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On the Influence Function for the Theil-Like Class of Inequality Measures

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ABSTRACT

On one hand, a large class of inequality measures, which includes the generalized entropy, the Atkinson, the Gini, etc., for example, has been introduced in P.D. Mergane, G.S. Lo, Appl. Math. 4 (2013), 986–1000. On the other hand, the influence function (IF) of statistics is an important tool in the asymptotics of a nonparametric statistic. This function has been and is being determined and analyzed in various aspects for a large number of statistics. We proceed to a unifying study of the *IF* of all the members of the so-called Theil-like family and regroup those *IF*'s in one formula. Comparative studies become easier.

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1. INTRODUCTION

Over the years, a number of measures of inequality have been developed. Examples include the generalized entropy, the Atkinson, the Gini, the quintile share ratio (QSR) and the Zenga measures (see e.g. [1–5]). Recently, [6] gathered a significant number of inequality measures under the name of Theil-like family. Such inequality measures are very important in capturing inequality in income distributions. They also have applications in many other branches of Science, e.g., in ecology (see e.g. [7]), sociology (see e.g. [8]), demography (see e.g. [9]) and information science (see e.g. [10]).

In order to make the above mentioned measures applicable, one often makes use of estimation. Classical methods unfortunately rely heavily on assumptions which are not always met in practice. For example, when there are outliers in the data, classical methods often have very poor performance. The idea in robust Statistics is to develop estimators that are not unduly affected by small departures from model assumptions, and so, in order to measure the sensitivity of estimators to outliers, the influence function (IF) was introduced (see [11,12]).

Let us begin by precising the objects and notation of our study, in particular the IF. To make the reading of what follows easier, we suppose that we have a probability space $(\Omega, \mathcal{A}, \mathbb{E})$ holding a random variable X associated with the cumulative distribution function (cdf) $F(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$, and a sequence of independent copies of X: X_1, X_2, etc. This random variable is considered as an income variable so that it is nonnegative and $F(0) = 0$. The absolute density distribution function (with respect to the Lebesgue measure on $\mathbb{R}$) of X (pdf), if it exists, is denoted by f. Its mean, we suppose finite and nonzero, and moments of order $\alpha \geq 1$ are denoted by

$$\mu_F = \int_0^{+\infty} y\, dF(y) \in (0, \infty) \text{ and } \mu_{F,\alpha} = \int_0^{+\infty} y^{\alpha}\, dF(y)\,,\ \mu_{F,1} = \mu_F.$$

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The quantile function associated to F, also called generalized inverse function is defined by

$$Q\left(p\right) \equiv F^{-1}\left(z\right) = \inf\{z \in \mathbb{R},\ F\left(z\right) \leq x\},\ p \in [0,1]$$

and the Lorentz curve of F is given by

$$L\left(F,p\right) = \frac{q\left(p\right)}{\mu_F},\ \text{with}\ q\left(p\right) = \int_0^p Q\left(s\right)\ ds,\ 0 \leq s \leq 1.$$

A nonparametric estimation $T(F)$ will studied as well as its plug-in nonparametric estimator of the form $T(F_n)$ which is based on the sample $X_1,\ldots, X_n$, $n \geq 1$.

The $IF(\circ, T(F))$ of $T(F)$ is the Gateaux derivative of T at F in the direction of Dirac measures in the form

$$IF\left(z, T\left(F\right)\right) = \lim_{\epsilon\to 0} \frac{T\left(F_\epsilon^{(z)}\right) - T\left(F\right)}{\epsilon} = \frac{\partial}{\partial\epsilon} T\left(F_\epsilon^{(z)}\right)|_{\epsilon=0}, \tag{1}$$

where

$$F_\epsilon^{(z)}\left(u\right) = \left(1-\epsilon\right)F\left(u\right) + \epsilon\Delta_Z\left(u\right), \epsilon \in [0;1],$$

Δ_z is the *cdf* of the δ_z, the Dirac measure with mass one at z and z is in the value domain of F.

