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Atlantis Studies in Uncertainty Modelling, volume 1

11th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2019)

Convex preferences as aggregation of orderings

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Abstract

Convexity of preferences is a canonical assumption in economic theory. In this paper we consider a generalized definition of convex preferences that relies on the abstract notion of convex space.

Keywords: Convex Space, Preference, Betweenness, Lattice.

1 Introduction

In economic theory it is often assumed for analytical convenience, but also in accordance with common intuition that consumer preferences are convex. Moreover convexity of preference is a conditions considered in other fields (see for example [3]).
The standard Euclidean notion of convex preferences is an algebraic property used to express the notion that agents exhibit an inclination for diversification and so they prefer a more balanced bundle to bundles with a more extreme composition.
Then a preference in a set X is convex if whenever $x, y \in X$ and $x \succeq y$ then

$$tx + (1-t)y \succeq y \quad \text{for all} \quad t, \quad 0 \leq t \leq 1.$$

This definition relies on the algebraic structure of the Euclidean space that is used to define a betweenness relation. When we say that a point z is "between" x and y in an Euclidean space we mean that z is a convex combination of x and y.
In this paper we consider abstract convex structures that are objects studied in various areas of mathematics and an abstract notion of betweenness. Then we propose a general definition of convex preferences in our setting and we study this notion when the convexity is a lattice convexity.
The paper is organized as follows. In section 2, we collect definitions and basic results about abstract convex spaces and we propose some examples of convex spaces. In section 3 we propose a definition of convex preference in our framework while in section 4 we study convex preferences in a lattice considered as a convex space.

2 Abstract convex structures

The general notion of abstract convexity structure studied in [10] is considered.
A family $\mathcal{C}$ of subsets of a set X is a convexity on a set X if $\emptyset$ and X belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersections and closed under unions of chains. The elements of $\mathcal{C}$ are called convex sets of X and the pair $(X, \mathcal{C})$ is called a convex space. A convex set with a convex complement is called a half-space.
The convexity notion allows us to define the notion of the convex hull operator, which is similar to that of the closure operator in topology. If X is a set with a convexity $\mathcal{C}$ and A is a subset of X, then the convex hull of $A \subseteq X$ is the set

$$\text{co}(A) = \bigcap \{C \in \mathcal{C} : A \subseteq C\}.$$

A convex structure is completely determined by its hull operator, or even by its effect on finite sets (see Proposition 2.1 of [10]). This operator enjoys certain properties that are identical to those of usual convexity: for instance $\text{co}(A)$ is the smallest convex set that contains set A. It is easy to prove that C is convex if and only if $\text{co}(C) = C$.
The convex hull of a set $\{x_1, ..., x_n\}$is called an n-polytope and is denoted by $[x_1, ..., x_n]$. A 2-polytope $[a, b]$ is called the segment joining a, b.
A convexity C is called N-ary ($N \in \mathbb{N}$) if $A \subseteq \mathcal{C}$ whenever $\text{co}(F) \subseteq A$ for all $F \subseteq A$ where F has at most N elements. A 2-ary convexity is called an interval convexity. We can also consider biconvex spaces i.e. triples of the form $(X, \mathcal{A}, \mathcal{B})$ where $\mathcal{A}, \mathcal{B}$ are two convexities on a set X, called the lower and the upper convexity. Obviously every convex space $(X, \mathcal{C})$ can be viewed as a biconvex space $(X, \mathcal{C}, \mathcal{C})$. For a general theory of convexity we refer to [5] and [10].

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2.1 Some examples

We present some examples and classes of convex spaces. First of all we note that every real vector space together with the collection of all convex sets in the usual meaning, is a 2-arity convex space.

Ordered spaces The usual convexity on $\mathbb{R}$ can be defined in terms of ordering as follows: a set C is convex if and only if when $a, b \in C$ and $a \leq x \leq b$ implies $x \in C$. We can define in the same way a convexity on a partially ordered set (see [10], pag 6). Such a convexity is called the order convexity.

Convexity generated by orderings Let X be a non empty set and $\mathcal{P}$ a set of orderings (reflexive, complete and transitive relations) on X. We refer [10] p.10 for a definition of a base and a subbase of a convexity. Then we define the convexity generated by the sets $\{x \in X : x \succeq_i z\}$ where $z \in X$ and $\succeq_i$ is an element of $\mathcal{P}$ and the complement sets of these sets that are the sets $\{x \in X : x \prec_i z\}$ where $z \in X$ and $\succeq_i$ is an element of $\mathcal{P}$. So this convexity is generated by half-spaces.

