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Stability of Fuzzy Cognitive Maps with Interval Weights

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Abstract

In fuzzy cognitive maps (FCMs) based modelling paradigm, the complex system's behaviour is gathered by the causal connections acting between its main characteristics or subsystems. The system is represented by a weighted, directed digraph, where the nodes represent specific subsystems or features, while the weighted and directed edges express the direction and strength of causal relations between them. The state of the complex system represented by the so-called activation values of the nodes, that is computed by an iterative method. The FCM based decision-making relies on the assumption that this iteration reaches an equilibrium point (fixed point), but other types of behaviour, namely limit cycles and chaotic patterns may also show up. In practice, the weights of connections are estimated by human experts or machine learning methods. Both cases have their own uncertainty, which can be represented by using intervals as weights instead of crisp numbers. In this paper, sufficient conditions are provided for the existence and uniqueness of fixed points of fuzzy cognitive maps that are equipped with interval weights, which also ensure the global asymptotic stability of the system.

Keywords: Fuzzy cognitive maps, Interval-valued fuzzy cognitive maps, Convergence of fuzzy cognitive maps, Stability, Equilibrium

1 Introduction

Fuzzy cognitive maps (FCMs) are network-based models [12] that are able to simulate the behaviour of complex systems, especially, when many interrelating factors should be considered by the decision maker [1]. These techniques form an effective way of interpreting and representing expert knowledge about complex systems [14], [3], including causalities and uncertainties.

1.1 Classical Fuzzy Cognitive Maps

The base of an FCM model is a directed digraph in which causal weights are assigned to the edges from the interval $[-1, 1]$. These weights are devoted to representing the strength and direction of causal relationships. The nodes (called concepts in the FCM literature) represent subsystems or specific factors of the modelled system. The current states (activation values) of the concepts are described by numbers from a given set, which is usually the $[0, 1]$ or the $[-1, 1]$ interval [15].

Formally a fuzzy cognitive map is a 4-tuple (C, W, A, f). Here C is the set of concepts $C = \{C_1, C_2, \dots, C_n\}$, W is the weight matrix, which assigns a causal value w_{ij} $(-1 \leq w_{ij} \leq 1)$ to each edge connecting the nodes C_i and C_j, describing how strongly influenced is concept C_i by concept C_j The sign of w_{ij} indicates whether the relationship between C_j and C_i is direct or inverse. Consequently, weight matrix $W \in \mathbb{R}^{n\times n}$ gathers the system causality. The function $A : C \to \mathbb{R}$ assigns an activation value $A_i \in \mathbb{R}$ to each node C_i at each time step during the simulation. A transformation or threshold function f calculates the activation value of concepts and keeps them in the allowed range, which is usually the $[0, 1]$ or the $[-1, 1]$ interval. The iteration which calculates the values of the concept may or may not include self-feedback. Self-feedbacks were not allowed in early FCMs, but there are many examples, where a concept has some kind of 'memory'. The intensity of that memory can be expressed by the weight of the self-feedback. The theoretical background of self-feedbacks is already laid down [29, 30], and several real-life examples can be found for their application [31, 32, 33] as well.

In general form, the iteration rule can be written as

$$A_i^{(k+1)} = f\left(\sum_{j=1, j\neq i}^{n} w_{ij}A_j^{(k)} + d_i A_i^{(k)}\right) \quad (1)$$

where $A_i^{(k)}$ is the value of concept C_i at discrete time k, w_{ij} is the weight of the connection from concept C_j to concept C_i and d_i expresses the strength of the self-feedback ($0 \leq d_i \leq 1$).

If we include the self-feedback into the weight matrix W (so $w_{ii} = d_i$) then the equation can be rewritten in a more compact form:

$$A_i^{(k+1)} = f\left(\sum_{j=1}^{n} w_{ij}A_j^{(k)}\right) = f(w_i A^{(k)}), \quad (2)$$

where $w_i = [w_{i1}, \ldots, w_{in}]$ is the ith row of W and $A^{(k)} = [A_1^{(k)}, \ldots, A_n^{(k)}]^T$ is the concept vector after k iterations, so $w_i A^{(k)}$ is a real number.