It is known that the asymptotic variance of the plug-in estimator $T(F_n)$ of statistic $T(F)$ is of the form $\sigma^2 = \int IF\left(x, T\left(F\right)\right)^2 dF\left(x\right)$ under specific condition, among them the Hadamard differentiability (see [13], Theorem 2.27, p. 19). So the IF gives an idea of what might be the variance of the Gaussian limit of the estimator if it exists. At the same time, the behavior of its tails (lower and upper) give indications on how lower extreme and/or upper extreme values impact on the quality of the estimation. For example, recently, the sensitivity of a statistic $T(F)$ and the impact of extreme observations of some IFs have been studied by, e.g. [1].

Another interesting fact is that the IF behaves in nonparametric estimation as the score function does in the parametric setting (see [13], p. 19).

An area of application of the IF is that of measures of inequality (see, e.g. [14–16]). Due to the importance of that key element in nonparametric estimation in Econometric and welfare studies, a collection of inequality measures is being actively made. To cite a few, the *IF*'s of the following measures are given in the Appendix section: the generalized entropy class of measures of inequality GE(α), where $\alpha > 0$, the mean logarithmic deviation (MDL), the Theil Measure, the Atkinson Class of Inequality Measures of parameter $\alpha \in (0,1]$, the Gini Coefficient, the QSR Measure of Inequality.

Fortunately, [6] introduced the so-called Theil-like family, in which are gathered the Generalized Entropy Measure, the MDL [17–19], the different inequality measures of Atkinson [20], Champernowne [21] and Kolm [22] in the following form:

$$T_n\left(F\right) = \tau\left(\frac{1}{h_1\left(\mu_n\right)}\ \frac{1}{n}\sum_{j=1}^{n} h\left(X_j\right)\ -\ h_2\left(\mu_n\right)\right), \tag{2}$$

where $\mu_n = \frac{1}{n}\sum_{j=1}^{n} X_j$ denotes the empirical mean while h, h_1, h_2, and τ are measurable functions.

The inequality measures mentioned above are derived from (2) with the particular values of α, τ, h, h_1 and h_2 as described below for all $s > 0$:

a. Generalized entropy

$$\alpha \neq 0,\ \alpha \neq 1,\ \tau\left(s\right) = \frac{s-1}{\alpha\left(\alpha-1\right)},\ h\left(s\right) = h_1\left(s\right) = s^\alpha,\ h_2\left(s\right) \equiv 0;$$

b. Theil's measure

$$\tau\left(s\right) = s,\ h\left(s\right) = s\log\left(s\right),\ h_1\left(s\right) = s,\ h_2\left(s\right) = \log\left(s\right);$$

c. Mean logarithmic deviation

$$\tau\left(s\right) = s,\ h\left(s\right) = h_2\left(s\right) = \log\left(s^{-1}\right),\ h_1\left(s\right) \equiv 1;$$

d. Atkinson's measure

$$\alpha < 1 \text{ and } \alpha \neq 0,\ \tau\left(s\right) = 1 - s^{1/\alpha},\ h\left(s\right) = h_1\left(s\right) = s^\alpha,\ h_2\left(s\right) \equiv 0;$$

e. Champernowne's measure

$$\tau(s) = 1 - \exp(s),\ h(s) = h_2(s) = \log(s),\ h_1(s) \equiv 1;$$

f. Kolm's measure

$$\alpha > 0,\ \tau(s) = \frac{1}{\alpha}\log(s),\ h(s) = h_1(s) = \exp(-\alpha s),\ h_2(s) \equiv 0.$$

This is simply the plug-in estimator of

$$T(F) = \tau\left(\frac{\mathbb{E}h(X)}{h_1(\mu_F)} - h_2(\mu_F)\right) = \tau(I). \tag{3}$$

The following conditions are required for the asymptotic theory.

B1 The functions τ admits a derivative τ' which is continuous at I and $\tau'(I) \neq 0$.

B2. The functions h_1 and h_2 admit derivatives h_1' and h_2' which are continuous at μ_F with $h_1(\mu_F) \neq 0$.

B3. $\mathbb{E}h^j(X) < +\infty, j = 1, 2$.

