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Solutions to HSOM Equation with Jordan Canonical Form

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Abstract. Three solutions to the high-order Sylvester observer matrix (HSOM) equation with Jordan canonical form are established. The obtained solutions contain numerical matrix calculations, which provide convenience for calculating these solutions in certain applications. The reliability of those solutions is shown by a numerical example and the improvements for the first two analytical solutions are proposed.

Introduction

Consider the HSOM equation

$$J^N U^{\mathrm{T}} A_N + \cdots + JU^{\mathrm{T}} A_1 + U^{\mathrm{T}} A_0 = J^M Z^{\mathrm{T}} C_M + \cdots + JZ^{\mathrm{T}} C_1 + Z^{\mathrm{T}} C_0, \tag{1}$$

where J takes following Jordan matrix form

$$J = \mathrm{diag}(J_1, J_2, \dots J_q) \in C^{p\times p},\ p \le nN, \tag{2}$$

$$J_i = \begin{bmatrix} s_i & 1 & & \\ & s_i & \ddots & \\ & & \ddots & 1 \\ & & & s \end{bmatrix} \in C^{p_i\times p_i}, i = 1, 2, \dots, q,$$

and $A_i \in R^{n\times n},\ i = 0,1,\dots,N,\ C_i \in R^{m\times n},\ i = 0,1,\dots,M$, are known matrices and satisfy following assumption.

Assumption 1: $\mathrm{rank}\begin{bmatrix} A^{\mathrm{T}}(s) & C^{\mathrm{T}}(s) \end{bmatrix} = n,\ \ \forall s \in C,$ where

$$A(s) = s^N A_N + \cdots + sA_1 + A_0,$$
$$C(s) = s^M C_M + \cdots + sC_1 + C_0.$$

If N=1, M=0, or N=2, M=0, (1) represents the first- or second-order Sylvester observer matrix equation

$$JU^{\mathrm{T}} A_1 + U^{\mathrm{T}} A_0 = Z^{\mathrm{T}} C_0,$$
$$J^2 U^{\mathrm{T}} A_2 + JU^{\mathrm{T}} A_1 + U^{\mathrm{T}} A_0 = Z^{\mathrm{T}} C_0,$$

which are found in the first-order[1-8] or second-order[9-10] linear system parametric observer design problems. In [11-12], the higher-order observer was designed and equation (1) was dealt with M=0. But all the above papers considered only J as a diagonal matrix rather than the Jordan canonical form.

It is obvious that HSOM equation (1) is equivalent to following Sylvester matrix equation if we take transport for it

$$A_N^{\mathrm{T}} U (J^N)^{\mathrm{T}} + \cdots + A_1^{\mathrm{T}} U J^{\mathrm{T}} + A_0^{\mathrm{T}} U = C_M^{\mathrm{T}} U (J^M)^{\mathrm{T}} + \cdots + C_1^{\mathrm{T}} U J^{\mathrm{T}} + C_0^{\mathrm{T}} U. \tag{3}$$

The above equation has been well studied in the eigenstructure assignment problem for various order system[13-17] with M=0. We can solve (3) or (1) using those methods when J is a diagonal matrix. But when J takes the Jordan form (2), those method are invalid.

This paper will study solutions for HSOM equation (1) when J is a Jordan matrix.

Solutions

A. The Vector Form of the Equation

Constructing matrices U and Z as follows according to the form of J:

$$U=\begin{bmatrix}U_1 & U_2 & \cdots & U_q\end{bmatrix}\in C^{n\times p},\ U_i=\begin{bmatrix}u_{i1} & u_{i2} & \cdots & u_{ip_i}\end{bmatrix},$$
$$Z=\begin{bmatrix}Z_1 & Z_2 & \cdots & Z_q\end{bmatrix}\in C^{m\times p},\ Z_i=\begin{bmatrix}z_{i1} & z_{i2} & \cdots & z_{ip_i}\end{bmatrix},$$

We first give the vector form of HSOM equation (1).