Moreover, if we couple the coordinates of the concept vector together and denote by G the mapping $\mathbb{R}^n \to \mathbb{R}^n$ that generates the concept vector $A^{(k+1)}$ from $A^{(k)}$, then we have that:

$$A^{(k+1)} = \begin{bmatrix} f(w_1 A^{(k)}) \\ \vdots \\ f(w_n A^{(k)}) \end{bmatrix} = G(A^{(k)}). \quad (3)$$

From the mathematical point of view, the analysis of the long term behaviour of the fuzzy cognitive map means the analysis of the iteration $A^{(k+1)} = G(A^{(k)})$, that is a nonlinear, multivariate difference (not differential) equation, a.k.a. nonlinear, multivariate discrete dynamical system (the term 'discrete' means that the time is discrete). A fuzzy cognitive map with continuous threshold function may produce one of the following long-term behaviours [15],[10], [11]:

- It may reache a fixed point, so the state vector stabilizes after a certain number of iterations.
- It may produce limit cycles, which means that a certain number of consecutive state vectors turn up repeatedly.
- It may show chaotic behaviour. Chaotic behaviour means that the activation vector never stabilizes.

The behaviour of an FCM may be influenced by the following factors:

- threshold function and its parameter(s);
- elements (weights) of the weight matrix and the topology of the map;
- initial state vector.

1.2 Fuzzy Cognitive Maps with Uncertain Weights

The weights of the causal connections in the fuzzy cognitive map are usually determined by human experts or computed by machine learning techniques using historical data. Although classical FCMs apply crisp weight values, in both cases there is some uncertainty about the precise value of the weights. Several extensions of FCMs have been introduced to handle these uncertainties, below we list a few of them.

Fuzzy grey cognitive maps (FGCMs) apply grey system theory for representing uncertainties [20]. This modelling paradigm was successfully applied in real-life problems, for example in reliability engineering [22], radiotherapy [21] and supplier selection [28].

Intuitionistic FCMs employ membership degree and non-membership degree, expressing the hesitancy of the decision maker [26], [27].

Interval-valued FCMs (IVFCMs) was introduced in [16] with a business decision problem and has many different applications: prediction of corporate financial distress [19], stock return prediction [18]. A synthesis of intuitionistic and interval-valued FCMs was applied in group decision making [17].

The inference provided by an IVFCM applies the usual threshold function, but instead of ordinary addition and multiplication, it applies the arithmetic operators of interval-valued fuzzy set theory [24], which differ from the standard interval arithmetic operations [25].

In the present article, we examine the case when the uncertainties of the weights are represented by intervals, but as a difference to IVFCMs, for addition and multiplication, we apply the standard interval arithmetic operations. Although this model is also an interval-valued fuzzy cognitive map, to avoid confusion we are going to use an ad-hoc name: fuzzy cognitive map with interval weights.

The dynamics of fuzzy cognitive maps with interval weights are similar to the dynamics of the classical FCMs: they may reach a fixed point attractor, may arrive at a limit cycle or may produce chaotic behaviour. Most of the applications rely on the assumption that the system converges to an equilibrium point. Moreover, systems' robustness and stability are also important features, which means that perturbations on the initial concept values should not cause different outputs. Consequently, the global stability, the existence and uniqueness of stable equilibrium points (fixed points) are essential problems of FCMs and its extensions, too.

The rest of the paper is organized as follows: In Section 2 we shortly summarize the notion of contraction mapping, stability and interval arithmetic. In Section 3 theorems and proofs are presented about the existence and uniqueness of fixed points of fuzzy cognitive maps with interval weights, which ensure the stability of the FCM model. Finally, in Section 4 we provide a short summary of the results.