This offers an opportunity to present a significant number of *IF*'s in a unified approach. This may be an asset for inequality measures comparison. By the way, it constitutes the main goal of this paper.

Let us add more notation. The lower endpoint and upper endpoint of *cdf* F are denoted by

$$lep(F) = \inf\{y \in F,\ F(x) > 0\} \text{ and } uep(F) = \sup\{y \in F,\ F(x) < 1\}.$$

So the domain of admissible values for X, denoted by $\mathcal{V}_X$, satisfies $\mathcal{V}_X \subset \mathcal{R}_X = [lep(F),\ uep(F)]$, the latter being the range of F.

The layout of this paper is as follows: In the next section we state our main result on the IF of the TLIM family members and some particularized forms related to each known members. For member whose *IF*'s are already given, we will make a comparison. In Section 3, we give the complete proofs. In Section 4 we provide a conclusion and some perspectives. Section 5 is an appendix gathering *IF*'s expressions of some members of the TLIM available in the literature.

2. MAIN RESULTS

(A) - The main theorem.

Theorem 2.1. *If conditions* $(B1) - (B2)$ *hold, then the IF of the TLIM index is given by*

$$IF(z, F) = \tau'(I)\left(-\left(\frac{h_1'(\mu_F)\,\mathbb{E}h(X)}{h_1(\mu_F)^2} + h_2'(\mu_F)\right)(z - \mu_F) + \frac{h(X) - \mathbb{E}h(X)}{h_1(\mu_F)}\right), \tag{4}$$

for $z \in \mathcal{V}_X$.

Remark on the asymptotic variance. It was said earlier that the plug-in estimator should give the asymptotic variance of the limiting Gaussian variable, if it exists, as

$$\sigma^2 = \int_{\mathcal{V}_X} IF(X)^2\, d\mathbb{P} = \mathbb{E}IF(X)^2.$$

This is exactly the case from the asymptotic normality of the plug-in estimator as established in Theorem 2 in [23].

Let us move to the illustrations of our results for particular cases.

(B) - Particular forms.

Let us proceed to the study of particular members of the TLIM class. We will have to compare our results with existent ones if any in the appendix. When the computation are simple, we only give the result without further details.

(1) Mean logarithmic deviation. We have

$$\tau(s) = s,\ h(s) = h_2(s) = \log\left(s^{-1}\right),\ h_1(s) \equiv 1$$

and next $\tau'(s) \equiv 1$, $h(s)' = h_2'(s) = -1/s$ and $h_1'(s) \equiv 0$. The application of Theorem 2.1 leads to

$$IF(z, DLM) = \mu_F^{-1}(z - \mu_F) + \left(\log z - \mathbb{E}\log X\right),\ z \in \mathcal{R}_F.$$

(2) Theil's index. We have

$$\tau(s) = s,\ h(s) = s\log s, h_1(s) = s,\ h_2(s) = \log s,$$

and next $\tau'(s) \equiv 1$, $h_1'(s) \equiv 1$ and $h_2'(s) = 1/s$. The application of Theorem 2.1 gives for $z \in \mathcal{R}_F$,

$$IF(z, DLM) = \mu_F^{-1}\left(z\log z - \mathbb{E}X\log X\right) - \mu_F^{-2}\left(\mu_F + \mathbb{E}\log X\right)(z-\mu_F)$$

(3) Class of generalized entropy Measures of parameter α, $\alpha \notin \{0,1\}$. We have

$$\tau(s) = \frac{s-1}{\alpha(\alpha-1)},\ \tau'(s) = \frac{1}{\alpha(\alpha-1)},\ h(s) = h_1(s) = s^{\alpha}, h_1' = \alpha s^{\alpha-1}\, h_2(s) \equiv 0.$$

The application of Theorem 2.1 gives

$$IF(z, GE(\alpha)) = \frac{z^{\alpha} - \mu_{F,\alpha}}{\alpha(\alpha-1)\mu_F^{\alpha}} - \mu_{F,\alpha}(\alpha-1)\mu_F^{\alpha+1}(z-\mu_F),\ z \in \mathcal{R}_F.$$

