

Application of Multilayer Networks to Detect Critical Energy Facilities

Elena Nosyreva
Irkutsk National Research Technical University
Irkutsk, Russia
nev-7@list.ru

Liudmila Massel^{1,2}
¹ *Melentiev Energy Systems Institute of Siberian Branch
of the Russian Academy of Sciences*
² *Irkutsk National Research Technical University*
Irkutsk, Russia
massel@isem.irk.ru

Abstract — One of the main components of national security is energy security. To ensure the protection of energy facilities, it is necessary to single out the most important facilities, the disruption of the functioning of which can cause irreparable consequences. The methods currently existing in Russia for identifying critical energy facilities do not take into account the interconnection of energy networks with other sectors of critical infrastructure. However, in practice, energy facilities are not only interconnected in a single network, but also connected with other critical infrastructures. The purpose of the study is to develop a formal approach to deal with this issue. To solve this problem, the author proposes to use the tool of the theory of complex networks. The relationships between objects of critical infrastructures can be described using multilayered networks, and centrality measures can be used as indicators of the importance of objects of critical infrastructure.

Keywords — critical infrastructures, critical objects, energy sector, complex networks, multi-layer networks, centrality measures.

I. INTRODUCTION

Ensuring national security is the most important task of any state. To solve this problem, it is necessary to create new methods and improve old ones and tools used to study security problems. This is caused, on the one hand, by the development of information technology, on the other hand, by the emergence of new approaches to the study and analysis of complex systems. One of the main components of national security is energy security, since the fuel and energy complex plays an important role in the economic and social life of the country.

To ensure the protection of energy facilities, it is necessary to single out among them the most important facilities, the disruption of the functioning of which can lead to irreparable consequences. The methods currently existing in Russia for identifying critical energy facilities do not take into account the interconnection of energy networks with other sectors of critical infrastructure. However, in practice, energy facilities are connected in a single network, not only among themselves, but also with objects of other interconnected networks. The purpose of the study considered in the article is to develop a formal approach to solving this problem.

To solve this problem, the author proposes to use such a tool of the theory of complex networks, as multilayer networks. The article discusses the concepts of critical infrastructure and critical objects, approaches to their research, tools of the theory of complex networks that can be used to solve the problem of identifying critical objects, and also provides examples of the application of this approach.

II. CRITICAL INFRASTRUCTURE

Recently, in order to ensure national security, the research has been actively conducted in the field of critical infrastructures. Critical infrastructure means a part of civil infrastructure, which is a combination of physical or virtual systems and tools that are important to the state to the extent that their failure or destruction can have devastating consequences in the field of defense, economy, health and national security. In the case of military conflicts, disabling or destroying civilian infrastructure can lead to damage comparable to strikes at armed forces [1]. In addition, elements of critical infrastructure can become targets of terrorist attacks.

In [1], the study of critical infrastructures is considered in military terms. However, it should be noted that this problem is much broader and includes not only military and terrorist threats, but also any threats, as a result of which critical infrastructure may be disabled or destroyed. For example, natural disasters, equipment deterioration, and others can become such threats.

The main purpose of the study of critical infrastructures is to protect them. At the same time, the main task is to identify the key objects (or their aggregate), the effects on which can have the most negative effect on a sector of the economy, a key resource or the entire infrastructure, as well as to assess the consequences of such an impact and to develop mechanisms in order to reduce such risks [1]. These key objects are referred to as critical objects. In [2], a critically important object is an object, the violation or termination of the functioning of which causes a loss of economic management of the Russian Federation, a subject of the Russian Federation or a municipality, an irreversible negative change or destruction of the economy of the Russian Federation, a subject of the Russian Federation or a municipality,

or a significant decrease of life safety of the population living in these territories for a long period.