2 Mathematical Tools

Fuzzy cognitive maps have a wide variety of applications, but the exact mathematical investigation of their behaviour has much more weaker literature. According to our knowledge, Boutalis, Kottas and Christodoulou [2] were the first, who studied the existence and uniqueness of fixed points of sigmoid FCM, namely for the case when the parameter of the log-sigmoid threshold function is $\lambda = 1$. Knight, Lloyd and Penn examined the possible number of fixed points of sigmoid FCMs [7]. Lyapunov stability of FCMs was also discussed by Lee and Kwon [8], [9]. Analytical conditions for the existence and uniqueness of fixed points of FCMs were introduced in [4]. The problem of fixed points of FCMs with fuzzy set weights was studied in [6], while the convergence of fuzzy grey cognitive maps was discussed in [5].

2.1 Contraction Mapping

The proofs of theorems of Section 3 are based on the so-called contraction property of the mapping that generates the iteration. Here we recall the definition of contraction mapping [34]:

Definition 1 *Let (X, d) be a metric space. A mapping $G\colon X \to X$ is a* contraction mapping *or* contraction *if there exists a constant c (independent from x and y), with $0 \leq c < 1$, such that*

$$d\left(G(x), G(y)\right) \leq cd(x, y). \tag{4}$$

The notion of contraction is related to the distance metric d applied. It may happen that a function is a contraction w.r.t. one distance metric, but not a contraction w.r.t. another distance metric. The iterative process of an FCM may end at an equilibrium point, which is a so-called fixed point. Let $G\colon X \to X$, then a point $x^* \in X$ such that $G(x^*) = x^*$ is a fixed point of G. The following theorem provides sufficient condition for the existence and uniqueness of a fixed point [34]. Moreover, if mapping that generates the iteration is a contraction, it ensures the stability of the iteration.

Theorem 2 *(Banach's fixed point theorem) If $G\colon X \to X$ is a contraction mapping on a nonempty complete metric space (X, d), then G has only one fixed point x^*. Moreover, x^* can be found as follows: start with an arbitrary $x_0 \in X$ and define the sequence $x_{n+1} = G(x_n)$, then $\lim_{n\to\infty} x_n = x^*$.*

Definition 3 *Let x^* be a fixed point of the iteration $x_{n+1} = G(x_n)$. x^* is* locally asymptotically stable *if there exist a neighborhood U of x^*, such that for each starting value $x_0 \in U$ we get that*

$$\lim_{n\to\infty} x_n = x^*. \tag{5}$$

If this neighborhood U is the entire domain of G, then x^* is a globally asymptotically stable *fixed point.*

Corollary 4 *If $G\colon X \to X$ is a contraction mapping on a nonempty complete metric space (X, d), then its unique fixed point x^* is globally asymptotically stable.*

Moreover, in the proofs of Section 3 the following well-known statement will be applied:

Lemma 5 *The derivative of the sigmoid function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 1/(1 + e^{-\lambda x})$, $(\lambda > 0)$ is bounded by $\lambda/4$. Moreover, for every $x, y \in \mathbb{R}$ the following inequality holds*

$$|f(x) - f(y)| \leq \lambda/4 \cdot |x - y| .$$

2.2 Interval arithmetic

Let us have two intervals, $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$. The standard rules of interval arithmetic between A and B are the following [25]:

1. $A + B = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$
2. $-A = [-\overline{a}, -\underline{a}]$
3. $A - B = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$
4. $A \times B = [\min(S), \max(S)]$ where $S = \left\{\underline{a}\cdot\underline{b}, \underline{a}\cdot\overline{b}, \overline{a}\cdot\underline{b}, \overline{a}\cdot\overline{b}\right\}$
5. If $\alpha > 0$, $\alpha \in \mathbb{R}$, then $\alpha \cdot A = [\alpha\underline{a}, \alpha\overline{a}]$

The Hausdorff distance between intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ is defined as (see [13]):

$$d_H(A, B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\} \tag{6}$$

Note that this distance of intervals is also known as Moore metric, since it was introduced by R. E. Moore [25]. Let K_c be the space of nonempty compact and convex sets of $\mathbb{R}$

$$K_c = \{[\underline{a}, \overline{a}] \colon \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\} \tag{7}$$

Space K_c with distance metric $d_H(A, B)$ form a complete metric space [13].