(4) Class of Atkinson measures with parameter $\beta \in (0,1)$. We have

$$\tau(s) = 1 - s^{1/\beta},\ h(s) = h_1(s) = s^{\beta},\ h_2(s) \equiv 0.$$

If we denote $\| X \|_{\beta} = \left(\mathbb{E}|X|^{\beta}\right)^{1/\beta}$, the application of Theorem 2.1 yields

$$IF(z, At(\beta)) = \frac{\| X \|_{\beta}}{\mu_F}\left(\frac{z-\mu_F}{\mu_F} - \frac{z^{\alpha} - \mu_{F,\beta}}{\beta\mu_{F,\beta}}\right),\ z \in \mathcal{R}_F.$$

(5) Champernowne's index. We have

$$\tau(s) = 1 - \exp(s),\ h(s) = h_2(s) = \log(s),\ h_1(s) \equiv 1.$$

The application of Theorem 2.1 implies that

$$IF\left(z, Champ\right) = \frac{\exp\left(\mathbb{E}\log X\right)}{\mu_F}\left(\frac{1}{\mu_F}(z-\mu_F) - \left(\log z - \mathbb{E}\log X\right)\right),\ z \in \mathcal{R}_F.$$

(6) Kolm's Famililly of inequality measure of parameter $\alpha \neq 0$. We have

$$\tau(s) = \frac{1}{\alpha}\log(s),\ h(s) = h_1(s) = \exp(-\alpha s),\ h_2(s) \equiv 0.$$

By Theorem 2.1, we have

$$IF(z,\ Kolm(\alpha)) = \frac{1}{\alpha}\left(\alpha(z-\mu_F) - \left(\frac{\exp(-\alpha z)}{\mathbb{E}\exp(-\alpha X)} - 1\right)\right),\ z \in \mathcal{R}_F.$$

3. PROOF OF THE MAIN THEOREM

In the following proof, we will use the method of finding the *IF* following argument as given in [24]. Suppose that we are interested in estimating $T(\mathbb{P}_X)$, where $\mathbb{P}_X$ the image measure is $d\mathbb{P}$ defined by $d\mathbb{P}_X(B) = d\mathbb{P}(X \in B)$ for $B \in \mathcal{B}(\mathbb{R})$ and is also Lebesgue–Stieltjes probability law associated F, that is $\mathbb{P}_X([a,b]) = F(b) - F(a)$ for all $-\infty \leq a \leq b \leq +\infty$. Here we use integrals based on measures and thus integrals in dF are integrals in $d\mathbb{P}_X$ in the following sense: for any nonnegative and measurable function $\ell : \mathbb{R} \to \mathbb{R}$, we have

$$\int \ell(X)\, d\mathbb{P} = \inf h(y)\, d\mathbb{P}_X \equiv \int h(y)\, dF(y).$$

Suppose that $T(\mathbb{P})$ is defined on a family of probability measures $\mathbb{P}_{\lambda}$, $\mathbb{P}_{\lambda}$ being associated with the random variable X_{λ} with $X = X_{\lambda_0}$ and $F = F_{\lambda_0}$. Suppose that T is independent of λ. If we have

$$\frac{\partial}{\partial\lambda}T(\mathbb{P}_{\lambda}) = \int \ell(y)\,\frac{\partial}{\partial\lambda}\mathbb{P}_{\lambda},$$

where ℓ is measurable and $\mathbb{P}_X$-integrable. Then the IF at $T\left(F_{\lambda_0}\right) = T(F)$ is given by

$$IF(z,F) = \ell(z) - \int \ell(y)\ dF(y) = \ell(z) - \mathbb{E}\ell(X).$$

Actually, the rule uses Gâteaux differentiations properties and constitutes one of the fastest methods of finding the *IF*. We are going to apply it.

Proof of Theorem 2.1.