III. METHODS OF CRITICAL INFRASTRUCTURES STUDY

The main approaches to the study of critical infrastructures are: the theory of Clausewitz's centers of gravity, a similar approach by Warden, the theory of complex systems, the theory of self-organizing networks (scale-free network) by A. Barabasi [1]. In all these approaches, the critical infrastructure is considered as a complex system that is a network of interconnected subsystems and objects. Currently in Russia, a technique based on the methods of analyzing hierarchies and expert assessments supported by a representative survey of experts and subsequent correct processing of its results is used to form a matrix of object importance indicators [2]. In power engineering, it is customary to refer an object to critical facilities if, when it fails, the supply of products to consumers decreases by 5%. This evaluation has been identified by expertise.

The task of building a quantitative indicator of an object importance with the formalized (mathematical) methods in many cases is difficult to solve because of the complexity of the objects and their systems under consideration. The method of expert assessments allows to circumvent these difficulties, however, it has several disadvantages. The disadvantage of this technique is the lack of reliability of expert estimates. The method of expert assessments is based on the subjective opinions of experts, which may depend on the expert's stereotypes and prejudices. In addition, the prediction result can be influenced by the phenomenon of groupthink that is suppressing the opinions of individual survey participants by more authoritative experts. All this may cause a biased conclusion about the object.

One of the main drawbacks of the existing methodology is the lack of consideration for the relationship and interdependence of objects. The importance is estimated for each object without regard to other objects. But objects are usually interconnected in a complex network, and the failure of one object can lead to disruption of the entire network as a whole, and even other interconnected networks. For example, such kind of cascading failures occurred in Italy on September 28, 2003, when damage to power plants caused the shutdown of some Internet nodes, which, in turn, caused a further shutdown of power plants. Fig. 1 illustrates an iterative process of cascading failures using actual data from the Italian electricity network and the Internet (shifted above the map). Networks are compiled using real geographic locations, and each Internet server is connected to the nearest power station [3].

In Figure 1a, one power plant (red node on the map) is removed from the power network, and as a result Internet nodes that depend on it are removed from the Internet (red nodes above the map). The nodes that will be disconnected from the giant cluster (the cluster that covers the entire network) in the next step are marked in green. In Figure 1b, additional nodes are removed that were disconnected from the giant Internet component (red nodes above the map). As a result, the power stations that depend on them are removed from the power grid (red nodes on the map). Again, the nodes that will be disconnected from the giant cluster in the next step are marked in green. In Figure 1c, additional nodes that were disconnected from the giant component of the power supply network (red nodes on the map), as well as nodes on the Internet that depend on them (red nodes above the map) [3] are removed.