3 Stability Conditions of Fuzzy Cognitive Maps with Interval Weights

As we have seen, the updating (iteration) process of an FCM can be viewed as a discrete dynamical system generated by a mapping G (that is defined coordinate-wise by the updating rules), which acts from the set of concept vectors to itself:

$$A^{(k+1)} = G(A^{(k)}) = [f(w_1 A^{(k)}), \ldots, f(w_n A^{(k)})]^T \tag{8}$$

In applications, the iteration stops when the concept vector reaches an equilibrium point or the number of iterations reaches the pre-defined maximum number of steps. Let us assume that the values of the concepts do not change any more after a number of iterations. It means that the iteration arrived to an equilibrium point A^*. For this point (concept vector) the following equality holds:

$$G(A^*) = A^*, \tag{9}$$

so it is a fixed point (equilibrium point, stationary point, steady-state point) of mapping G.

If the weights are intervals, then the updating rules ensure that the concept values will be also intervals, regardless of nature (crisp or interval) of the initial concept values. Consequently, the concept vector A has interval coordinates. The long-term dynamical behaviour of a fuzzy cognitive map with interval weights is similar to the behaviour of the classical FCM: it may reach an equilibrium point, may arrive at a limit cycle or may show a chaotic pattern. Of course, the fixed point is a vector with interval coordinates, but these coordinates (i.e. the endpoints of the intervals) are stabilized during the iteration (similarly to the classical FCM's fixed point). In other words, the fixed point still carries uncertainties in its coordinates, but the limits of uncertainty (endpoints of the intervals) become stabilized, do not change after a few (or large) number of iteration steps. So the equality

$$G(A^*) = A^*, \tag{10}$$

holds with $A^* = \left[[\underline{A_1^*}, \overline{A_1^*}], \ldots, [\underline{A_n^*}, \overline{A_n^*}]\right]^T$.

Theorem 2 states that if G is a contraction, then it has only one fixed point, i.e. if G is a contraction, then the FCM with interval weights has exactly one equilibrium point. Moreover, it ensures the model's stability in the sense that the iteration will converge to this fixed point, regardless of the initial concept values. In the following, we provide two theorems about the existence and uniqueness of fixed points of FCMs with interval weights. The philosophy of the theorems are the same: if the weights of the FCM with interval weights fulfil some conditions, then mapping G is a contraction, so it has one and only one equilibrium point.

We apply the assumption that the weights have their proper sign, which means that the expert knows the sign of the relationship, but uncertain about the magnitude. Let w_{ij} be a weight describing the connection between concepts C_j and C_i. Due to our assumptions, w_{ij} is an interval, which is subset of the interval $[-1, 0]$ or the interval $[0, 1]$. Let us introduce the following notation:

$$w_{ij}^* = \max\left\{|\underline{w_{ij}}|,\ |\overline{w_{ij}}|\right\} \tag{11}$$

I.e. w_{ij}^* is the Hausdorff distance between w_{ij} and the crips 0, or in other words, it is the absolute value of the interval $[\underline{w_{ij}}, \overline{w_{ij}}]$ as it was defined by Moore [25].

Theorem 6 *Let W be the weight matrix of the fuzzy cognitive map (including possible feedback), where the weights w_{ij} are nonnegative or nonpositive intervals and let $\lambda > 0$ be the parameter of the sigmoid function $f(x) = 1/(1 + e^{-\lambda x})$ applied for the iteration. Let w_{ij}^* be defined as in Eq. 11. If the inequality*

$$\sum_{i=1}^{n} \max_j \left\{w_{ij}^*\right\} < \frac{4}{\lambda} \tag{12}$$

holds, then the FCM has one and only one fixed point, regardless of the initial concept values.