We remind the notation.

$$I = \frac{\mathbb{E}h(X)}{h_1(\mu_F)} - h_2(\mu_F).$$

We have

$$\frac{\partial}{\partial\lambda} TLIM(\mathbb{P}_X) = \frac{\partial}{\partial\lambda}\tau\left(\left(\frac{1}{h_1\left(\int Xd\mathbb{P}\right)}\right)\int h(X)\,d\mathbb{P} - h_1\left(\int Xd\mathbb{P}\right)\right).$$

We get

$$\begin{aligned}
\frac{1}{\tau'(I)} TLIM(\mathbb{P}_X) &= -\frac{h_1'(\mu_F)\,\mathbb{E}h(X)}{h_1(\mu_F)^2}\int X\frac{\partial}{\partial\lambda}d\mathbb{P} \\
&\quad + \frac{1}{h_1(\mu_F)}\int h(X)\frac{\partial}{\partial\lambda}\mathbb{P} \\
&\quad - h_2'(\mu_F)\int X\frac{\partial}{\partial\lambda}d\mathbb{P} \\
&= \int\left(-\left(\frac{h_1'(\mu_F)\,\mathbb{E}h(X)}{h_1(\mu_F)^2} + h_2'(\mu_F)\right)X + \frac{h(X)}{h_1(\mu_F)}\right)\frac{\partial}{\partial\lambda}d\mathbb{P}.
\end{aligned}$$

By centering at expectations, we have

$$IF(z,F) = \tau'(I)\left(-\left(\frac{h_1'(\mu_F)\,\mathbb{E}h(X)}{h_1(\mu_F)^2} + h_2'(\mu_F)\right)(z-\mu_F) + \frac{h(X)-\mathbb{E}h(X)}{h_1(\mu_F)}\right), z \in \mathcal{V}_X.$$

4. CONCLUSION AND PERSPECTIVES

I this paper, we studied the Theil-like family of inequality measures introduced in [23]. Following the paper on the asymptotic finite-distribution normality, we focus on the IF of that family. Results are compared with those of some authors in particular. We think that this unified and compact approach will serve as general tools for comparison purpose. In addition, in computation packages, it allows more compact programs resulting in more efficiency. A paper on computational aspects will follow soon.

5. APPENDIX: A LIST OF SOME IFS

Here, we list a number of inequality measures and the corresponding *IFs*.

The generalized entropy measures of inequality GE(α), which depends of a parameter $\alpha > 0$ and defined by

$$\begin{aligned}
I_E^\alpha &= \int_0^\infty \frac{1}{\alpha(\alpha-1)}\left[\left(\frac{y}{\mu_F}\right)^\alpha - 1\right]dF(y) \\
&= \frac{1}{\alpha(\alpha-1)}\left(\frac{\mu_{F,\alpha}}{\mu_F^\alpha} - 1\right),\ \alpha > 0,\ \alpha \notin \{0,1\},
\end{aligned}$$

has the *IF* (see e.g. [1])

$$IF\left(z; I_E^\alpha\right) = \frac{1}{\alpha(\alpha-1)\,\mu_F^\alpha}\left(z^\alpha - \mu_\alpha\right) - \frac{\mu_\alpha}{(\alpha-1)\,\mu_F^{\alpha+1}}\left[z-\mu_F\right], \alpha \notin \{0,1\}. \tag{5}$$

Important remark. Our result on the *IF* of the $GE(\alpha)$ is different from that of [1] by the multiplicative coefficient $\frac{1}{\alpha(\alpha-1)\mu_F^\alpha}$. In other words, that coefficient is missing in [1]. We also find the same result by the computations below which is a direct proof.

$$\begin{aligned}
\frac{\partial}{\partial\lambda}GE(\alpha) &= \frac{\partial}{\partial\lambda}GE(\alpha) = \frac{\partial}{\partial\lambda}\frac{1}{\alpha(\alpha-1)}\left(\frac{\int X^\alpha\,d\mathbb{P}}{\left(\int X\,d\mathbb{P}\right)^\alpha} - 1\right) \\
&= \frac{1}{\alpha(\alpha-1)}\int\frac{\mu_F^\alpha X^\alpha - \alpha\mu_F^{\alpha-1}X}{\mu_F^{2\alpha}}\frac{\partial}{\partial\lambda}d\mathbb{P}.
\end{aligned}$$

By the method described in the proof, we may center the integrand to get

$$IF(X, GE(\alpha)) = \frac{1}{\alpha(\alpha-1)} \frac{\mu_F^{\alpha}(X^{\alpha} - \mathbb{E}X^{\alpha}) - \alpha\mu_F^{\alpha-1}(X - \mathbb{E}X)}{\mu_F^{2\alpha}}.$$

which again gives the result.