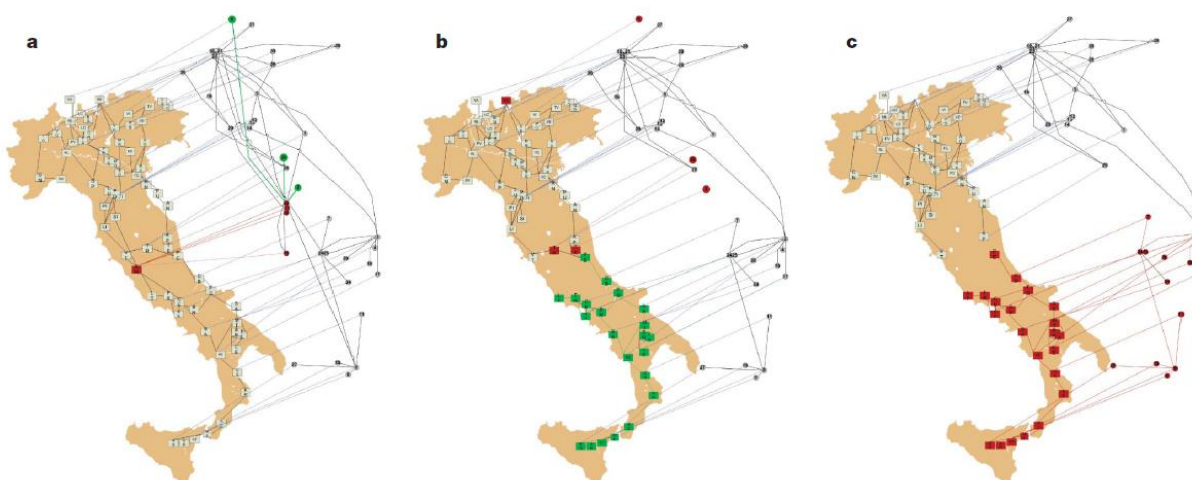


Fig. 1. Modeling a blackout in Italy. Illustration of an iterative process of a cascade of failures using real-world data from a power network (located on the map of Italy) and an Internet network (shifted above the map) that were implicated in an electrical blackout that occurred in Italy in September 2003²⁰. The networks are drawn using the real geographical locations and every Internet server is connected to the geographically nearest power station. a. One power station is removed (red node on map) from the power network and as a result the Internet nodes depending on it are removed from the Internet network (red nodes above the map). The nodes that will be disconnected from the giant cluster (a cluster that spans the entire network) [3].

To illustrate the complexity of the relationship of elements of critical infrastructures, we can consider an accident that occurred on July 19, 2001, when a train of 62 tanks carrying hazardous chemicals derailed in a tunnel on Howard Street in Baltimore, USA. In addition to the violation of rail and road communication, there was a cascading destruction of the infrastructure: water supply, electrical distribution systems and telecommunications [1].

One such example in the power systems of Russia can be a technological accident on May 25, 2005 in the power system of Moscow, which, according to some estimates, affected about 2 million people. The accident began at the power substation number 510 "Chagino". It is believed that the immediate cause of the accident was a combination of several factors: wear and tear of equipment, lack of reserve capacity and high temperature for several days (over 30 ° C). As a result of the accident, the power supply was turned off for several hours in several districts of Moscow, the Moscow region, as well as Tula, Kaluga and Ryazan regions. Several tens of thousands of people were blocked in the stopped trains of the Moscow metro and elevators, the railway communication was disrupted and the work of many organizations was paralyzed. As a result, Moscow estimated the losses from the "blackout" at 1.708 billion rubles, and the Moscow region at 503.94 million rubles [3].

Due to the fact that the methods used have significant drawbacks, there is a need to develop new methods for identifying critical objects based on objective criteria. One of the approaches to the development of such methods is the application of the theory of complex networks for analyzing the interconnection of objects and determining their importance in a network structure. The approach to the analysis of complex systems based on complex networks is very effective in many scientific fields, for example, in sociology, biology, technology, etc. [5, 6]. The effectiveness of this approach is due primarily to the fact that it allows to abstract from unnecessary details, simplifying the consideration of elements and interactions and focusing on the behavior of the system as a whole. Currently, mathematical methods for describing and analyzing complex networks are developing very actively. Thus, it is possible to build formal indicators of the importance of objects based on this theory.

IV. MULTILAYER NETWORKS

The examples above clearly show that the various networks that are part of the critical infrastructure are often interconnected and form a multilayer network. The resulting multilayer network will have a non-trivial complex structure and be a complex network. Multilayer networks are a fairly new direction in the theory of complex networks. One type of multilayer network is stem networks. A stem network is a collection of networks defined on the same set of

nodes. Methods of formal description and analysis of stem networks are considered in [7, 8].

In the most common multilayer network structure, each node can belong to any subset of layers, and edges can be considered that cover pairwise connections between all possible combinations of nodes and layers. One can further generalize this structure and consider the hyper-edges that connect more than two nodes. In addition, multidimensional layered structures can be considered. For example, the type of edge can be considered as one dimension, and the time of existence of this node as another.

Multi-layer network is called a quadruplet $M = (V_M, E_M, V, L)$, where V – is the set of nodes (as in ordinary graphs), $L = \{L_1, L_2, \dots, L_d\}$ – is a sequence of sets of possible layers (where there is one set for each additional aspect $d \geq 0$ beyond an ordinary graph) (where L_d is the set of layers in aspect a), $V_M \in V \times (L_1 \times L_2 \times \dots \times L_d)$ – is a set of tuples that represent which node belongs to which layer, $E_M \in V_M \times V_M$ – is the multilayer edge set that connects these tuples [9].