Proof:

We are going to show that for a suitable distance metric d and under certain conditions the inequality

$$d(G(A), G(A')) \leq c \cdot d(A, A') \tag{13}$$

holds for every A and A' with $c < 1$ (independent from A and A'), so mapping G is a contraction, consequently it has one and only one fixed point. In this case for metric d we apply the sum of the Hausdorff distances of the coordinates, i.e. :

$$d(A, A') = \sum_{i=1}^{n} d_H(A_i, A_i') = \tag{14}$$

$$= \sum_{i=1}^{n} \max\left\{\left|\underline{A_i} - \underline{A_i'}\right|, \left|\overline{A_i} - \overline{A_i'}\right|\right\} \tag{15}$$

Note that in the degenerated case, when all of the intervals are real numbers, so if $\underline{A_i} = \overline{A_i}$ for A and A' and for all of the coordinates, then it becomes the sum of the absolute differences of the coordinates, so it gives the so-called 1-norm or Manhattan norm of vector $A - A'$. Using this distance metric, the distance of $G(A)$ and $G(A')$ is the following:

$$\begin{aligned} &d(G(A), G(A')) \\ &= \sum_{i=1}^{n} \max \left\{ \left|\underline{G(A)_i} - \underline{G(A')_i}\right|, \left|\overline{G(A)_i} - \overline{G(A')_i}\right| \right\} \end{aligned} \tag{16}$$

The left and right (lower and upper) endpoints of the interval $G(A)_i$ can be expressed by the threshold function. The threshold function is strictly monotone increasing, which implies that:

$$\underline{G(A)_i} = \underline{f(w_i A)} = f(\underline{w_i A}) \tag{17}$$

$$\overline{G(A)_i} = \overline{f(w_i A)} = f(\overline{w_i A}) \tag{18}$$

Moreover, according to Lemma 5, the following inequalities hold:

$$\left|\underline{G(A)_i} - \underline{G(A')_i}\right| \le \frac{\lambda}{4} \left|\underline{w_i A} - \underline{w_i A'}\right| \tag{19}$$

$$\left|\overline{G(A)_i} - \overline{G(A')_i}\right| \le \frac{\lambda}{4} \left|\overline{w_i A} - \overline{w_i A'}\right| \tag{20}$$

The next step is to provide an upper estimation for the distance of $G(A)$ and $G(A')$, applying the previous inequalities:

$$\begin{aligned} &d(G(A), G(A')) \\ &= \sum_{i=1}^{n} \max \left\{ \left|\underline{G(A)_i} - \underline{G(A')_i}\right|, \left|\overline{G(A)_i} - \overline{G(A')_i}\right| \right\} \\ &\le \sum_{i=1}^{n} \frac{\lambda}{4} \max \left\{ \left|\underline{w_i A} - \underline{w_i A'}\right|, \left|\overline{w_i A} - \overline{w_i A'}\right| \right\} \\ &\le \sum_{i=1}^{n} \frac{\lambda}{4} \max \left\{ \sum_{j=1}^{n} w_{ij}^* \left|\underline{A_j} - \underline{A_j'}\right|, \sum_{j=1}^{n} w_{ij}^* \left|\overline{A_j} - \overline{A_j'}\right| \right\} \end{aligned} \tag{21}$$