Median spaces A median space is a convexity space X with a 2-arity convexity such that for each $a, b, c \in X$ there exists a unique point in $[a, b] \cap [a, c] \cap [b, c]$. We call it the median of a, b, c and denote by $m(a, b, c)$. This defines a map $m\colon X^3 \to X$, called the median operator on X. In any convexity space, every point in $[a, b] \cap [a, c] \cap [b, c]$ is called a median of a, b, c. There is a natural way to define the structure of a median space by means of the median operator (see [10]).

Property-based domains A property-based domain (as defined in [1]) is a pair $(X, \mathcal{H})$ where X is a non-empty set and $\mathcal{H}$ is a collection of non-empty subsets of X and if $x, y \in X$ and $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$. The elements of $\mathcal{H}$ are referred to as properties and if $x \in H$ we say that x has property represented by the subset H. This definition is slightly more general than that of [4], [7] and of [8], in fact it is not assumed that the set X is finite and we do not consider that the set H^c is a property if H is a property.
The "property space" model provides a very general framework for representing preferences and then aggregation of preferences. In every property-based domain we can define a convexity defined as follows. A subset $S \subseteq X$ is said to be convex if it is intersection of properties.

2.2 Betweenness

The notion of a point lying between two given points on a geometric line has been generalized in a number of directions. In all of these, betweenness is taken to be a ternary relation that satises certain conditions.
The ternary relation of betweenness comes up in a large variety of structures on a given set as it is well known reflecting intuitions that range from order-theorethic to the geometrical and topological.
These relations have been introduced in the context of abstract convexity in [10], in the context of property spaces (see for example [7] and [8]).
A convex space $(X, \mathcal{C})$ induces a ternary betweenness relation $B \subseteq X^3$ according to

$$B(x, z, y) \iff \text{for all } C \in \mathcal{C} : \{x, y\} \subseteq C \implies z \in C.$$

Thus z is between x and y of B if z possess all basic properties that are common to x and y and possibly some more. This ternary relation satisfies the following properties.
[B1] (Reflexivity) If $z \in \{x, y\}$ then $B(x, z, y)$.
[B2] (Symmetry) If $B(x, z, y)$ then $B(y, z, x)$.
[B3] (Transitivity) If $B(x, x', y)$, $B(x, y', y)$ and $B(x', z, y')$ then $B(x, z, y)$.

2.3 Separation axioms

Two sets are separated by a set A if one is contained in A and the other one is disjoint from A.
We shall consider the following separation axioms (see [10] p.53):
S_0: For every two distinct points there exists a convex set which contains exactly one of them.
S_1: Every one-point subset is convex.
S_2: Distinct points are separated by half-spaces
S_3: Every convex set is an intersection of half-spaces
S_4: For every two disjoint convex sets there exists a half- space which contains exactly one of them.
A S_1 convex space is called point convex while axiom S_4 is called the Kakutani separation property. In fact the classical theorem of Kakutani says that each two disjoint in a real vector space can be separated by a half-space.
It is also clear that $S_4 \implies S_3$, $S_2 \implies S_1 \implies S_0$, and $S_1 + S_3 \implies S_2$.
We denote by $\mathcal{H} \subseteq \mathcal{C}$ the set of half-spaces of the convex space $(X, \mathcal{C})$.

3 Convex preferences

In this section we consider convex spaces $(X, \mathcal{C})$ that satisfy axioms S_1 and S_4. The following definition extends the definition of convex preferences proposed in [9]. We propose a definition of convex preferences in abstract convex structures and we do not consider

only complete relations as in [9].
We consider preorders on a nonempty set X i.e. transitive and reflexive binary relations on X and we use the term "preorder" and "preference" interchangeably throughout the present paper.
A preference relation on a convex space $(X, \mathcal{C})$ is $\succeq$ on X is said to be a convex preference if $\{x \in X : x \succeq z\}$ is a convex set for every $z \in X$.

Remark 1 If the convex space is an Euclidean space our definition is the well known definition of convexity of a preference.

Remark 2 Consider the definition of preference relation proposed in [9]. The set X is a convex space with respect to the convexity generated by the a set of orderings $\mathcal{P}$. Then if $x \succeq z$ for every $\succeq_i \in \mathcal{P}$ there exists $z_i \in X$ such that $z \in \{x \in X : x \succeq_i z_i\}$ and then x is an element of a convex sets contained in $\{x \in X : x \succeq z\}$. Then we can easily prove that the convex order as defined in [9] are convex also with respect to our definition.

Remark 3 If $(X, \mathcal{C})$ is a convex space we consider the relation $\succeq$ in X such that $x \succeq y$ if and only if

$$\{H \in \mathcal{H}' : x \in H\} \supseteq \{H \in \mathcal{H}' : y \in H\}.$$

where $\mathcal{H}' \subseteq \mathcal{H}$. This relation is transitive and complete and it can be proved that the set $\{x \in X : x \succeq z\}$ is an intersection of half-spaces and so is a convex set for every $z \in X$. Hence the considered relation is a convex preference.