If $d = 0$, then the multilayer network is reduced to a single-layer network, so that the set $V_M = V$ is redundant. The first two elements in the multilayer network M are the graph $G_M = (V_M, E_M)$, so that one can interpret the multilayer network as a graph, the nodes of which are labeled in a certain way. Using this remark, one can summarize some of the basic concepts of monolayer networks for multilayer networks. For example, you can determine a weighted multilayer network by matching the real number of edges to the network using the function $\omega: E_M \rightarrow \mathbb{R}$. A multilayer network is non-oriented if $((u, \alpha), (v, \beta)) \in E_M \Rightarrow ((v, \beta), (u, \alpha)) \in E_M$, where $\alpha = \alpha_1, \dots, \alpha_d$. One can also use the usual loop ban convention, so that $((u, \alpha), (u, \alpha)) \notin E_M$, but this is not necessary [9].

Many real networks can be described using multilayer networks. Energy infrastructure can also be viewed as a multi-layer network. For example, the main branches of the energy industry — oil, gas, and power engineering — can be considered as layers.

Another division into layers is also possible as well as the presentation of this network as a network of networks, including meta-layers and meta-nodes. Meta-layers are top-level layers that contain meta-nodes as nodes. Meta-nodes are some kind of network, which can also be multi-layered.

V. CENTRALITY MEASURES

To determine the relative importance of nodes in the theory of complex networks, the concept of centrality is introduced. Centrality is a characteristic that shows the "importance" or "influence" of a particular node within the graph. However, the concept of "importance" has a wide range of meanings and can be interpreted differently depending on the situation; therefore there are various measures of centrality.

Two main categories of node importance can be distinguished: importance in the network flow and importance for maintaining network connectivity. Obviously, the measures of centrality used in one category may be incorrect when used in another category. Note that all centrality measures estimate the position of the node when wandering along network walks. One can distinguish measures by walk type. The variants of measures here are distinguished by restrictions on which walk are calculated: shortest paths, all paths of a certain length, edge-independent, node-independent. Another classification dimension is walk property. All measures characterize the walk length or their number. On this basis, they can be called measures length measures or volume measures. Centrality measures are also divided into radial and medial measures. Radial centralities count paths that have a beginning or end at a given vertex. The degree is an example of radial centrality. Medial centrality calculates the paths that pass through this node. An example is the load node [10].

In [11], Paolo Boldi gives the following classification of centrality measures:

- measures based on degree;
- measures based on the number of paths or shortest paths (geodesics) passing through the vertex;
- measures based on distances from a vertex to other vertices;
- spectral measures (spectral indices).

Boldi calls the first three classes geometric measures. The latter class is based on the calculation of the eigenvector of the transformed adjacency matrix. Some of the geometric indices also have an equivalent spectral definition. Let us consider some measures belonging to each of these classes. The first class includes the degree of the vertex d , i.e. the number of its connections with other peaks. This also includes the half degree of entry d^+ (the number of arcs entering the top) and the half degree of outcome d^- (the number of outgoing arcs from the top) for directed graphs.

Measures based on the number of paths or shortest paths (geodesics) that pass through the peak include betweenness centrality, Katz centrality, etc. Betweenness centrality x in the network is defined as the fraction of the shortest paths between all other nodes that pass through node x . Katz centrality is a generalization of degree centrality. The difference is that the degree of centrality takes into account the number of direct neighbors of a node, and the Katz centrality takes into account the number of all nodes that can be connected by a path to node x , but the long routes are fined, i.e. short routes are expensive, and long ones are cheap.

Measures based on distance include closeness centrality, harmonic centrality, and others. Closeness centrality shows how closely a node is located in relation to other ones. The greater the centrality of the

node is, the lower its total distance is from all other nodes. Harmonious centrality is reciprocal of denormalized harmonic mean of distances.

Spectral measures include eigenvector centrality, PageRank, HITS, etc. Let A be the adjacency matrix of the graph $G(V, E)$, that is, $a_{ij} = 1$ if the node i is connected with the node j and $a_{ij} = 0$ otherwise. The dominant eigenvector of the matrix A is called eigenvector centrality.

