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Concomitants of Order Statistics and Record Values from Generalization of FGM Bivariate-Generalized Exponential Distribution

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ABSTRACT

We introduce the generalized Farlie–Gumbel–Morgenstern (FGM) type bivariate-generalized exponential distribution. Some distributional properties of concomitants of order statistics as well as record values for this family are studied. Recurrence relations between the moments of concomitants are obtained, some of these recurrence relations were not publishes before for Morgenstern type bivariate distributions. Moreover, most of the paper results are extended to arbitrary distributions.

1. INTRODUCTION

Let (X, Y) be a bivariate absolutely continuous random variable (rv), formally defined by the distribution function (df)

$$F_{X,Y}(x,y) = F_X(x) F_Y(y)\left\{1 + \lambda A(F_x(x)) B(F_y(y))\right\}, \tag{1}$$

where, $F_X(x)$ and $F_Y(y)$ are the marginals df's of X and Y, respectively. Moreover, the two kernel $A(x) \to 0$ and $B(y) \to 0$, as $x \to 1$ and $y \to 1$, satisfy certain regularity conditions ensuring that $F_{X,Y}(x,y)$ is a df with absolutely continuous marginals $F_X(x)$ and $F_Y(y)$. The model (1) was originally introduced by [1] for $A(x) = 1-x$ and $B(y) = 1-y$ and investigated by [2] for exponential marginals. Subsequent generalizations for this model is due to Farlie (1960) [3–6]. The successive generalizations of this model aims generally to enlarge the range of its correlation. If $A(x) = 1 - F_X(x)$ and $B(y) = 1 - F_Y(y)$ in model (1) then we have the classical Farlie–Gumbel–Morgenstern (FGM) for arbitrary continuous marginals $F_X(x)$ and $F_Y(y)$ with df

$$F_{X,Y}(x,y) = F_X(x) F_Y(y)\left\{1 + \lambda(1 - F_X(x))(1 - F_Y(y))\right\}, \ -1 \le \lambda \le 1. \tag{2}$$

The extension of the model (2), due to [4] and denoted by HK-FGM, has the df and probability density function (pdf)

$$F_{X,Y}(x,y) = F_X(x) F_Y(y)\left\{1 + \lambda\left(1 - F_X^p(x)\right)\left(1 - F_Y^p(y)\right)\right\}, \ -1 \le \lambda \le 1, \ p \ge 1 \tag{3}$$

and

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)\left\{1 + \lambda\left((1+p) F_X(x) - 1\right)\left((1+p) F_Y(y) - 1\right)\right\}, \ -1 \le \lambda \le 1, \ p \ge 1, \tag{4}$$

respectively, where $f_X(x)$ and $f_Y(y)$ are the pdf's of the rv's X and Y respectively. The admissible range of the associated parameter λ is $-\max(1,p)^{-2} \le \lambda \le p^{-1}$, and since $p \ge 1$, this admissible rang is $p^{-2} \le \lambda \le p^{-1}$.

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Remark 1.1. It is worth mentioning that, under the conditions $p > 0$ and $-\max\left(1, p\right)^{-2} \leq \lambda \leq p^{-1}$, the model (4) is a bona fide (i.e., $F_{X,Y}\left(x, y\right)$ is a genuine bivariate df. However, when $0 < p < 1$ the model (4) becomes very poor and is not allowing any improvement of the positive correlation compared to the classical FGM model (see, [6]). Therefore most of authors who tackle the model (4) postulate the condition $p > 1$.

[7] studied some properties of the model (3) and (4)) for bivariate-generalized exponential (GE) distribution (denoted by MTBGED). Also, they studied some distributional properties of concomitants of order statistics as well as record values of this model. Moreover, they obtained some recurrence relations between moments of concomitants of order statistics. Recently, [8] extended all the results of [7] to HK-FGM family for bivariate-GE distribution (denoted by HK-FGMGE). Moreover, some new results, which were not be obtained by [7], for FGM family, were given. Finally, [8] studied the asymptotic behavior of the concomitants of order statistics and made some corrections of [7].

[9] proposed a new generalization of FGM model (3), with marginals $F_U(u) = u$ and $F_V(v) = v$, $0 \leq u, v \leq 1$, by

$$F_{U,V}(u, v) = uv[1 + \lambda(1 - u^p)(1 - v^p)]^m, \ -1 \leq \lambda \leq 1, \ p > 0, \tag{5}$$

where the admissible range of the parameter λ is $-\min\left(1, \frac{1}{mp^2}\right) \leq \lambda \leq \frac{1}{mp}$. The Spearman's correlation coefficient of the model (5) is

$$\rho^\star = 12\sum_{j=1}^{n} \lambda^j \binom{n}{j} \left[\frac{\Gamma(j+1)\Gamma\left(\frac{2}{p}\right)}{p\Gamma\left(j+1+\frac{2}{p}\right)}\right]^2. \tag{6}$$

Clearly, if put $m = p = 1$ in (6) then we get $\rho^\star$ for the model (2), while if put $m = 1$ then we get $\rho^\star$ for the model (3).

A *rv* X is a two-parameter GE if it has the df

$$F_X(x) = (1 - \exp(-\theta x))^\alpha, \ x > 0; \ \theta > 0, \ \alpha > 0, \tag{7}$$

and the pdf

$$f_X(x) = \alpha\theta(1 - \exp(-\theta x))^{\alpha-1} \exp\left(-\theta x\right). \tag{8}$$

This distribution is a generalization of the exponential distribution and is more flexible, for being that, the hazard function of the exponential distribution is constant, but the hazard function of GE distribution can be constant, increasing or decreasing. [10] showed that the kth moment of $GE(\theta; \alpha)$ is

$$\mu_k(\theta, \alpha) = \frac{\alpha k!}{\theta^k} \sum_{i=0}^{\aleph(\alpha-1)} \frac{(-1)^i}{(i+1)^{k+1}} \binom{\alpha-1}{i},$$

where $\aleph(x) = \infty$, if x is non-integer and $\aleph(x) = x$, if x is integer. Moreover, the mean, variance and moment generating function of $GE(\theta; \alpha)$ are given, respectively, by

$$\mu_1(\theta, \alpha) = \mathrm{E}(X) = \frac{B(\alpha)}{\theta}, \ \mathrm{Var}(X) = \frac{C(\alpha)}{\theta^2} \text{ and } M_X(t) = \alpha\beta\left(\alpha, 1 - \frac{t}{\theta}\right), \tag{9}$$

where $B(\alpha) = \Psi(\alpha+1) - \Psi(1)$, $C(\alpha) = \Psi'(1) - \Psi'(\alpha+1)$, $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Psi(.)$ is the digamma function, while $\Psi'(.)$ is its derivation (the trigamma function).