Lemma 1: The equivalent form of HSOM equation (1) is

$$\sum_{h=0}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i)=0,\ k=1,2,\ldots,p_i,\ i=1,2,\ldots,q, \tag{4}$$

where

$$\alpha_{ij}^{\mathrm{T}}=\begin{bmatrix}u_{ij}^{\mathrm{T}} & z_{ij}^{\mathrm{T}}\end{bmatrix},\ X^{\mathrm{T}}(s)=\begin{bmatrix}A^{\mathrm{T}}(s) & -C^{\mathrm{T}}(s)\end{bmatrix},$$

and $X^{(h)}(s)$ represents the h-th derivative of matrix polynomial $X(s)$.

B. Solution Based on the Smith Form Reduction (SFR)

If Assumption 1 holds, there exist unimodular matrices $P(s)\in R^{(n+m)\times(n+m)}[s],\ Q(s)\in R^{n\times n}[s]$, satisfying following Smith form reduction:

$$P(s)X(s)Q(s)=\begin{bmatrix}0 & I_n\end{bmatrix}^{\mathrm{T}}. \tag{5}$$

Then we obtain following conclusion based on Lemma 1.

Theorem 1: HSOM equation (1) have following parametric solution, if Assumption 1 holds,

$$\alpha_{ik}^{\mathrm{T}}=\begin{bmatrix}f_{ik}^{\mathrm{T}} & -\sum_{h=1}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i)Q(s_i)\end{bmatrix}P(s_i), \tag{6}$$

where $P(s)$, $Q(s)$ satisfy (5), and $f_{ik}\in C^m,\quad k=1,2,\ldots,p_i,\quad i=1,2,\ldots,q,$ are parametric vectors.

Proof: We first prove (6) is a solution to HSOM equation (1). Changing (4) and substituting (6) and (5) into it yields

$$\begin{aligned}&\sum_{h=0}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i)\\ =&\sum_{h=1}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i)\\ &+\begin{bmatrix}f_{ik}^{\mathrm{T}} & -\sum_{h=1}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i)Q(s_i)\end{bmatrix}\begin{bmatrix}0\\ I_n\end{bmatrix}Q^{-1}(s_i)=0,\end{aligned}$$

which shows that (6) is a solution to (1).

Next we prove that any solution to HSOM equation (1) can be express by (6). Let (4) hold, thus

$$\alpha_{ik}^{\mathrm{T}}X(s_i)=-\sum_{h=1}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i). \tag{7}$$

Denoting

$$\begin{bmatrix}f_{ik}^{\mathrm{T}} & g_{ik}^{\mathrm{T}}\end{bmatrix}=\alpha_{ik}^{\mathrm{T}}P^{-1}(s_i),\ k=1,2,\ldots,p_i,\ i=1,2,\ldots,q, \tag{8}$$

we have, according to (5) and (7),

$$\begin{bmatrix}f_{ik}^{\mathrm{T}} & g_{ik}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}0\\ I_n\end{bmatrix}=-\sum_{h=1}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i)Q(s_i).$$

Then we obtain

$$g_{ik}^{\mathrm{T}}=-\sum_{h=1}^{p_i-k}\frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}}X^{(h)}(s_i)Q(s_i).$$

Substituting this into (8), and post-multiplying $P(s_i)$ on both sides of the obtained equation, we have (6).

Thus, the conclusion holds.

Corollary 1: HSOM equation (1) have following parametric solution, if M=0,

$$\alpha_{ik}^{\mathrm{T}} = \left[f_{ik}^{\mathrm{T}} \quad -\sum_{h=1}^{p_i-k} \frac{1}{h!} u_{i(k+h)}^{\mathrm{T}} A^{(h)}(s_i) Q(s_i) \right] P(s_i),$$

where $P(s)$, $Q(s)$ satisfy (5), and $f_{ik} \in C^m$, $k = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$, are parametric vectors.