Since

$$\sum_{j=1}^{n} w_{ij}^* \left|\underline{A_j} - \underline{A_j'}\right| \le \sum_{j=1}^{n} w_{ij}^* \max_j \left\{ \left|\underline{A_j} - \underline{A_j'}\right| \right\} \tag{22}$$

$$\sum_{j=1}^{n} w_{ij}^* \left|\overline{A_j} - \overline{A_j'}\right| \le \sum_{j=1}^{n} w_{ij}^* \max_j \left\{ \left|\overline{A_j} - \overline{A_j'}\right| \right\} \tag{23}$$

we get further upper estimation:

$$\begin{aligned} &d(G(A), G(A')) \\ &\le \sum_{i=1}^{n} \frac{\lambda}{4} \sum_{j=1}^{n} w_{ij}^* \max \left\{ \left|\underline{A_j} - \underline{A_j'}\right|, \left|\overline{A_j} - \overline{A_j'}\right| \right\} \\ &\le \sum_{i=1}^{n} \frac{\lambda}{4} \max_j \{w_{ij}^*\} \sum_{j=1}^{n} \max \left\{ \left|\underline{A_j} - \underline{A_j'}\right|, \left|\overline{A_j} - \overline{A_j'}\right| \right\} \\ &= \sum_{i=1}^{n} \frac{\lambda}{4} \max_j \{w_{ij}^*\} d(A, A') \end{aligned} \tag{24}$$

Finally, we get that

$$d(G(A), G(A')) \le \frac{\lambda}{4} d(A, A') \sum_{i=1}^{n} \max_j \{w_{ij}^*\} \tag{25}$$

If $\sum_{i=1}^{n} \max_j \{w_{ij}^*\} < 4/\lambda$, then the coefficient of $d(A, A')$ is less than one, consequently mapping G is a contraction. It implies that it has exactly one fixed point, which completes the proof.

Remark 7 *In other words, Theorem 6 states that if $\lambda \le 4/\sum_{i=1}^{n} \max_j \{w_{ij}^*\}$, then the FCM whose weights are intervals has one and only one fixed point.*

Different choice of the distance between concept vectors yields different and sometimes better conditions for the existence and uniqueness of fixed points of FCMs with interval weights. One condition is better than the other, if it ensures the convergence for a larger set of parameter λ.

Theorem 8 *Let W be the weight matrix of a fuzzy cognitive map (including possible feedback), where the weights w_{ij} are nonnegative or nonpositive intervals and let $\lambda > 0$ be the parameter of the sigmoid function $f(x) = 1/(1 + e^{-\lambda x})$ applied for the iteration. Let w_{ij}^* be defined as in Eq. 11. If the inequality*

$$\max_i \left\{ \sum_{j=1}^{n} w_{ij}^* \right\} < \frac{4}{\lambda} \tag{26}$$

holds, then the FCM has one and only one fixed point, regardless of the initial concept values.

Proof:

Similarly to the proof of Theorem 6, we are going to show that for a suitable distance metric d and under the conditions stated in the theorem the inequality

$$d(G(A), G(A')) \le c \cdot d(A, A') \tag{27}$$

holds for every A and A' with $c < 1$ (independent from A and A'), so mapping G is a contraction, consequently it has one and only one fixed point.

Let the distance of the concept vectors A and A' be defined as the maximum of Hausdorff distances of the coordinates:

$$d(A, A') = \max_i d_H(A_i, A_i') = \tag{28}$$

$$= \max_i \max \left\{ \left|\underline{A_i} - \underline{A_i'}\right|, \left|\overline{A_i} - \overline{A_i'}\right| \right\} \tag{29}$$

Using this metric, the distance of $G(A)$and $G(A')$ is the following:

$$\begin{aligned} &d(G(A), G(A')) \\ &= \max_i \max \left\{ \left|\underline{G(A)_i} - \underline{G(A')_i}\right|, \left|\overline{G(A)_i} - \overline{G(A')_i}\right| \right\} \end{aligned}$$

As we have seen previously, the following upper estimation can be given for the distance:

$$d(G(A), G(A')) \\ \leq \max_i \max \left\{ \frac{\lambda}{4} \left| \underline{w_i A} - \underline{w_i A'} \right|, \frac{\lambda}{4} \left| \overline{w_i A} - \overline{w_i A'} \right| \right\} \tag{30}$$