In this paper we studied some properties of the model (5) for bivariate-GE distribution (denoted by GFGM-GE). Also, we studied some distributional properties of concomitants of order statistics as well as record values of this model. Moreover, some recurrence relations between moments of concomitants of order statistics are obtained. It is more suitable for achievement our aim to put the model (5) in the following form (by using binomial expansion):

$$F_{X,Y}\left(x, y\right) = F_X(x) F_Y\left(y\right) \sum_{i=0}^{m} \lambda^i \binom{m}{i} (1 - F_X^p(x))^i \ (1 - F_Y^p(y))^i. \tag{10}$$

In this case the pdf of the model (1.10) is given by

$$\begin{aligned} f_{X,Y}\left(x, y\right) &= f_X(x) f_Y\left(y\right)\left[1 + \sum_{i=1}^{m} \lambda^i \binom{m}{i} (1 - F_X^p(x))^{i-1} \left(1 - \left(1 + ip\right) F_X^p(x)\right)\right. \\ &\left. .(1 - F_Y^p(y))^{i-1} \left(1 - \left(1 + ip\right) F_Y^p\left(y\right)\right)\right]. \end{aligned}$$

2. SOME PROPERTIES OF GFGM-GE $(\theta_1, \alpha_1; \theta_2, \alpha_2)$

In this section we determined the correlation coefficient the model GFGM-GE $(\theta_1, \alpha_1; \theta_2, \alpha_2)$. By using the Hoeffding formula (see [11]), we get

$$\begin{aligned} \mathrm{COV}\left(X, Y : \lambda; \alpha_1, \alpha_2; p; m\right) &= \int_0^\infty \int_0^\infty \left[F_{X,Y}\left(x, y\right) - F_X\left(x\right) F_Y\left(y\right)\right] dxdy \\ &= \sum_{i=1}^{m} \lambda^i \binom{m}{i} \int_0^\infty \int_0^\infty F_X\left(x\right) F_Y\left(y\right) (1 - F_X^p(x))^i (1 - F_Y^p(y))^i dxdy \\ &= \frac{1}{\theta_1 \theta_2} \sum_{i=1}^{m} \lambda^i \binom{m}{i} I_1 I_2, \end{aligned} \tag{11}$$

where

$$\begin{aligned} I_t &= \int_0^1 \xi^{\alpha_t} (1 - \xi^{\alpha_t p})^i \frac{1}{1 - \xi} d\xi = \sum_{j=0}^{\infty} \int_0^1 \xi^{\alpha_t + j} (1 - \xi^{\alpha_t p})^i d\xi \\ &= \frac{1}{\alpha_t p} \sum_{j=0}^{\infty} \int_0^1 \xi^{\frac{\alpha_t + j + 1}{\alpha_t p} - 1} (1 - \xi)^i d\xi = \frac{1}{\alpha_t p} \sum_{j=1}^{\infty} \beta\left(\frac{\alpha_t + j + 1}{\alpha_t p}, i + 1\right), \ t = 1, 2 \end{aligned}$$

and $\beta\left(a, b\right)$ is the usual Beta function. Therefore, the correlation coefficient of the model GFGM-GE $(\theta_1, \alpha_1; \theta_2, \alpha_2))$ is given by

$$\rho\left(X, Y : \lambda; \alpha_1, \alpha_2; p; m\right) = \frac{1}{p^2} \sum_{i=1}^{m} \lambda^i \binom{m}{i} \prod_{t=1}^{2} \frac{1}{\sqrt{C\left(\alpha_t\right)} \alpha_t} \sum_{j=1}^{\infty} \beta\left(\frac{\alpha_t + j + 1}{\alpha_t p}, i + 1\right). \tag{12}$$

Remark 2.1. For $m = 1$, the correlation coefficient (12) reduces to

$$\rho\left(X, Y : \lambda; \alpha_1, \alpha_2; p; 1\right) = \lambda \frac{D\left(\alpha_1, p\right) D\left(\alpha_2, p\right)}{\sqrt{C\left(\alpha_1\right) C\left(\alpha_2\right)}},$$

where $D\left(\alpha_i, p\right) = B\left(\alpha_i\left(1 + p\right)\right) - B\left(\alpha_i\right),\ i = 1, 2$. The preceding formula was derived by [8].

The following theorem gives some interesting properties of the model GFGM-GE $(\theta_1, \alpha_1; \theta_2, \alpha_2)$.

Theorem 2.1. Let $-\min\left\{1, \frac{1}{mp^2}\right\} \leq \lambda \leq \frac{1}{mp}$. Then,

$$\rho\left(X, Y : \lambda; \alpha_1, \alpha_2; p; m\right) > \rho\left(X, Y : \lambda; \alpha_1, \alpha_2; p, 1\right), \forall m > 1. \tag{13}$$

Moreover,

$$\lim_{\substack{\alpha_1 \to 0 \\ \alpha_2 \to 0}} \rho\left(X, Y : \lambda; \alpha_1, \alpha_2; p; m\right) = 0$$

and

$$\begin{aligned} \lim_{\substack{\alpha_1 \to \infty \\ \alpha_2 \to \infty}} \rho\left(X, Y : \lambda; \alpha_1, \alpha_2; p; m\right) &= \rho\left(X, Y : \lambda; p; m\right) \\ &= \frac{6}{\pi^2} \sum_{i=1}^{m} \lambda^i \binom{m}{i} \left(\int_0^1 (1 - z^p)^{2i} \frac{1}{\log z} dz\right)^2. \end{aligned} \tag{14}$$

Finally, a more accessible formula for $\rho\left(X, Y : \lambda; p; m\right)$ is given by

$$\rho\left(X, Y : \lambda; p; m\right) = \frac{6}{\pi^2} \sum_{i=1}^{m} \lambda^i \binom{m}{i} \left(\sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} \log(1 + pj)\right)^2. \tag{15}$$

Proof of Theorem 2.1. The proof of the relation (13) follows immediately from the fact that the function $\beta\left(x,y\right)$ is non-increasing in both x and y, and $i \geq 1$. In order to prove the relation (14), we start with the relation (11), where

$$I_t = \int_0^1 \xi^{\alpha_t}\left(1-\xi^{\alpha_t p}\right)^i \frac{1}{1-\xi} d\xi = \frac{1}{\alpha_t}\int_0^1 (1-z^p)^i \frac{z^{\frac{1}{\alpha_t}}}{1-z^{\frac{1}{\alpha_t}}} dz \tag{16}$$

(by using the transformation $z=\xi^{\alpha_t}$). On the other hand, for any $0<z<1$, we have

$$\lim_{\alpha_t\to\infty} \frac{z^{\frac{1}{\alpha_t}}}{\alpha_t\left(1-z^{\frac{1}{\alpha_t}}\right)} = \lim_{\theta\to 0}\frac{\theta z^{\theta}}{1-z^{\theta}} = -\frac{1}{\log z} \tag{17}$$

and

$$\lim_{\alpha_t\to\infty} C\left(\alpha_t\right) = \frac{\pi^2}{6}. \tag{18}$$

Combining (16), (17) and (18), we get the relation (14). By using a result of [12] we get

$$\int_0^1 (z^p-1)^i \frac{1}{\log z} dz = \sum_{j=1}^{i}(-1)^{i-j}\binom{i}{j}\log\left(1+pj\right),$$

which immediately proves (15).

Remark 2.2. From (15) with $m=1$, we have

$$\rho\left(X,Y:\lambda;p;1\right) = \frac{6}{\pi^2}\log^2\left(1+p\right),$$

which coincides with the result of [8].