Corollary 2: If J is a non-defective matrix, that is, $J = \mathrm{diag}(s_1, s_2, \ldots s_q)$, HSOM equation (1) have following parametric solution:

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}, \; Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_m \end{bmatrix},$$
$$\begin{bmatrix} u_i^{\mathrm{T}} & z_i^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} f_i^{\mathrm{T}} & 0 \end{bmatrix} P(s_i),$$

where $P(s)$ satisfies (5), and $f_i \in C^m$, $i = 1, 2, \ldots, p$, are parametric vectors.

C. Solution Based on Left Coprime Factorization

Let

$$P(s) = \begin{bmatrix} H(s) & L(s) \\ * & * \end{bmatrix}, \; H(s) \in R^{m \times n}[s], \; L(s) \in R^{m \times m}[s], \tag{9}$$

we can easily prove that

$$H(s)A(s) - L(s)C(s) = 0, \tag{10}$$

which is the left coprime factorization of matrix pair ($A(s)$, $C(s)$). Then we have following conclusion based on Lemma 1.

Theorem 2: HSOM equation (1) have following parametric solution, if Assumption 1 holds ,

$$\alpha_{ik}^{\mathrm{T}} = -\sum_{l=0}^{p_i-k} \frac{1}{l!} f_{i(k+l)}^{\mathrm{T}} \begin{bmatrix} H^{(l)}(s_i) & L^{(l)}(s_i) \end{bmatrix}, \; k = 1, 2, \ldots, p_i, \; i = 1, 2, \ldots, q, \tag{11}$$

where $H(s)$, $L(s)$ satisfy (10), and $f_{ik} \in C^m$, $k = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$, are parametric vectors.

Proof: Taking the l-th derivative of (10) yields,

$$\sum_{h=0}^{l} \frac{l!}{h!(l-h)!} \begin{bmatrix} H^{(l-h)}(s_i) & L^{(l-h)}(s_i) \end{bmatrix} X^{(h)}(s) = 0.$$

Substituting s as s_i in it, pre-multiplying $\frac{1}{l!} f_{i(k+l)}^{\mathrm{T}}$ and adding all the obtained equations, we have

$$\begin{aligned} 0 &= \sum_{l=0}^{p_i-k} \sum_{h=0}^{l} \frac{1}{h!(l-h)!} f_{i(k+l)}^{\mathrm{T}} \begin{bmatrix} H^{(l-h)}(s_i) & L^{(l-h)}(s_i) \end{bmatrix} X^{(h)}(s) \\ &= \sum_{h=0}^{p_i-k} \frac{1}{h!} \left(\sum_{l=0}^{p_i-k-h} \frac{1}{l!} f_{i(k+h+l)}^{\mathrm{T}} \begin{bmatrix} H^{(l)}(s_i) & L^{(l)}(s_i) \end{bmatrix} \right) X^{(h)}(s). \end{aligned}$$

Substituting (11) into the above equation, we get (4), which states that U and Z given by (11) is a solution to equation (1).

Furthermore, noting that (6) and (11) both are solutions to (1), and the numbers of the free parameters $\{f_{ik}\}$ in them are equal, we know that (11) is a complete solution of (1) in view of the completeness of (6).

Corollary 3: If J is a non-defective matrix, that is, $J = \mathrm{diag}(s_1, s_2, \ldots s_q)$, HSOM equation (1) have following parametric solution:

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}, \; Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_m \end{bmatrix},$$
$$\begin{bmatrix} u_i^{\mathrm{T}} & z_i^{\mathrm{T}} \end{bmatrix} = f_i^{\mathrm{T}} \begin{bmatrix} H(s_i) & L(s_i) \end{bmatrix}, \; i = 1, 2, \ldots, q,$$

where $H(s)$ and $L(s)$ satisfy (10), and $f_i \in C^m$, $i = 1, 2, \ldots, p$, are parametric vectors.

