) \xrightarrow{\Phi} (A, \wedge, \vee, ^-, 0, 1) \xrightarrow{\Psi} (A, \Rightarrow, ^-, 1)$.
Then, for all $x, y \in A$, we have:
$x \Rightarrow y = x^- \vee y = (x^-)^- \to y \overset{(DN)}{=} x \to y$,
hence $\Psi \circ \Phi = 1_{(A,\to,^-,1)}$.

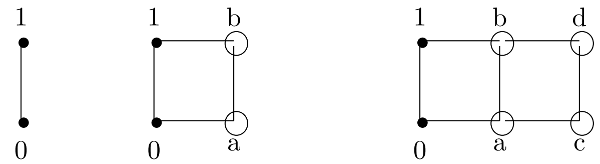
Conversely, let
$(A, \wedge, \vee, ^-, 0, 1) \overset{\Psi}{\to} (A, \to, ^-, 1) \overset{\Phi}{\to} (A, \bigwedge, \bigvee, ^-, \mathbf{0}, 1)$.
Then, for all $x, y \in A$, we have:
$x \bigwedge y = (x \to y^-)^- = (x^- \vee y^-)^- \overset{(ii),(DN)}{=} x \wedge y$,
$x \bigvee y = x^- \to y = (x^-)^- \vee y = x \vee y$, $\mathbf{0} = 1^- \overset{(iii)}{=} 0$,
hence $\Phi \circ \Psi = 1_{(A,\wedge,\vee,^-,0,1)}$. □

3.4 Examples

Consider the standard i-Boolean algebra

$$\mathcal{L}_2 = (L_2 = \{0, 1\}, \to, ^-, 1),$$

represented by the Hasse diagram given in Figure 1 and with the tables of $\to$ and $^-$ (involutive negation) recalled below:



$\mathcal{L}_2$ $\quad$ $\mathcal{L}_2^1$ $\quad$ $\mathcal{L}_2^{2^{1,2}}$

Boolean algebra $\quad$ quasi-i-Boolean algebras

Figure 1: The Boolean algebra $\mathcal{L}_2$ and the quasi-i-Boolean algebras $\mathcal{L}_2^1$, $\mathcal{L}_2^{2^{1,2}}$

$\mathcal{L}_2$

$\to$	**0**	**1**
0	**1**	**1**
1	**0**	**1**

,

x	$x^-(= x \to 0)$
0	**1**
1	**0**

.

By Theorem 2.3, the standard i-Boolean algebra $\mathcal{L}_2$ is term-equivalent to the standard Boolean algebra

$$\mathcal{L}_2^m = (L_2 = \{0, 1\}, \wedge, \vee, ^-, 0, 1),$$

represented by the same Hasse diagram given in Figure 1 and with the tables of $\wedge$, $\vee$ given below and the same table of $^-$ (involutive negation):

$\mathcal{L}_2^m$

$\wedge$	**0**	**1**
0	**0**	**0**
1	**0**	**1**

,

$\vee$	**0**	**1**
0	**0**	**1**
1	**1**	**1**

.

We can build an infinity of quasi-i-Boolean algebras whose regular algebra be the standard i-Boolean algebra $\mathcal{L}_2$, by [10]. We present two examples.

3.4.1 Example 1

By adding two quasi-elements, a, b, to the above regular elements $0, 1$, such that $a \parallel 0$ and $b \parallel 1$ (see [10]), we obtain the quasi-i-Boolean algebra

$$\mathcal{L}_2^1 = (L_2^1 = \{0, a, b, 1\}, \to^1, ^-, 1),$$

represented by the quasi-Hasse diagram (see [10]) given also in Figure 1 and with the following tables of $\to^1$ and $^-$ (involutive quasi-negation):

$\mathcal{L}_2^1$

$\to^1$	**0**	a	b	**1**
0	**1**	1	1	**1**
a	1	1	1	1
b	0	0	1	1
1	**0**	0	1	**1**

,

x	x^-
0	**1**
a	b
b	a
1	**0**

.

Note that the line/column of a (b) coincides with the line/column of 0 (1, respectively).