The MDL, which is a special case of the GE class where $\alpha = 0$, defined by

$$I_E^0 = -\int_0^{\infty} \log\left(\frac{y}{\mu_F}\right) dF(y) = \log\mu_1 - \nu,\ \nu = \mathbb{E}\log X, \tag{6}$$

is associated to the *IF*

$$IF\left(z, I_E^0\right) = -\left[\log z - \nu\right] + \frac{1}{\mu_1}\left[z - \mu_F\right]. \tag{7}$$

The Theil measure, which also is a special case of the GE class for $\alpha = 1$,

$$I_E^1 = \int_0^{\infty} \frac{y}{\mu_F} \log\left(\frac{y}{\mu_F}\right) dF(y) = \frac{\nu}{\mu_F} - \log\mu_F,\ \nu = \mathbb{E}X\log X, \tag{8}$$

has the *IF*

$$IF\left(z; I_E^1\right) = \frac{1}{\mu_F}\left[z\log z - \nu\right] - \frac{\nu + \mu_F}{\mu_1^2}\left[z - \mu_F\right]. \tag{9}$$

The Atkinson class of inequality measures of parameter $\alpha \in (0, 1]$, defined by (see [1])

$$\begin{aligned} I_A^{\alpha} &= 1 - \left[\int_0^{\infty} \left(\frac{y}{\mu_F}\right)^{1-\alpha} dF(y)\right]^{1/(1-\alpha)} \\ &= 1 - \frac{\mu_{F,1-\alpha}^{1/(1-\alpha)}}{\mu_F},\ \alpha > 0, \alpha \neq 1, \end{aligned}$$

and its IF is given by

$$IF\left(z; I_A^{\alpha}\right) = -\frac{\nu^{(1/(1-\epsilon))-1}}{(1-\epsilon)\mu_F}\left(z^{1-\epsilon} - \nu\right) + \frac{\nu^{1/(1-\epsilon)}}{\mu_F^2}(z - \mu_F), \tag{10}$$

where $\nu = \mathbb{E}X^{1-\epsilon}$.

We notice that for $\alpha = 1$, we have

$$I_A^1 = 1 - \frac{e^{\int_0^{\infty}(\log y)dy}}{\mu} = 1 - e^{-I_E^0}, \tag{11}$$

The Gini coefficient, defined by (see e.g. [1]):

$$I_G = 1 - 2\int_0^1 L\left(F, p\right) dp, \tag{12}$$

has the *IF*

$$IF(z, I_G) = 2\left[R(F) - C(F, F(z)) + \frac{z}{\mu_F}(R(F) - (1 - F(z)))\right], \tag{13}$$

where

$$R(F) = \int_0^1 L\left(F, p\right) dp \tag{14}$$

and C is is the cumulative functional defined by

$$C\left(F, p\right) = \int_0^{Q(p)} x dF(x),\ 0 \leq p \leq 1. \tag{15}$$

The QSR measure of inequality, defined by

$$\eta = \frac{\int_{Q(0.8)}^{\infty} y dF(y)}{\int_{0}^{Q(0.2)} y dF(y)} = \frac{EX1_{\{X>Q(0.8)\}}}{EX1_{\{X\leq Q(0.2)\}}}, \tag{16}$$

where 1_A is an indicator function of a set A, is associated with the *IF* described below (see [16]). Let

$$N(F) = \int_{Q(0.8)}^{\infty} x dF(x) \tag{17}$$

and

$$D(F) = \int_{0}^{Q(0.2)} x dF(x). \tag{18}$$

And define the subdivision of $\mathbb{R}_+ : A_1 = [0,\ Q(0.2)]$, $A_2 = (Q(0.2),\ Q(0.8))$, $A_3 = (Q(0.8),\ 1]$ and set