D. Solution Based on Singular Value Decomposition

When Assumption 1 holds, we know that, for the given s_i, $i = 1, 2, \ldots, q$,

$$\mathrm{rank} X(s_i) = n, \; i = 1, 2, \ldots, q,$$

and matrices $X(s_i)$, $i=1,2,\ldots,q$, are constant ones. In this case, we use orthogonal matrices $\bar{P}_i \in C^{(n+m)\times(n+m)}$, $\bar{Q}_i \in C^{n\times n}$, satisfying the singular value decomposition (SVD)

$$\bar{P}_i X(s_i)\bar{Q}_i = \begin{bmatrix} 0 & \mathrm{S}_i \end{bmatrix}^{\mathrm{T}}. \tag{12}$$

Then we obtain following conclusion based on Lemma 1.

Theorem 3: HSOM equation (1) have following parametric solution, if Assumption 1 holds,

$$\alpha_{ik}^{\mathrm{T}} = \begin{bmatrix} f_{ik}^{\mathrm{T}} & -\sum_{h=1}^{p_i-k} \frac{1}{h!}\alpha_{i(k+h)}^{\mathrm{T}} X^{(h)}(s_i)\bar{Q}_i\Sigma_i^{-1} \end{bmatrix}\bar{P}_i, \tag{13}$$

where $\bar{P}_i$, $\bar{Q}_i$ are given by (12), and $f_{ik} \in C^m$, $k=1,2,\ldots,p_i$, $i=1,2,\ldots,q$, are free parameters.

Corollary 4: HSOM equation (1) have following parametric solution, if M=0,

$$\alpha_{ik}^{\mathrm{T}} = \begin{bmatrix} f_{ik}^{\mathrm{T}} & -\sum_{h=1}^{p_i-k} \frac{1}{h!}u_{i(k+h)}^{\mathrm{T}} A^{(h)}(s_i)\bar{Q}_i\Sigma_i^{-1} \end{bmatrix}\bar{P}_i,$$

where $\bar{P}_i$, $\bar{Q}_i$ are given by (12), and $f_{ik} \in C^m$, $k=1,2,\ldots,p_i$, $i=1,2,\ldots,q$, are free parameters.

Corollary 5: If matrix J is non-defective, that is, $J=\mathrm{diag}(s_1,s_2,\ldots s_q)$, HSOM equation (1) have following parametric solution:

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix},\ Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_m \end{bmatrix},$$
$$\begin{bmatrix} u_i^{\mathrm{T}} & z_i^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} f_i^{\mathrm{T}} & 0 \end{bmatrix}\bar{P}_i,$$

where $\bar{P}_i$ satisfies (12) and $f_i \in C^m$, $i=1,2,\ldots,p$, are free parameters.

Numerical Reliability and Improvement

When we find matrix pair $P(s) \in R^{(n+m)\times(n+m)}[s]$, $Q(s) \in R^{n\times n}[s]$ satisfying Smith form reduction (5), and let

$$P(s) = \begin{bmatrix} P_1^{\mathrm{T}}(s) & P_2^{\mathrm{T}}(s) \end{bmatrix}^{\mathrm{T}},\ P_1(s) \in R^{m\times(n+m)}[s],\ P_2(s) \in R^{n\times(n+m)}[s],$$

it can be easily verified that for an arbitrary scalar α,

$$P(\alpha,s) = \begin{bmatrix} \alpha P_1^{\mathrm{T}}(s) & P_2^{\mathrm{T}}(s) \end{bmatrix}^{\mathrm{T}} \tag{14}$$

satisfies (5) too. Therefore the reduction (5) is not unique, which promotes two questions for us. The first one is whether the solution is affected by α. And the second is how we choose the parameters if the answer to the first question is sure.

Similarly, the left coprime factorization (10) is not unique too, since for arbitrary scalar β, $\beta H(s)$ and $\beta L(s)$ satisfy (10). The choice of β also must be considered.