In Table 1 , we give some values of the correlation coefficient $\rho_{\lambda_{max}} = \rho\left(X,Y:\lambda_{max};p;m\right)$ defined in (15), where $\lambda_{max}=\frac{1}{mp}$. The result of this table shows that, when $0<p<1$, the model becomes very poor and is not allowing any improvement of the positive correlation compared to the HK-FGM model or even the classical model FGM. Therefore, for the practical purposes, we always take $p>1$ (see also Remark 1.1). Moreover, Table 1 shows that the values of $\rho_{\lambda_{max}}$ are irregularly fluctuated with changing the values of the parameters p and m. However the maximum value of $\rho_{\lambda_{max}}$, which is 0.4214, reaches at $m=25$ and $p=4.5$. This maximum value is a significant improvement comparing with the upper bound “0.2921”obtained by [7] for the model MTBGED and is also a satisfactory improvement comparing with the upper pound “0.3937”obtained by [8] for the model HK-FGMGE. This fact gives a satisfactory motivation to deal with GFGM-GE rather than MTBGED and HK-FGMGE.

Table 1 Some different values of the correlation coefficient for the family GFGM-GE.

m	p	$\lambda_{\max}$	$\rho_{\lambda_{\max}}$	m	p	$\lambda_{\max}$	$\rho_{\lambda_{\max}}$
1	0.01	100	0.006	1	3	0.33333	0.389
3	0.01	33.333	0.0062	50	3	0.0067	0.42
1	0.1	10	0.055	1	4	0.25	0.394
25	0.1	0.4	0.057	50	4	0.005	0.4210
1	1	1	0.29	25	4.5	0.0089	0.4214
50	1	0.02	0.32	1	5	0.2	0.399
100	1	0.01	0.33	25	5	0.008	0.411
3	1.5	0.2222	0.3513	50	5	0.004	0.417
1	2	0.5	0.366	1	6	0.16667	0.3836
3	2	0.16667	0.3849	50	6	0.0033	0.3983
50	2	0.01	0.399	500	6	0.00033	0.3988
100	2	0.005	0.3999	1000	6	0.000166	0.3998

3. CONCOMITANTS OF ORDER STATISTICS BASED ON GFGM-GE

In the last two decades much attention has been paid to the concomitants of order statistics models, see, e.g. [7,8,13,14]. The importance of these models increased owing to the rigorous demand of natural science, social science and economics to study the problems which generally depend on two different dependent characteristics. Let $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ be a random sample from a bivariate df $F_{X,Y}(x, y)$. If we arrange the X—variate in ascending order as $X_{1:n} \le X_{2:n} \le .... \le X_{n:n}$, then, the Y—variate paired with these order statistics are denoted by $Y_{[1:n]}, Y_{[2:n]}, ..., Y_{[n:n]}$ and termed the concomitants of order statistics. The concept of concomitants of order statistics was first introduced by [15] and almost simultaneously under the name of induced order statistics by [16]. These concomitant order statistics are of interest in selection and prediction problems based on the ranks of the X's. Another application of concomitants of order statistics is in ranked-set sampling. It is a sampling scheme for situations where measurement of the variable of primary interest for sampled items is expensive or time-consuming while ranking of a set of items related to the variable of interest can be easily done. A comprehensive review of ranked-set sampling can be found in [17]. For a recent comprehensive review of possible applications of the concomitants of order statistics, see [18].

Let $X \sim GE(\theta_1; \alpha_1)$ and $Y \sim GE(\theta_2; \alpha_2)$. Since the conditional pdf of $Y_{[r:n]}$ given $X_{[r:n]} = x$ is $f_{Y_{[r:n]}|X_{r:n}}(y|x) = f_{Y|X}(y|x)$, then the pdf of $Y_{[r:n]}$ is given by

$$f_{[r:n]}(y) = \int_0^\infty f_{Y|X}(y|x) f_{r:n}(x)\, dx, \tag{19}$$

where $f_{r:n}(x) = \dfrac{1}{\beta(r, n-r+1)} F_X^{r-1}(x)(1 - F_X(x))^{n-r} f_X(x)$ is the pdf of the rth order statistic $X_{r:n}$ and $f_{Y|X}(y|x)$ can be computed by using (7), (8) and (10). The following theorem gives the useful representation of the pdf $f_{[r:n]}(y)$.

Theorem 3.1. Let $U_j \sim GE(\theta_2, \alpha_2(jp+1))$ and $V_i \sim GE(\theta_2, \alpha_2((j+1)p+1))$. Then

$$f_{[r:n]}(y) = f_Y(y) + \sum_{i=1}^{m} \lambda^i \binom{m}{i} \mathcal{S}_{r,n}^{(t)}(p, i)$$

$$\times \sum_{j=0}^{\aleph(i-1)} \binom{i-1}{j} (-1)^j \left\{ \frac{1}{jp+1} f_{U_j}(y) - \frac{1+pi}{(j+1)p+1} f_{V_j}(y) \right\}, t = 1, 2,$$

where

$$\mathcal{S}_{r,n}^{(1)}(p, i) = \frac{1}{p} \sum_{l=0}^{n-r} \binom{n-r}{l} (-1)^l \Delta_{l:r,n}^{\star}(p, i),$$

$$\Delta_{l:r,n}^{\star}(p, i) = \frac{\beta\left(\frac{r+l}{p}, i\right) - (1+pi)\,\beta\left(\frac{r+l}{p}+1, i\right)}{\beta(r, n-r+1)},$$

$$\mathcal{S}_{r,n}^{(2)}(p, i) = \sum_{l=0}^{\aleph(i-1)} \binom{i-1}{l} (-1)^l \Delta_{l:r,n}^{\star\star}(p, i)$$

and

$$\Delta_{l:r,n}^{\star\star}(p, i) = \frac{\beta(lp+r, n-r+1) - (1+pi)\,\beta((l+1)p+r, n-r+1)}{\beta(r, n-r+1)}.$$

Proof. Clearly, the relation (19), can be written in the form

$$f_{[r:n]}(y) = f_Y(y) \left[1 + \sum_{i=1}^{m} \lambda^i \binom{m}{i} (1 - F_Y^p(y))^{i-1} (1 - (1+pi) F_Y^p(y)) (J_1 - (1+pi) J_2) \right]$$

$$= f_Y(y) + \sum_{i=1}^{m} \lambda^i \binom{m}{i} (J_1 - (1+pi) J_2) \sum_{j=0}^{\aleph(i-1)} \binom{i-1}{j} (-1)^j \left\{ \frac{f_{U_j}(y)}{jp+1} - \frac{(1+pi) f_{V_j}(y)}{(j+1)p+1} \right\},$$

where

$$
\begin{aligned}
J_1\beta(r,n-r+1) &= \int_0^\infty \left(1-F_X^p(x)\right)^{i-1} F_X^{r-1}(x)(1-F_X(x))^{n-r} f_X(x)\,dx \\
&= \sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \int_0^\infty (1-F_X^p(x))^{i-1}F_X^{\ell+r-1}(x) f_X(x)\,dx \\
&= \sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \alpha_1\theta_1 \int_0^\infty (1-(1-e^{-\theta_1 x})^{\alpha_1 p})^{i-1(1-e^{-\theta_1 x})^{\alpha_1(\ell+r)-1}} e^{-\theta_1 x}dx \\
&= \sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \alpha_1 \int_0^1 (1-\xi^{\alpha_1 p})^{i-1}\xi^{\alpha_1(\ell+r)-1}d\xi \\
&= \frac{1}{p}\sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \int_0^1 t^{\frac{\ell+r}{p}-1}(1-t)^{i-1}dt = \frac{1}{p}\sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \beta\left(\frac{r+\ell}{p}, i\right)
\end{aligned} \tag{20}
$$