The following result generalizes in our framework the well known property of convex preferences in Euclidean spaces.

Proposition 1. *Let $(X, \mathcal{C})$ be a convex space. If $\succeq$ is a convex preference in X then for every $x, y \in X$*

$$\text{if} \quad x \succeq y \quad \text{and} \quad B(x, z, y) \quad \text{then} \quad z \succeq y.$$

If $\mathcal{C}$ is an interval convexity a transitive and complete relation $\succeq$ in X such that for every $x, y \in X$

$$\text{if} \quad x \succeq y \quad \text{and} \quad B(x, z, y) \quad \text{then} \quad z \succeq y$$

is a convex preference.

Proof. If we consider a convex preference $\succeq$ in $(X, \mathcal{C})$ and two elements x, y of X such that $x \succeq y$ then obviously x, y belong to the convex set $\{t \in X : t \succeq y\}$. Moreover if $B(x, z, y)$ is satisfied for an an element of X z is if and only if $z \in \text{co}\{x, y\}$ that is the smallest convex set that contains x and y. Then if $B(x, z, y)$ is satisfied z is an element of the convex set $\{t \in X : t \succeq y\}$ and then we get $z \succeq y$.
Conversely let $\mathcal{C}$ be an interval convexity on X and $\succeq$ a transitive and reflexive relation in X such that for every $x, y \in X$, if $x \succeq y$ and $B(x, z, y)$ then $z \succeq y$. Then if x_1, x_2 are elements of X such that $x_1 \succeq y$ and $x_2 \succeq y$ we can suppose that $x_1 \succeq x_2$. Hence $[x_1, x_2]$ is contained in the set $\{t \in X : t \succeq y\}$ since if $z \in [x_1, x_2]$ then $z \succeq x_2 \succeq x$. □

4 Convex preferences in lattices

An element z of a lattice L is called *join irreducible* if $z = x \vee y$ for $x, y \in L$ implies that $z = x$ or $z = y$. The notion of meet-irreducible element is defined dually. The set of join-irreducible elements of a lattice L is denoted by $J(L)$.
An element z of a lattice L is called *join prime* i if $z \leq x \vee y$ for $x, y \in L$ implies that $z \leq x$ or $z \leq y$. The notion of meet-prime element is defined dually. The set of join-prime elements of a lattice L is denoted by $JI(L)$. It can be proved that a join-prime element is join-irreducible and if L is a finite distributive lattice we have that $J(L) = JI(L)$.
A *filter* of a lattice L is a nonempty subset F such that

(i) if $x \in F$ and $x \leq y$ then $y \in F$,

(ii) $x, y \in F$ then $x \wedge y \in F$.

Sets satisfying Condition (i) of a filter are called upsets. The dual notation is that of an *ideal.* If $x \in L$ we define the *principal filter* generated by x as $\uparrow x = \{y \in L : y \geq x\}$. It is easy to prove that $\uparrow x$ is a filter for every $x \in L$. It can be proved that in a finite lattice each filter and each ideal are principal.
A *proper filter* is a filter that is neither empty nor the whole lattice while a *prime filter* is a proper filter P such that if $x \vee y \in P$ then $x \in P$ or $y \in P$. An element x of a lattice L is join-prime if and only if $\uparrow x$ is prime. A filter F is prime if and only if $L \setminus F$ is an ideal, which is then a prime ideal.
Throughout this paper lattice means bounded and distributive lattice. We note that if L is a bounded and distributive lattice every element is characterized by the set of prime filters which contain the given element since a duality between the lattice and the power set of the s et of prime filters of L ordered by inclusion can be defined as is proved in [6].
If $\langle L, \wedge, \vee \rangle$ is a lattice we denote by $\mathcal{L}$ and $\mathcal{U}$ the collections of all ideals and all filters respectively (the empty set and the whole lattice are treated as (non-proper) ideals and filters). Since the union of a chain of filters (ideals) is a filter (ideal), these are two convexities on L that will be called the lower and the upper lattice convexity respectively. Moreover there exists a convexity

$\mathcal{C}$ generated by $\mathcal{L}\bigcup\mathcal{U}$ the least convexity containing all ideals and filters. This convexity will be called the lattice convexity on L.
Note that if L is linearly ordered then $\mathcal{C}$ equals the order convexity. The convexity of the dual lattice is the same as the original one.
It is possible to consider lattices as convex spaces (with the lattice convexity) as well as bi-convex spaces (with the lower and upper lattice convexities). It is easy to check that a proper half-space is either a prime filter or a prime ideal. It can be proved also that

$$[x,y] = \{z \in L : x \wedge y \leq z \leq x \vee y\}$$

and that in a lattice L the ternary betweenness relation is defined by

$$B(x,z,y) \iff x \wedge y \leq z \leq x \vee y$$

(see [5] and [10]).
A lattice satisfies Kakutani property if and only if it is a distributive lattice (see [10]). The following result characterizes lattice convexity as a convexity defined by orderings.