Centrality can be calculated using the PageRank reference ranking algorithm that is used in the Google search engine. It is based on the principle of "importance" of a web page: the more links to a page, the more important it is. In addition, the weight of page X depends on the weight of page Y , if from page Y there is a link to page X , i.e. the greater the weight of Y is, the greater the weight of X will be. The PageRank algorithm can be applied not only to web pages, but also to any network. In terms of an arbitrary network, PageRank is a method for calculating the weight of a network node by calculating the importance of its edges, i.e. a node whose sources of incoming connections have more weight itself receives more weight.

The predecessor of the PageRank algorithm is the HITS algorithm. In the HITS algorithm, two estimates are calculated for each web page: an authority score and a hub score, that is, each page is recursively calculated as its significance as an "author" and a "hub". In general, authority score determines the importance of a node as a source, and hub score defines the importance of a node as a target in the network.

The centrality measures listed here can be attributed to the main and most frequently used in practice. There are other measures, but they are not used as often. Formulas for calculating the centrality measures of single-layer networks can be found in [11]. Generalization of the above measures of centrality in the case of a multilayered network is, in general, a non-trivial task. This is due to the fact that such concepts as distance, route, and path are determined for multilayer networks ambiguously. A generalization of centrality measures for multilayer networks is given in [12-16].

The following are examples of the use of centrality measures to identify critical objects in the energy sector.

VI. THE RESULTS OF THE ANALYSIS OF THE POWER SUPPLY NETWORK OF ITALY

Let us return to the example of the cascade shutdown of power plants in Italy, which was discussed above. Obviously, the node from which the cascading failures began should be a critical network object. We will show that measures of centrality can serve as an indicator of the importance of the object in this case. For the example under consideration, a two-layer network was constructed containing a layer with power stations (nodes highlighted in black) and a layer

with Internet nodes (nodes highlighted in blue) (Fig. 2).

Since the layers of this network do not contain the same nodes, the resulting two-layer network can be transformed into a single-layer network by combining two layers into one. Thus, formulas for single-layer networks can be used to calculate the centrality measures of this network.

The following measures of centrality were calculated for this network: node degree, closeness centrality, harmonic centrality, betweenness centrality, HITS, PageRank, and eigenvector centrality. The measures of centrality were calculated first separately for the network of power plants, and then as a whole for the whole two-layer network. Further, all nodes were ranked in ascending order by each of the measures. For all nodes the average rank was also calculated (by the method of arithmetic average ranks). Figure 3 shows the average ranks of the nodes (nodes with a higher rank have a larger size). The figure illustrates that the ranks of the nodes in Fig. 3a (excluding communication with the Internet) and Fig. 3b (with regard to communication with the Internet) differ.

Table 1 shows the numbers of 15 nodes of the power supply network with the highest ranks for each of the measures for the case when the rank of the node was determined without regard to communication with Internet nodes. The bold font identifies the node number (45) from which the cascade trip began.

Table 2 shows the numbers of 15 nodes of the power supply network with the highest ranks for each of the measures for the case when the rank of the node was determined taking into account the connection with the Internet nodes, i.e. across the entire two-layer network. It can be seen that node 45, from which the cascade shutdown began, in the first table as a whole has a rank lower than in the second one. When calculating measures of centrality, taking into account connections to the Internet, node 45 has the maximum rank in all measures, except for the load of the node. According to the average rank, node 45 was ranked 10th in the first table and 1st in the second one. If in the first case, the critical importance of node 45 is in doubt, in the second case, node 45 is the most critical node in the network. Thus, this example clearly shows that the lack of consideration of interconnections between all infrastructure facilities, including interconnections between different networks and levels, can lead to incorrect results when determining critical objects.