(by using the substitute $\xi = 1 - e^{-\theta_1 x}$ and then use the substitution $t = \xi^{\alpha_1 p}$) and

$$
\begin{aligned}
J_2\beta(r,n-r+1) &= \int_0^\infty \left(1-F_Y^p(x)\right)^{i-1} F_X^{p+r-1}(x)(1-F_X(x))^{n-r} f_X(x)\,dx \\
&= \sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \int_0^\infty (1-F_X^p(x))^{i-1}F_X^{\ell+p+r-1}(x) f_X(x)\,dx \\
&= \sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \alpha_1\theta_1 \int_0^\infty (1-(1-e^{-\theta_1 x})^{\alpha_1 p})^{i-1}(1-e^{-\theta_1 x})^{\alpha_1(\ell+p+r)-1} e^{-\theta_1 x}dx \\
&= \frac{1}{p}\sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \alpha_1 \int_0^1 (1-\xi^{\alpha_1 p})^{i-1}\xi^{\alpha_1(\ell+p+r)-1}d\xi = \frac{1}{p}\sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \beta\left(\frac{r+\ell}{p}+1, i\right)
\end{aligned} \tag{21}
$$

(by using the substitute $\xi = 1 - e^{-\theta_1 x}$ and then use the substitution $t = \xi^{\alpha_1 p}$). Therefore, by combining (20) and (21), we get

$$
\mathcal{S}_{r,n}^{(1)}(p,i) = J_1 - (1+pi)J_2 = \frac{1}{p}\sum_{\ell=0}^{n-r}\binom{n-r}{\ell}(-1)^\ell \Delta^\star_{\ell:r,n}(p,i),
$$

where

$$
\Delta^\star_{\ell:r,n}(p,i) = \frac{\beta\left(\frac{r+\ell}{p}, i\right) - (1+pi)\beta\left(\frac{r+\ell}{p}+1, i\right)}{\beta(r,n-r+1)}.
$$

On the other hand,

$$
\begin{aligned}
J_1\beta(r,n-r+1) &= \int_0^\infty \left(1-F_X^p(x)\right)^{i-1} F_X^{r-1}(x)(1-F_X(x))^{n-r} f_X(x)\,dx \\
&= \alpha_1\theta_1 \int_0^\infty (1-(1-e^{-\theta_1 x})^{\alpha_1 p})^{i-1}(1-e^{-\theta_1 x})^{\alpha_1 r-1}(1-(1-e^{-\theta_1 x})^{\alpha_1})^{n-r}e^{-\theta_1 x}dx.
\end{aligned}
$$

Upon substituting $\xi = 1 - e^{-\theta_1 x}$, we get

$$
J_1\beta(r,n-r+1) = \alpha_1 \int_0^1 (1-\xi^{\alpha_1 p})^{i-1}\xi^{\alpha_1 r-1}(1-\xi^{\alpha_1})^{n-r}d\xi.
$$

Moreover, by using the substitution $u = \xi^{\alpha_1}$, we get

$$
J_1\beta(r,n-r+1) = \frac{1}{p}\int_0^1 u^{r-1}(1-u)^{n-r}(1-u^p)^{i-1}du,
$$

after simple calculations, we get

$$J_1\beta\left(r,n-r+1\right)=\sum_{\ell=0}^{\aleph(i-1)}\left(\ell^{i-1}\right)(-1)^{\ell}\beta\left(\ell p+r,n-r+1\right). \tag{22}$$

Similarly

$$J_2\beta\left(r,n-r+1\right)=\sum_{\ell=0}^{\aleph(i-1)}\left(\ell^{i-1}\right)(-1)^{\ell}\beta\left((\ell+1)p+r,n-r+1\right). \tag{23}$$

Therefore, by combining (22) and (23), we get

$$\mathcal{S}^{(2)}_{r,n}\left(p,i\right)=\left(J_1-\left(1+pi\right)J_2\right)=\sum_{j=0}^{\aleph(i-1)}\binom{i-1}{\ell}(-1)^{\ell}\Delta^{\star\star}_{\ell:r,n}\left(p,i\right),$$

where

$$\Delta^{\star\star}_{\ell:r,n}\left(p,i\right)=\frac{\beta\left(\ell p+r,n-r+1\right)-\left(1+pi\right)\beta\left((\ell+1)p+r,n-r+1\right)}{\beta\left(r,n-r+1\right)}.$$

This completes the proof of the theorem. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.1. Let $\mu^{(k)}_{[r:n]}=\mathrm{E}\left(Y^k_{[r:n]}\right)$, $k=1,2,\ldots$. Then,

$$\begin{aligned}\mu^{(k)}_{[r:n]}&=\mu_k\left(\theta_2,\alpha_2\right)+\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{S}^{(t)}_{r,n}\left(p,i\right)\mathcal{D}\left(k;p,i\right)\\&=\mu_k\left(\theta_2,\alpha_2\right)+\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{S}^{(t)}_{r,n}\left(p,i\right)\sum_{j=0}^{\aleph(i-1)}\binom{i-1}{j}(-1)^j\left\{\frac{\mathrm{E}\left(U^k_j\right)}{jp+1}-\frac{\left(1+pi\right)\mathrm{E}\left(V^k_j\right)}{\left(j+1\right)p+1}\right\},\ t=1,2,\end{aligned} \tag{24}$$

where, $\mathrm{E}\left(U^k_j\right)$ and $\mathrm{E}\left(V^k_j\right)$ can be easily computed by using the relation (9). Therefore, the mean $\mu_{[r:n]}=\mathrm{E}\left(Y_{[r:n]}\right)$ is given by

$$\mu_{[r:n]}=\frac{B\left(\alpha_2\right)}{\theta_2}+\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{S}^{(t)}_{r,n}\left(p,i\right)\mathcal{D}\left(1,p,i\right),t=1,2, \tag{25}$$

where

$$\mathcal{D}\left(1,p,i\right)=\frac{1}{\theta_2}\sum_{j=0}^{\aleph(i-1)}\binom{i-1}{j}(-1)^j\left[\frac{B\left(\alpha_2\left(jp+1\right)\right)}{jp+1}-\frac{\left(1+pi\right)B\left(\alpha_2\left(\left(j+1\right)p+1\right)\right)}{\left(j+1\right)p+1}\right].$$

Corollary 3.2. When $m=1$, we get $\mu_{[r:n]}$ for the family HK-FGM-GE, (see [8])

$$\mu_{[r:n]}=\frac{1}{\theta_2}\left[\left(1+\lambda\Delta^{\star\star}_{0:r,n}\left(p,1\right)\right)B\left(\alpha_2\right)-\lambda\Delta^{\star\star}_{0:r,n}\left(p,1\right)B\left(\alpha_2\left(p+1\right)\right)\right]. \tag{26}$$

Remark 3.1. It is worth mentioning that, if we replace $\mu_k\left(\theta_2,\alpha_2\right)$ by $\mathrm{E}\left(Y^k\right)$. Moreover, U_j and V_j in $\mathcal{D}\left(k;p,i\right)$ are taken to be such that $U_j\sim F^{jp+1}_Y\left(y\right)$ and $V_j\sim F^{(j+1)p+1}_Y\left(y\right)$, then the representation (24) holds for any two arbitrary distributions $F_X\left(x\right)$ and $F_Y\left(y\right)$.