A. Example

Considering an HSOM equation in the form of (1) with following parameters:

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix},\ A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix},\ J = \begin{bmatrix} s & 1 & 0 \\ 0 & s & 1 \\ 0 & 0 & s \end{bmatrix}.$$

We first calculate $P(s)$ and $Q(s)$, then obtain $H(s)$, $L(s)$ as (9). Let

$$f_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ f_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and solve equation (1) using Theorems 1-3, respectively, for $s_j = -50+j$, $j=0,1,\ldots,30$, to check the effectiveness of the conclusions. We define the mean square deviation (MSD) of the error matrix $\begin{bmatrix} e_{ij} \end{bmatrix}_{3\times 3}$ as

$$E_j = \sqrt{\sum_{k=1}^{3}\sum_{l=1}^{3} e_{ij}^2 \Big/ 9},\ j = 0,1,\ldots,30,$$

and plot the MSD E_j with respect to the eigenvalue s_j in Figure I, which shows that the solution given by Theorem 1 has the largest error with the increase of $|s_j|$, while Theorem 3 provide us with a most reliable solution.

B. Improvement of Theorem 1

If we choose $\alpha = 1/\|P(s)\|$ in (14), the MSD decreases when $|s_j|$ is not very large (see Figure II). But it is invalid when $|s_j|$ increases to certain numbers.

C. Improvement of Theorem 2

If we choose $\beta = 1/\|[H(s) \quad L(s)]\|$ in the left coprime factorization, the MSD decreases dramatically (Figure III).

Conclusion

The paper gives three solutions for the HSOM equation with Jordan canonical form, while the first two are analytical solutions and the last is a numerical one.

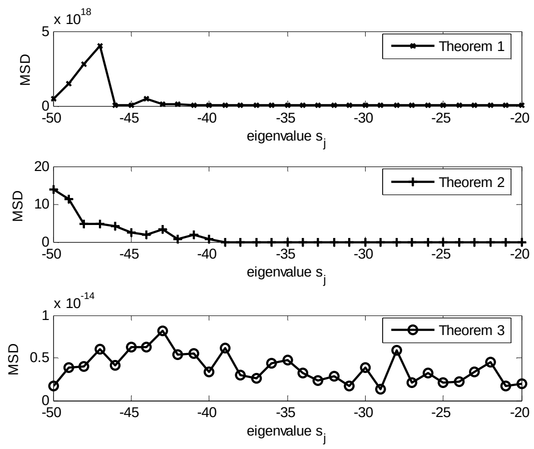

Figure 1. MSDs of the solutions acquired using THeorems 1-3, plotted with respect to the eigenvalue s_j

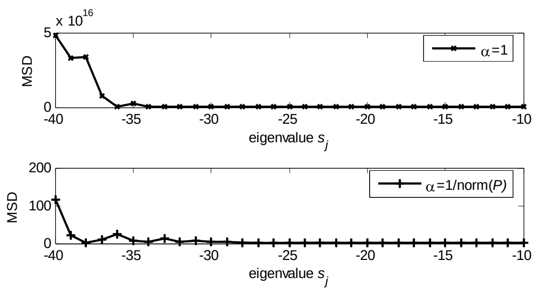

Figure 2. MSDs of the solutions acquired using THeorem 1 with different coefficients α

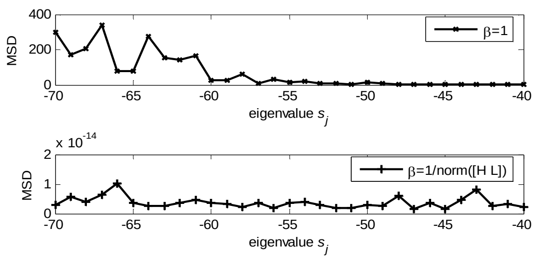

Figure 3. MSDs of the solutions acquired using THeorem 2 with different coefficients β

By comparing the MSDs for every solution, we know that the third conclusion provides us a reliable solution, and the first two do not. But we can improve the methods by multiplying a coefficient for the polynomial matrices, and this is especially effective for Theorem 2.

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