TABLE 1. RANKS OF THE NODES OF THE ELECTRICAL NETWORK IN ITALY (WITHOUT REGARD TO LINKS WITH INTERNET NODES)

Rank	The number of the node with the corresponding rank according to the centrality measures:							
	Degree	Closeness centrality	Harmonious centrality	Betweenness centrality	HITS	PageRank	Eigenvector centrality	Average for all measures
6 7	1	32	27	27	10	1	10	27
6 6	10	39	10	45	1	48	1	10
6 5	26	33	32	48	7	26	27	26
6 4	48	27	33	33	27	10	18	32
6 3	3	40	26	59	18	8	7	1
6 2	8	38	39	32	3	54	26	18
6 1	11	45	18	57	23	11	3	39
6 0	18	31	23	38	26	45	23	33
5 9	27	23	1	40	11	3	33	48
5 8	32	37	45	39	21	64	11	45
5 7	39	26	40	26	20	39	32	3
5 6	45	18	48	60	33	32	39	11
5 5	54	48	3	28	6	18	21	23
5 4	5	28	31	10	5	27	48	40
5 3	7	41	11	31	24	13	20	8

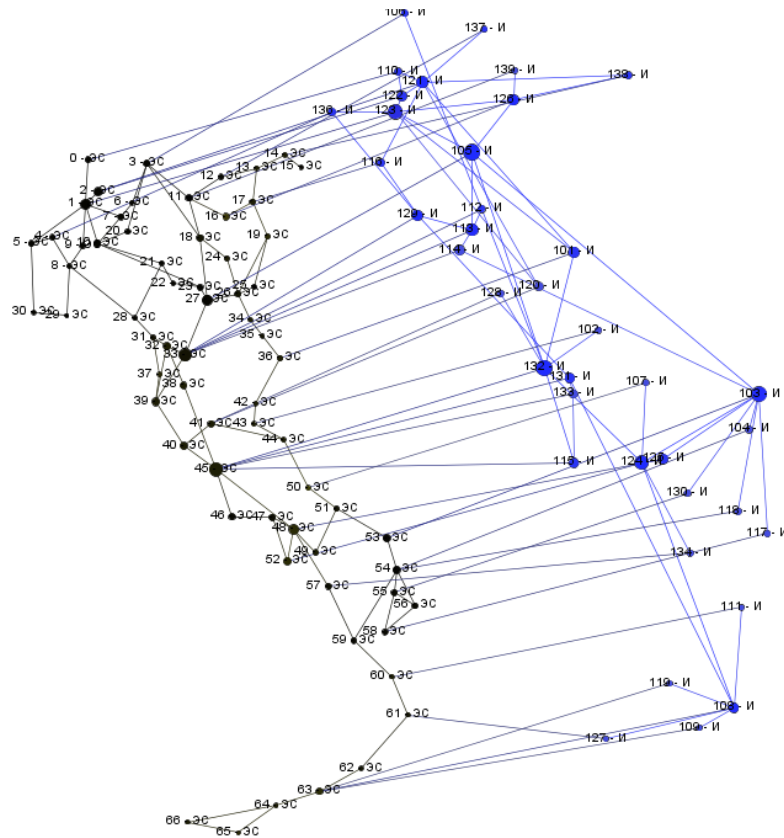


Fig. 2. Two-layer electricity (nodes highlighted in black) and Internet (nodes highlighted in blue) network in Italy