Now, by using the two representations in relation (24), as well as (25), at $t=1$ and $t=2$, we can derive some useful recurrence relations satisfied by the moments $\mu^{(k)}_{[r:n]}$, $k=1,2,\ldots$. The following theorem give a new recurrence relation by using the representation at $t=1$. It is worth mentioning that this recurrence relation was not proved even for the model FGM-GE. Moreover, in view of Remark 3.1, all the next recurrence relations are satisfied for arbitrary distributions $F_X\left(x\right)$ and $F_Y\left(y\right)$, if only we would consider the obvious changes illustrated in Remark 3.1.

Theorem 3.2. Let p be an integer and $k = 1, 2, ...,$ then

$$\frac{1}{Q^{(2)}_{r,n}(p)}\mu^{(k)}_{[r+2p:n+2p]} + \frac{1}{Q^{(1)}_{r,n}(p)}\mu^{(k)}_{[r+p:n+p]} - 2\mu^{(k)}_{[r:n]} =$$

$$\left(\frac{1}{Q^{(2)}_{r,n}(p)} + \frac{1}{Q^{(1)}_{r,n}(p)} - 2\right)\mu_k(\theta_2, \alpha_2) - \frac{1}{p}\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{D}(k,p,i)\sum_{l=0}^{n-r}\binom{n-r}{l}(-1)^l\eta_{l:r,n}(p,i),$$

where

$$Q^{(j)}_{r,n}(p) = \frac{\Gamma(r)\Gamma(n+jp+1)}{\Gamma(r+jp)\Gamma(n+1)}, j = 1, 2,$$

and

$$\eta_{l:r,n}(p,i) = \left(3 - \frac{p(1+i)}{r+p(1+i)+l}\right)\frac{\beta\left(\frac{r+l}{p}, i+1\right)}{\beta(r, n-r+1)}$$
$$- (1+pi)\left(3 - \frac{p(1+i)}{r+p(2+i)+l}\right)\frac{\beta\left(\frac{r+l}{p}+1, i+1\right)}{\beta(r, n-r+1)}.$$

Proof. Starting with $\Delta^{\star}_{l:r,n}(p,i)$, after simple calculations, we can show that

$$\Delta^{\star}_{l:r+p,n+p}(p,i) = \frac{\beta\left(\frac{r+l}{p}+1, i\right)}{\beta(r+p, n-r+1)} - (1+pi)\frac{\beta\left(\frac{r+l}{p}+2, i\right)}{\beta(r+p, n-r+1)}$$
$$= \frac{(n+p)!(r-1)!}{(r+p-1)!(n!)}\left[\frac{\frac{r+l}{p}}{\frac{r+l}{p}+i}\frac{\beta\left(\frac{r+l}{p}, i\right)}{\beta(r, n-r+1)} - (1+pi)\frac{\frac{r+l}{p}+1}{\frac{r+l}{p}+i+1}\frac{\beta\left(\frac{r+l}{p}+1, i\right)}{\beta(r, n-r+1)}\right].$$

On the other hand, since $Q^{(1)}_{r,n}(p) = \dfrac{(n+p)!(r-1)!}{(r+p-1)!(n!)}$, we get $\Delta^{\star}_{l:r+p,n+p}(p,i) = Q^{(1)}_{r,n}(p)\left[\Delta^{\star}_{l:r,n}(p,i) - \xi_1\right]$, where

$$\xi_1 = \frac{\beta\left(\frac{r+l}{p}, i+1\right) - (1+pi)\beta\left(\frac{r+l}{p}+1, i+1\right)}{\beta(r, n-r+1)}.$$

Therefore,

$$\frac{1}{Q^{(1)}_{r,n}(p)}\Delta^{\star}_{l:r+p,n+p}(p,i) - \Delta^{\star}_{l:r,n}(p,i) = -\xi_1. \tag{27}$$

Similarly, after some calculations, we get

$$\Delta^{\star}_{l:r+2p,n+2p}(p,i) = \frac{\beta\left(\frac{r+l}{p}+2, i\right)}{\beta(r+2p, n-r+1)} - (1+pi)\frac{\beta\left(\frac{r+l}{p}+3, i\right)}{\beta(r+2p, n-r+1)}$$
$$= \frac{(n+2p)!(r-1)!}{(r+2p-1)!(n!)}\left[\frac{\frac{r+l}{p}\left(\frac{r+l}{p}+1\right)}{\frac{r+l}{p}+i\left(\frac{r+l}{p}+i+1\right)}\frac{\beta\left(\frac{r+l}{p}, i\right)}{\beta(r, n-r+1)}\right.$$
$$\left. - (1+pi)\frac{\left(\frac{r+l}{p}+1\right)\left(\frac{r+l}{p}+2\right)}{\left(\frac{r+l}{p}+i+1\right)\left(\frac{r+l}{p}+i+2\right)}\frac{\beta\left(\frac{r+l}{p}+1, i\right)}{\beta(r, n-r+1)}\right].$$

Thus, we get

$$\Delta^{\star}_{l:r+2p,n+2p}\left(p,i\right)=Q^{(2)}_{r,n}\left(p\right)\left[\Delta^{\star}_{l:r,n}\left(p,i\right)-\xi_2\right], \tag{28}$$

where

$$\xi_2=\xi_1+\frac{\frac{l+r}{p}\beta\left(\frac{r+l}{p},i+1\right)}{\left(\frac{r+l}{p}+i+1\right)\beta\left(r,n-r+1\right)}-\frac{\left(1+pi\right)\left(1+\frac{l+r}{p}\right)\beta\left(\frac{r+l}{p}+1,i+1\right)}{\left(\frac{l+r}{p}+i+2\right)\beta\left(r,n-r+1\right)}.$$

Therefore,

$$\frac{1}{Q^{(2)}_{r,n}\left(p\right)}\Delta^{\star}_{j:r+2p,n+2p}\left(p,i\right)-\Delta^{\star}_{l:r,n}\left(p,i\right)=-\xi_2. \tag{29}$$

Put $x=\dfrac{r+l}{p}$, we can write

$$\begin{aligned}\xi_2&=\frac{i\left(2x+i+1\right)}{\left(x+i\right)\left(x+i+1\right)}\frac{\beta\left(x,i\right)}{\beta\left(r,n-r+1\right)}-\left(1+pi\right)\frac{i\left(2x+i+3\right)}{\left(x+i+1\right)\left(x+i+2\right)}\frac{\beta\left(x+1,i\right)}{\beta\left(r,n-r+1\right)}\\&=\frac{\left(2x+i+1\right)}{\left(x+i+1\right)}\frac{\beta\left(x,i+1\right)}{\beta\left(r,n-r+1\right)}-\left(1+pi\right)\frac{\left(2x+i+3\right)}{\left(x+q+2\right)}\frac{\beta\left(x+1,q+1\right)}{\beta\left(r,n-r+1\right)}\\&=\xi_1+\frac{x}{\left(x+i+1\right)}\frac{\beta\left(x,i+1\right)}{\beta\left(r,n-r+1\right)}-\left(1+pi\right)\frac{\left(x+1\right)}{\left(x+i+2\right)}\frac{\beta\left(x+1,i+1\right)}{\beta\left(r,n-r+1\right)}.\end{aligned}$$