$$\begin{aligned} I_1(z,\eta) &= -zN(F) + 0.2Q(0.8)\,D(F) + 0.8Q(0.2)\,N(F)]\,/D^2(F)\,; \\ I_2(z,\eta) &= 0.2\,Q(0.8)\,D(F) - 0.2Q(0.2)\,N(F)]\,/D^2(F)\,; \\ I_3(z,\eta) &= zD(F) - 0.8Q(0.8)\,D(F) - 0.2Q(0.2)\,N(F)]\,/D^2(F)\,. \end{aligned}$$

The *SQR* IF is defined by

$$I_1(z,\eta) = I_1(z,\eta)\,1_{A_1}(z) + I_2(z,\eta)\,1_{A_2}(z) + I_3(z,\eta)\,1_{A_3}(z)\,.$$

AUTHORS' CONTRIBUTIONS

Each author contributed of 25% of the paper.

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REFERENCES

1. F.A. Cowell, E. Flachaire, J. Econom. 141 (2007). 1044–1072.
2. F.A. Cowell, E. Flachaire, S. Bandyopadhyay, Goodness-of-Fit: An Economic Approach, Distributional Analysis Research Programme (DARP 101), The Toyota Centre, London School of Economics and Political Science, London, 2009. http://sticerd.lse.ac.uk/_new/publications/series.asp?prog=DARP.
3. B. Hulliger, T. Schoch, in Presented at the Swiss Statistics Meeting, Geneva, 2009.
4. M. Zenga, Giornale degli Economisti e Annali di Economia, 43 (1984), 301–326.
5. M.Zenga, in: C. Dagum, M. Zenga (Eds.), Income and Wealth Distribution, Inequality and Poverty, Springer-Verlag, New York, (1990).
6. P.D. Mergane, G.S. Lo, Appl. Math. 4 (2013), 986–1000. http://www.scirp.org/journal/am.
7. A.E. Magurran, Ecological Diversity and its Measurement, Chapman and Hall, 1991.
8. P.D. Allison, Am. Sociol. Rev. 43 (1978), 865–880.
9. M.J. White, Popul. Index. 52 (1986), 198–221.
10. R. Rousseau, Scientometrics. 28 (1993), 3–14.
11. F.R. Hampel, J. Am. Stat. Assoc. 69 (1974), 383–393.
12. F.R. Hampel, E.M. Ronchetti, P.J. Rousseeuw, W.A. Stahel, Robust Statistics: The Approach Based on Influence Functions, John Wiley, New York, 1986.
13. L. Wasserman, All of Nonparametric Statistics, Springer Science + Business Media, Inc., 2006.
14. B. Van Praag, A. Hagenaars, W. Van Eck, Econometrica. 51 (1983), 1093–1108.
15. M.-P. Victoria-Feser, Int. Stat. Rev. 68 (2000), 277–293.
16. T.A. Kpanzou, Commun. Stat. Simul. Comput. 44 (2015), 2492–2499.
17. F.A. Cowell, Theil, Inequality and the Structure of Income Distribution, London School of Economics and Political Sciences, 2003. http://eprints.lse.ac.uk/2288/

18. H. Theil, Economics and Information Theory, North Holland, Amsterdam, 1967.
19. F.A. Cowell, Eur. Econ. Rev. 13 (1980), 147–159.
20. A.B. Atkinson, J. Econ. Theory. 2 (1970), 244–263.
21. D.G. Champernowne, F. A. Cowell, Economic Inequality and Income Distribution, Cambridge University Press, Cambridge, 1998.
22. S. Kolm, J. Econ. Theory. 12 (1976), 416–442.
23. P.D. Mergane, T.A. Kpanzou, B. Diam, G.S. Lo, Afr. Stat. 13 (2018), 1699–1715.
24. J. Kahn, Influence Functions for Fun and Profit, Working Paper, Ann Arbor, 2015. http://j-kahn.com/files/influencefunctions.pdf