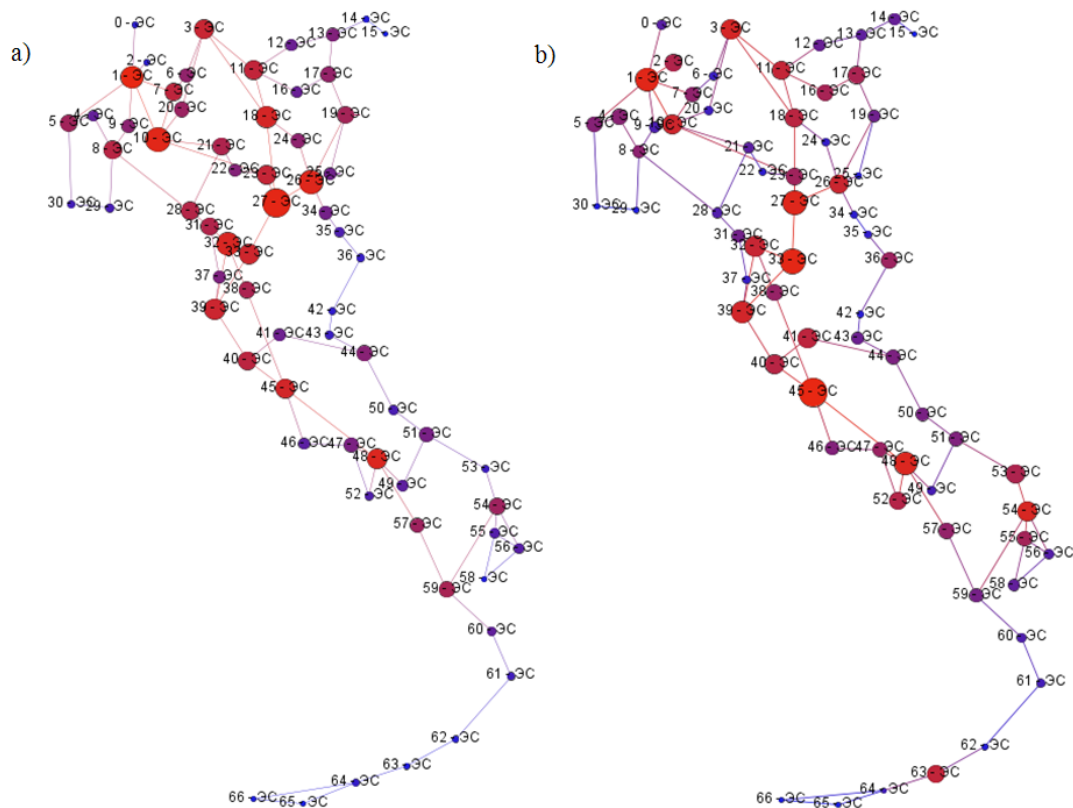


Fig.3. Ranking of the nodes of the electrical power supply network in Italy: a - without regard to communication with the Internet nodes, b - taking into account communication with the Internet node

TABLE II. RANKS OF THE NODES OF THE ELECTRICAL NETWORK IN ITALY (TAKING INTO ACCOUNT THE LINKS WITH THE INTERNET NODES)

Rank	The number of the node with the corresponding rank according to the centrality measures:							
	Degree	Closeness centrality	Harmonious centrality	Betweenness centrality	HITS	PageRank	Eigenvector centrality	Average for all measures
6 7	45	45	45	27	45	45	45	45
6 6	33	27	27	1	33	1	33	33
6 5	1	33	33	45	27	33	27	27
6 4	48	40	1	33	48	54	1	1
6 3	54	1	48	63	1	48	48	48
6 2	27	48	40	26	52	63	2	39
6 1	3	41	3	48	2	26	52	32
6 0	26	53	41	3	39	3	39	3
5 9	11	3	39	64	32	10	54	41
5 8	10	39	53	10	40	11	32	54
5 7	63	32	32	23	53	27	53	10
5 6	41	38	18	32	54	8	40	40
5 5	39	23	26	41	38	17	10	53
5 4	32	52	10	54	47	55	18	26
5 3	18	18	52	36	46	41	38	18

VII. CONCLUSIONS

. The study of critical infrastructures is one of the most important tasks of ensuring national security. At the same time, the energy sector is the most important among all sectors of critical infrastructure. The main objective of this study is to identify the most important objects of critical infrastructures. The main disadvantage of existing methods in Russia for identifying critical facilities is the lack of consideration of the relationship between critical infrastructure facilities both within sectors and between them. To overcome this drawback, it is proposed to use the apparatus of the theory of complex networks. Consideration of interconnections of objects (including those between infrastructure sectors) can be taken using the apparatus of multilayer networks. As a formal indicator of the importance of objects, one can use the centrality measures of network nodes. Thus, the theory of complex networks in combination with other methods can be effectively applied to analyze the

interconnections of objects of critical infrastructures and identify critical ones among them.

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