Therefore, we can easily show that

$$\xi_2+\xi_1=\left(2+\frac{x}{x+i+1}\right)\frac{\beta\left(x,i+1\right)}{\beta\left(r,n-r+1\right)}-\left(1+pi\right)\left(2+\frac{x+1}{x+i+2}\right)\frac{\beta\left(x+1,i+1\right)}{\beta\left(r,n-r+1\right)}=\eta_{l:r,n}\left(p,i\right).$$

Thus by combining this equality with (27), (28), (29) and (24), at $t=1$, the proof of the theorem follows immediately. □

Corollary 3.3. For $m=1$, we get for the family HK-FGM-GE, (see [8])

$$\begin{aligned}&\frac{1}{Q^{(2)}_{r,n}\left(p\right)}\mu^{(k)}_{[r+2p:n+2p]}+\frac{1}{Q^{(1)}_{r,n}\left(p\right)}\mu^{(k)}_{[r+p:n+p]}-2\mu^{(k)}_{[r:n]}=\\&\left(\frac{1}{Q^{(2)}_{r,n}\left(p\right)}+\frac{1}{Q^{(1)}_{r,n}\left(p\right)}-2\right)\mu_k\left(\theta_2,\alpha_2\right)-\frac{\lambda\mathcal{D}\left(k,p,1\right)}{p\beta\left(r,n-r+1\right)}\sum_{l=0}^{3}c_l\left(p\right)\beta\left(r+lp,n-r+1\right),\end{aligned} \tag{30}$$

where $c_0\left(p\right)=2p$, $c_1\left(p\right)=-p\left(2p+3\right)$, $c_2\left(p\right)=p^2$, $c_3\left(p\right)=p\left(1+p\right)$ and $\mathcal{D}\left(k,p,1\right)=\mu_k\left(\theta_2,\alpha_2\right)-\mu_k\left(\theta_2,\alpha_2\left(1+p\right)\right)$.
The following theorem, which is relying on the representation (27), at $t=2$, given some recurrence relations satisfied by the kth moments of concomitants of order statistics for any arbitrary distributions.

Theorem 3.3. For any $k=1,2,....$, we have

$$\frac{\mu^{(k)}_{[r+2:n]}-\mu^{(k)}_{[r:n]}}{\mu^{(k)}_{[r+1:n]}-\mu^{(k)}_{[r:n]}}=\frac{\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(1)}_{r,n}\left(p,i\right)}{\left(r+1\right)\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(2)}_{r,n}\left(p,i\right)} \tag{31}$$

and

$$\frac{\mu^{(k)}_{[r:n-2]}-\mu^{(k)}_{[r:n]}}{\mu^{(k)}_{[r:n-1]}-\mu^{(k)}_{[r:n]}}=\frac{\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(1)}_{r,n}\left(p,i\right)}{\left(n-1\right)\sum_{i=1}^{m}\lambda^i\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(2)}_{r,n}\left(p,i\right)}, \tag{32}$$

where

$$\Omega^{(1)}_{r,n}\left(p,i\right)=\sum_{l=0}^{\aleph(i-1)}\binom{i-1}{l}(-1)^{l}\left[lp\left(lp+2r+1\right)\beta\left(lp+r,n-r+1\right)\right.$$
$$\left.-\left(1+pi\right)(1+l)p\left((l+1)p+2r+1\right)\beta\left((l+1)p+r,n-r+1\right)\right],$$

and

$$\Omega^{(2)}_{r,n}\left(p,i\right)=\sum_{l=0}^{\aleph(i-1)}\binom{i-1}{l}(-1)^{l}\left[lp\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)(1+l)p\beta\left((l+1)p+r,n-r+1\right)\right].$$

Proof. It is easy to check that

$$\begin{aligned}\Delta^{\star\star}_{l:r+1,n}\left(p,i\right)&=\frac{\beta\left(lp+r+1,n-r\right)-\left(1+pi\right)\beta\left((l+1)p+r+1,n-r\right)}{\beta\left(r+1,n-r\right)}\\&=\frac{\frac{lp+r}{n-r}\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)\frac{(l+1)p_1+r}{n-r}\beta\left((l+1)p+r,n-r+1\right)}{\frac{r}{n-r}\beta\left(r,n-r+1\right)}\\&=\Delta^{\star\star}_{l:r,n}\left(p,i\right)+\frac{lp\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)(l+1)p\beta\left((l+1)p+r,n-r+1\right)}{r\beta\left(r,n-r+1\right)}.\end{aligned}$$

Therefore, we get

$$\mathcal{S}^{(2)}_{r+1,n}\left(p,i\right)-\mathcal{S}^{(2)}_{r,n}\left(p,i\right)=\frac{1}{r\beta\left(r,n-r+1\right)}$$

$$\times\sum_{l=0}^{\aleph(i-1)}\binom{i-1}{l}(-1)^{l}\left[lp\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)(1+l)p\beta\left((l+1)p+r,n-r+1\right)\right].$$

Moreover, we have

$$\begin{aligned}\Delta^{\star\star}_{l:r+2,n}\left(p,i\right)&=\frac{\beta\left(lp+r+2,n-r-1\right)-\left(1+pi\right)\beta\left((l+1)p+r+2,n-r-1\right)}{\beta\left(r+2,n-r-1\right)}.\\&=\frac{\frac{(lp+r)(lp+r+1)}{(n-r)(n-r-1)}\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)\frac{((l+1)p+r)((l+1)p+r+1)}{(n-r)(n-r-1)}\beta\left((l+1)p+r,n-r+1\right)}{\frac{r(r+1)}{(n-r)(n-r-1)}\beta\left(r,n-r+1\right)}.\end{aligned}$$

Thus, we get

$$\mathcal{S}^{(2)}_{r+2,n}\left(p,i\right)-\mathcal{S}^{(2)}_{r,n}\left(p,i\right)=\frac{\lambda}{r(r+1)\beta\left(r,n-r+1\right)}$$

$$\times\left[\sum_{l=0}^{\aleph(i-1)}\binom{i-1}{l}(-1)^{l}\left[lp\left(lp+2r+1\right)\beta\left(lp+r,n-r+1\right)\right.\right.$$

$$\left.\left.-\left(1+pi\right)(1+l)p\left((l+1)p+2r+1\right)\beta\left((l+1)p+r,n-r+1\right)\right]\right].$$

Therefore,

$$\frac{\mu^{(k)}_{[r+2:n]}-\mu^{(k)}_{[r:n]}}{\mu^{(k)}_{[r+1:n]}-\mu^{(k)}_{[r:n]}}=\frac{\sum_{i=1}^{m}\lambda^{i}\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(1)}_{r,n}\left(p,i\right)}{(r+1)\sum_{i=1}^{m}\lambda^{i}\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(2)}_{r,n}\left(p,i\right)}.$$