By Theorem 3.11, the quasi-i-Boolean algebra $\mathcal{L}_2^1$ is term-equivalent to the quasi-m Boolean algebra

$$\mathcal{L}_2^{1m} = (L_2^1 = \{0, a, b, 1\}, \wedge^1, \vee^1, ^-, 0, 1),$$

with the tables of $\wedge^1$, $\vee^1$ given below and the same table of $^-$ (involutive quasi-m negation, this time):

$\mathcal{L}_2^{1m}$

$\wedge^1$	**0**	a	b	**1**
0	**0**	0	0	**0**
a	0	0	0	0
b	0	0	1	1
1	**0**	0	1	**1**

,

$\vee^1$	**0**	a	b	**1**
0	**0**	0	1	**1**
a	0	0	1	1
b	1	1	1	1
1	**1**	1	1	**1**

.

Note that the line/column of a (b) coincides with the line/column of 0 (1, respectively).

Note that $\mathcal{R}(\mathcal{L}_2^1) = \mathcal{L}_2$ and $\mathcal{R}m(\mathcal{L}_2^{1m}) = \mathcal{L}_2^m$.

3.4.2 Example 2

By adding four quasi-elements, a, b, c, d, to the above regular elements $0, 1$, such that $c \parallel a \parallel 0$ and $d \parallel b \parallel 1$ (see [10]), we obtain two quasi-i-Boolean algebras:
$\mathcal{L}_2^{2^1} = (L_2^2 = \{0, a, b, c, d, 1\}, \to^2, ^{-^1}, 1)$ and
$\mathcal{L}_2^{2^2} = (L_2^2 = \{0, a, b, c, d, 1\}, \to^2, ^{-^2}, 1)$, respectively,
(with the same table of $\to^2$, but with different involutive quasi-negations),
represented by the quasi-Hasse diagram (see [10]) given also in Figure 1 and with the following tables of $\to^2$ and $^{-^1}$, $^{-^2}$:

$\to^2$	**0**	a	b	c	d	**1**
0	**1**	1	1	1	1	**1**
a	1	1	1	1	1	1
b	0	0	1	0	1	1
c	1	1	1	1	1	1
d	0	0	1	0	1	1
1	**0**	0	1	0	1	**1**

,

x	x^{-^1}	x^{-^2}
0	**1**	**1**
a	b	d
b	a	c
c	d	b
d	c	a
1	**0**	**0**

.

Note that the lines/columns of a, c (b, d) coincide with the line/column of 0 (1, respectively).

By Theorem 3.11, the quasi-i-Boolean algebras $\mathcal{L}_2^{2^{1,2}}$ are term-equivalent to the quasi-m Boolean algebras:
$\mathcal{L}_2^{2^{1m}} = (L_2^2 = \{0, a, b, c, d, 1\}, \wedge^2, \vee^2, ^{-^1}, 0, 1)$ and
$\mathcal{L}_2^{2^{2m}} = (L_2^2 = \{0, a, b, c, d, 1\}, \wedge^2, \vee^2, ^{-^2}, 0, 1)$, respectively, with the tables of $\wedge^2$, $\vee^2$ given below and the same tables of $^{-^1}$ and $^{-^2}$ (involutive quasi-m negations, this time):

$\mathcal{L}_2^{2^{1,2m}}$

$\wedge^2$	**0**	a	b	c	d	**1**
0	**0**	0	0	0	0	**0**
a	0	0	0	0	0	0
b	0	0	1	0	1	1
c	0	0	0	0	0	0
d	0	0	1	0	1	1
1	**0**	0	1	0	1	**1**

,

$\vee^2$	**0**	a	b	c	d	**1**
0	**0**	0	1	0	1	**1**
a	0	0	1	0	1	1
b	1	1	1	1	1	1
c	0	0	1	0	1	1
d	1	1	1	1	1	1
1	**1**	1	1	1	1	**1**

.

Note that the lines/columns of a, c (b, d) coincide with the line/column of 0 (1, respectively).

Note that $\mathcal{R}(\mathcal{L}_2^{2^{1,2}}) = \mathcal{L}_2$ and $\mathcal{R}m(\mathcal{L}_2^{2^{1,2m}}) = \mathcal{L}_2^m$.

Other examples will be presented in the book in preparation, on *quasi-algebras vs. quasi-m algebras.*

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