Applying similar upper estimations as in the proof of the previous theorem, we get that

$$\max \left\{ \left| \underline{w_i A} - \underline{w_i A'} \right|, \left| \overline{w_i A} - \overline{w_i A'} \right| \right\} \\ \leq \sum_{j=1}^{n} w_{ij}^* \max \left\{ \left| \underline{A_j} - \underline{A'_j} \right|, \left| \overline{A_j} - \overline{A'_j} \right| \right\} \tag{31}$$

Coupling these inequalities together we arrive to the following upper estimation of $d(G(A), G(A'))$:

$$\begin{aligned} & d(G(A), G(A')) \\ & \leq \frac{\lambda}{4} \max_i \max \left\{ \left| \underline{w_i A} - \underline{w_i A'} \right|, \left| \overline{w_i A} - \overline{w_i A'} \right| \right\} \\ & \leq \frac{\lambda}{4} \max_i \left\{ \sum_{j=1}^{n} w_{ij}^* \max \left\{ \left| \underline{A_j} - \underline{A'_j} \right|, \left| \overline{A_j} - \overline{A'_j} \right| \right\} \right\} \\ & \leq \frac{\lambda}{4} \max_i \left\{ \sum_{j=1}^{n} w_{ij}^* \right\} \max_j \max \left\{ \left| \underline{A_j} - \underline{A'_j} \right|, \left| \overline{A_j} - \overline{A'_j} \right| \right\} \\ & = \frac{\lambda}{4} \max_i \left\{ \sum_{j=1}^{n} w_{ij}^* \right\} \cdot d(A, A') \end{aligned} \tag{32}$$

Here the last equality comes from the definition of the distance between concept vectors A and A'. The coefficient of $d(A, A')$ is less than one in the inequalty above if and only if $\max_i \left\{ \sum_{j=1}^{n} w_{ij}^* \right\} < 4/\lambda$ and this completes the proof.

4 Summary

Fuzzy cognitive maps with interval weights are modifications of the original FCM model, that are able to represent the uncertainties in the estimation of weights of causal connections between the concepts. Just like the original FCMs, these models may produce stable or unstable behaviour. In a large class of applications, stability is an important and required property of the model. In this paper, mathematical conditions have been introduced that ensure the stability of fuzzy cognitive maps when the weights are represented by intervals. Roughly speaking, without formulae, these statements tell that if the weights of connections are not so large, then the fuzzy cognitive map equipped with interval weights with given parameter λ will converge to a unique fixed point (activation vector with interval coordinates). In other words, it means that if the parameter of the threshold function remains under a limit computed from the weight matrix, then the FCM will converge to a unique concept vector, whose coordinates are intervals, but the endpoints of these intervals are stabilized. In this case, the system is stable since it converges to a unique fixed point (equilibrium point) and perturbations on the initial values (initial concept vector) are not able to cause deviations in the final behaviour of the system.

The operation rules applied for computation with intervals do matters: different arithmetic rules yield different FCM models with different properties. In this paper, we applied the standard interval arithmetic. Among other features, it means that (just like in the case of the classical FCM) in-between computational values may fall outside the interval $[0, 1]$, but the threshold function f transforms the result into the required range.

We should emphasize that the model examined here is different from that was introduced by Hajek and Prochazka [16] under the name of interval-valued fuzzy cognitive maps (IVFCMs) since they applied the arithmetic rules of interval-valued fuzzy sets [24].

Sufficient conditions have been introduced for the existence and uniqueness of fixed points of fuzzy cognitive maps with interval weights. These are sufficient, but not necessary conditions: if at least one of them is fulfilled, then the FCM has exactly one fixed point, which means that the model is globally asymptotically stable. On the other hand, there may exist cases, when the conditions do not hold, but the FCM has exactly one (globally asymptotically stable) fixed point.

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