Similarly, we have

$$\Delta^{\star\star}_{l:r,n-1}\left(p,i\right)=\frac{\beta\left(lp+r,n-r\right)-\left(1+pi\right)\beta\left(\left(l+1\right)p+r,n-r\right)}{\beta\left(r,n-r\right)}.$$
$$=\frac{\frac{lp+n}{n-r}\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)\frac{(l+1)p+n}{n-r}\beta\left(\left(l+1\right)p+r,n-r+1\right)}{\frac{n}{n-r}\beta\left(r,n-r+1\right)}$$
$$=\Delta^{\star\star}_{l:r,n}\left(p,i\right)+\frac{lp\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)\left(l+1\right)p\beta\left(\left(l+1\right)p+r,n-r+1\right)}{n\beta\left(r,n-r+1\right)}.$$

Consequently, we get

$$\mathcal{S}^{(2)}_{r,n-1}\left(p,i\right)-\mathcal{S}^{(2)}_{r,n}\left(p,i\right)=\frac{\lambda}{n\beta\left(r,n-r+1\right)}$$

$$\times\sum_{l=0}^{\aleph(i-1)}\binom{i-1}{j}(-1)^{l}\left[lp\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)(1+l)p\beta\left(\left(l+1\right)p+r,n-r+1\right)\right].$$

Moreover,

$$\Delta^{\star\star}_{l:r,n-2}\left(p,i\right)=\frac{\beta\left(lp+r,n-r-1\right)-\left(1+pi\right)\beta\left(\left(l+1\right)p+r,n-r-1\right)}{\beta\left(r,n-r-1\right)}$$
$$=\frac{\frac{(lp+n)(lp+n-1)}{(n-r)(n-r-1)}\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)\frac{((l+1)p+n)((l+1)p+n-1)}{(n-r)(n-r-1)}\beta\left(\left(l+1\right)p+r,n-r+1\right)}{\frac{n(n-1)}{(n-r)(n-r-1)}\beta\left(r,n-r+1\right)}$$
$$=\Delta^{\star\star}_{l:r,n}\left(p,i\right)$$
$$+\frac{lp\left(lp+2n-1\right)\beta\left(lp+r,n-r+1\right)-\left(1+pi\right)(l+1)p\left((l+1)p+2n-1\right)\beta\left(\left(l+1\right)p+r,n-r+1\right)}{n\left(n-1\right)\beta\left(r,n-r+1\right)}.$$

Thus,

$$\mathcal{S}^{(2)}_{r+2,n}\left(p,i\right)-\mathcal{S}^{(2)}_{r,n}\left(p,i\right)=\frac{\lambda}{n\left(n-1\right)\beta\left(r,n-r+1\right)}$$

$$\times\left[\sum_{l=0}^{\aleph(i-1)}\binom{i-1}{j}(-1)^{l}\left[lp\left(lp+2n-1\right)\beta\left(lp+r,n-r+1\right)\right.\right.$$

$$\left.\left.-\left(1+pi\right)(1+l)p\left((l+1)p+2n-1\right)\beta\left(\left(l+1\right)p+r,n-r+1\right)\right]\right].$$

Therefore,

$$\frac{\mu^{(k)}_{[r:n-2]}-\mu^{(k)}_{[r:n]}}{\mu^{(k)}_{[r:n-1]}-\mu^{(k)}_{[r:n]}}=\frac{\sum_{i=1}^{m}\lambda^{i}\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(1)}_{r,n}\left(p,i\right)}{(n-1)\sum_{i=1}^{m}\lambda^{i}\binom{m}{i}\mathcal{D}\left(k,p,i\right)\Omega^{(2)}_{r,n}\left(p,i\right)}.$$

This completes the proof of the theorem. □

Corollary 3.4. At $m=1$, we get for the HK-FGM-GE family (see [8])

$$(r+1)\,\mu^{(k)}_{[r+2:n]}=\left(2r+p+1\right)\mu^{(k)}_{[r+1:n]}-\left(r+p\right)\mu^{(k)}_{[r:n]}$$

and

$$\left(n+p\right)\mu^{(k)}_{[r:n]}=\left(2n+p-1\right)\mu^{(k)}_{[r:n-1]}-(n-1)\,\mu^{(k)}_{[r:n-2]}.$$

Theorem 3.4. For any $k = 1, 2, \ldots$, we have

$$\mu^{(k)}_{[r+2:n]} + \mu^{(k)}_{[r+1:n]} - 2\mu^{(k)}_{[r:n]} = \sum_{i=1}^{m} \lambda^i \binom{m}{i} \mathcal{D}\left(k, p, i\right) \Phi^{(1)}_{r,n}\left(p, i\right),$$

where

$$\Phi^{(1)}_{r,n}\left(p, i\right) = \frac{p}{r\left(r+1\right)\beta\left(r, n-r+1\right)} \sum_{\ell=0}^{\aleph(i-1)} \binom{i-1}{\ell} (-1)^{\ell} \left[\ell\left(\ell p + 3r + 2\right)\beta\left(\ell p + r, n-r+1\right)\right.$$

$$\left. - \left(1 + pi\right)\left(1 + \ell\right)\left(\left(\ell+1\right)p + 3r + 2\right)\beta\left(\left(\ell+1\right)p + r, n-r+1\right)\right].$$

Moreover,

$$\mu^{(k)}_{[r:n-2]} + \mu^{(k)}_{[r:n-1]} - 2\mu^{(k)}_{[r:n]} = \sum_{i=1}^{m} \lambda^i \binom{m}{i} \mathcal{D}\left(k, p, i\right) \Phi^{(2)}_{r,n}\left(p, i\right),$$

where

$$\Phi^{(2)}_{r,n}\left(p, i\right) = \frac{p}{n\left(n-1\right)\beta\left(r, n-r+1\right)} \sum_{\ell=0}^{\aleph(i-1)} \binom{i-1}{\ell} (-1)^{\ell} \left[\ell\left(\ell p + 3n - 2\right)\beta\left(\ell p + r, n-r+1\right)\right.$$

$$\left. - \left(1 + pi\right)\left(1 + \ell\right)\left(\left(\ell+1\right)p + 3n - 2\right)\beta\left(\left(\ell+1\right)p + r, n-r+1\right)\right].$$

Proof. The proof of the theorem is similar to the proof of Theorem 3.3, with the exception that the addition operation supersedes the subtraction operation. □

Corollary 3.5. At $m = 1$, we have for HK-FGM-GE family (see [8]),

$$\mu^{(k)}_{[r+2:n]} + \mu^{(k)}_{[r+1:n]} - 2\mu^{(k)}_{[r:n]} = -\frac{\lambda p\left(1+p\right)\left(p + 3r + 2\right)\beta\left(r + p, n-r+1\right)}{r\left(r+1\right)\beta\left(r, n-r+1\right)} \mathcal{D}\left(k, p, 1\right).$$

Moreover, for $m = 1$ we have for the HK-FGM-GE family (see [8]),

$$\mu^{(k)}_{[r:n-2]} + \mu^{(k)}_{[r:n-1]} - 2\mu^{(k)}_{[r:n]} = -\frac{\lambda p\left(1+p\right)\left(p + 3n - 2\right)\beta\left(r + p, n-r+1\right)}{n\left(n-1\right)\beta\left(r, n-r+1\right)} \mathcal{D}\left(k, p, 1\right).$$