Proposition 2. *If $\langle L, \wedge, \vee\rangle$ is a bounded and distributive lattice, the lattice convexity is a convexity generated by a set of orderings on L.*

Proof. We consider the set $\mathcal{L}$ of prime filter of L (that are half-spaces of the lattice convexity) and we define an equivalence relation in L defined by

$$F_1 \sim F_2 \quad \text{if} \quad F_1 \subseteq F_2 \quad \text{or} \quad F_2 \subseteq F_1$$

where $F_1, F_2 \in \mathcal{L}$. If E is an equivalence class with respect to the equivalence relation defined above E is a chain ordered by inclusion. Then the relation $\succeq_E$ such that $x \succeq_E y$ if and only if

$$\{F \in \mathcal{L} : x \in E\} \supseteq \{F \in \mathcal{L} : y \in F\}$$

is a complete and transitive relation. Let $\mathcal{E}$ the set of orderings $\succeq_E$ where E is an equivalence class with respect to the equivalence relation defined above.
Observe that the set $\{x \in L : x \succeq_E z\}$ where $z \in L$ and $\succeq_E$ is an element of $\mathcal{E}$ is a prime filter and every filter can be represented as $\{x \in L : x \succeq_E z\}$ where $z \in L$ and $\succeq_E$ is an element of $\mathcal{E}$. So we have proved that the lattice convexity is defined by the orderings in $\mathcal{E}$. □

A relation $\succeq$ in a lattice L is said to be compatible with the order $\geq$ of L if

$$\text{if} \quad x,y \in L \quad \text{and} \quad x \geq y \quad \text{then} \quad x \succeq y.$$

We introduce a property of a reflexive and transitive relation in a lattice that in [2] is named meet dominance and we prove that this property characterizes convex preferences defined on a lattice.

Proposition 3. *If $\langle L, \wedge, \vee\rangle$ is a bounded and distributive lattice a reflexive and transitive relation $\succeq$ in L is a convex preference with respect to the lattice convexity if and only is a compatible relation such that for all $x, y \in L$ if $x \succeq y$ then $x \wedge y \succeq y$.*

Proof. Note that by Proposition 8.2 in [5] the lattice convexity is an interval convexity and then a set A is convex if and only if if x, y are elements of A then the segment $[x,y] = \{z \in L : x \wedge y \leq z \leq x \vee y\}$.
Let $\succeq$ a convex preference in a lattice L.
By Priestley duality (see [6]) we get that $\succeq$ is compatible with $\geq$ that is

$$\text{if} \quad x \geq y \quad \text{then} \quad y \in F \quad \text{implies that} \quad x \in F$$

for every prime filter of L. Then if $x, y \in L$ and $x \succeq y$ then x, y belong to the convex set $\{z \in L : z \succeq y\}$. Then also $x \wedge y$ is an element of $\{z \in L : z \succeq y\}$ hence $x \wedge y \succeq y$.
Conversely assume that $\succeq$ is a compatible relation in a lattice L that satisfies the property above.
If x_1, x_2 are two elements of L that belong to the set $\{x \in L : x \succeq z\}$ then, since $\succeq$ is a compatible and transitive relation, we have that $x_2 \succeq z \succeq x_1 \wedge z$ and so $x_2 \succeq x_1 \wedge z$. By meet dominance $x_1 \wedge x_2 \wedge z \succeq x_1 \wedge z$. Hence we can prove that $x_1 \wedge x_2 \succeq x_1 \wedge x_2 \wedge z \succeq x_1 \wedge z \succeq z$. Then we can easily prove that the segment $[x_1, x_2]$ is contained in $\{x \in L : x \succeq z\}$, so the set $\{x \in L : x \succeq z\}$ is convex for every $z \in L$ and $\succeq$ is a convex preference. □

5 Concluding remarks

This paper is a first step toward the study of a general notion of convex preferences. We consider an abstract notion of convexity in a base set and we study this notion when the convexity is a lattice convexity.
We prove that in a lattice a convex structure is generated by a set of orderings that in some sense play a role of linear functions in Euclidean spaces and then we characterize convex preferences in a lattice.
There are many examples of economic models that consider convex spaces (see [9]). We plan to study convex preferences in general convex spaces, and to find more applications of our results in future work.

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