4. CONCOMITANTS OF RECORD VALUES BASED ON GFGM-GE MODEL

A new topic in record values theory is concomitants of record values as analogue to concomitants of order statistics, which was suggested for the first time and studied by [19]. The most important use of concomitants of record values arises in experiments in which a specified characteristic's measurements of an individual are made sequentially, and only values that exceed or fall below the current extreme value are recorded. So the only observations are bivariate record values, i.e., records and their concomitants. Let $\{(X_i, Y_i)\}, i = 1, 2, \ldots$ be a random sample from the model GFGM-GE $(\theta_1, \alpha_1; \theta_2, \alpha_2)$. When the experimenter interests in studying just the sequence of records of the first component X_i's the second component associated with the record value of the first one is termed as the concomitant of that record value. The concomitants of record values has many applications, e.g., see [20] and [21]. Some properties from concomitants of record values can be found in [14] and [22]. Let $\{R_n, n \geq 1\}$ be the sequence of record values in the sequence of X's while $R_{[n]}$ be the corresponding concomitant. [19] obtained the pdf of concomitant of nth record value for $n \geq 1$, as $\mathcal{R}_{[n]}\left(y\right) = \int_0^\infty f_Y\left(y|x\right) h_n(x)\,dx$, where $h_n(x) = \frac{1}{\Gamma(n)}\left(-\log\left(1 - F_X(x)\right)\right)^{n-1} f_X(x)$ is the pdf of R_n. The following theorem gives a useful representation for the pdf $\mathcal{R}_{[n]}\left(y\right)$, as well as the kth moments concomitants of record values based on the GFGM-GE model.

Theorem 4.1. Let U_j and V_j be defined as in Theorem 3.1. Then,

$$\mathcal{R}_{[n]}\left(y\right) = f_Y\left(y\right) + \sum_{i=1}^{m} \lambda^i \binom{m}{i} \left[\mathcal{S}^{\star}\left(p, i\right) - \left(1 + pi\right)\mathcal{S}^{\star\star}\left(p, i\right)\right]$$

$$\times \sum_{j=0}^{\aleph(i-1)} \binom{i-1}{i} (-1)^j \left\{\frac{1}{jp+1} f_{U_j}\left(y\right) - \frac{1+pi}{\left(j+1\right)p+1} f_{V_j}\left(y\right)\right\}, \tag{33}$$

where

$$\mathcal{S}^{\star}\left(p,i\right)=\sum_{\ell=0}^{\aleph(i-1)}\sum_{c=0}^{\aleph(\ell p)}(-1)^{\ell+c}\frac{\binom{i-1}{j}\binom{\ell p}{c}}{(c+1)^{n}}$$

and

$$\mathcal{S}^{\star\star}\left(p,i\right)=\sum_{\ell=0}^{\aleph(i-1)}\sum_{c=0}^{\aleph(p(\ell+1))}(-1)^{\ell+c}\frac{\binom{i-1}{\ell}\binom{p\left(\ell+1\right)}{\ell}}{(\ell+1)^{n}}.$$

Moreover, if $\mu_{R_n}^{(k)}=\mathrm{E}\left(R_n^k\right),\ k=1,2,\ldots,$ then,

$$\mu_{R_n}^{(k)}=\mu_k\left(\theta_2,\alpha_2\right)+\sum_{i=1}^{m}\lambda^i\binom{m}{i}\left[\mathcal{S}^{\star}\left(p,i\right)-\left(1+pi\right)\mathcal{S}^{\star\star}\left(p,i\right)\right]\mathcal{D}\left(k;p,i\right). \tag{34}$$

Proof. Clearly, (34) is a simple consequence of (33). Therefore, we have only to prove the relation (33). Now, we have

$$\begin{aligned}
\mathcal{R}_{[n]}\left(y\right)=&f_Y\left(y\right)+\sum_{i=1}^{m}\lambda^i\binom{m}{i}(1-F_Y(y)^p(y))^{i-1}\left[1-\left(1+pi\right)F_Y^p\left(y\right)\right]\\
&\times\int_0^\infty(1-F_X^p(x))^{i-1}\left[1-\left(1+pi\right)F_X^p\left(x\right)\right]\frac{(-\log(1-F_X(x)))^{n-1}}{\Gamma\left(n\right)}f_X\left(x\right)dx\\
=&f_Y\left(y\right)+\sum_{i=1}^{m}\lambda^i\binom{m}{i}\sum_{j=0}^{\aleph(i-1)}\binom{i-1}{i}(-1)^j\left\{\frac{1}{jp+1}f_{U_j}\left(y\right)-\frac{1+pi}{\left(j+1\right)p+1}f_{V_j}\left(y\right)\right\}\\
&\sum_{\ell=0}^{\aleph(j-1)}\binom{i-1}{\ell}(-1)^{\ell}\frac{1}{\Gamma\left(n\right)}\int_0^\infty F_X^{\ell p}\left(x\right)(-\log(1-F_X(x)))^{n-1}f_X\left(x\right)dx\\
&-\left(1+pi\right)\sum_{\ell=0}^{\aleph(i-1)}\binom{i-1}{j}(-1)^{\ell}\int_0^\infty F_X^{(\ell+1)p}\left(x\right)(-\log(1-F_X(x)))^{n-1}f_X\left(x\right)dx.
\end{aligned}$$

Upon using the transformation $-\log\left(1-F_X\left(x\right)\right)=t$ in the above two integrations and applying the binomial theorem on the terms $\left(1-e^{-t}\right)^{\ell p}$ and $\left(1-e^{-t}\right)^{(\ell+1)p}$, in the first and second integrations, respectively, we get the representation (33). □

Corollary 4.1. For $m=1$, i.e., for HK-FGM-GE model, we get

$$\begin{aligned}
\mu_{R_n}^{(k)}&=\mu_k\left(\theta_2,\alpha_2\right)+\lambda\left[\mathcal{S}^{\star}\left(p,1\right)-\left(1+p\right)\mathcal{S}^{\star\star}\left(p,1\right)\right]\mathcal{D}\left(k;p,1\right)\\
&=\mu_k\left(\theta_2,\alpha_2\right)+\lambda\left[\mu_k\left(\theta_2,\alpha_2\right)-\left(1+p\right)\mu_k\left(\theta_2,\alpha_2p\right)\right]\left[1-\left(1+p\right)\sum_{\ell=0}^{\aleph(p)}(-1)^{\ell}\frac{\binom{p}{\ell}}{(\ell+1)^n}\right].
\end{aligned}$$

Moreover, For $m=1$ and $p=1$, i.e., for FGM-GE family, we get

$$\begin{aligned}
\mu_{R_n}^{(k)}&=\mu_k\left(\theta_2,\alpha_2\right)+\lambda\left[\mathcal{S}^{\star}\left(1,1\right)-\left(1+p\right)\mathcal{S}^{\star\star}\left(1,1\right)\right]\mathcal{D}\left(k;1,1\right)\\
&=\mu_k\left(\theta_2,\alpha_2\right)\left[1-\lambda\left(2^{-(n-1)}-1\right)\right].
\end{aligned}$$

Proof. The proof is obvious, since it follows after simple algebra. □

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