International Journal of Computational Intelligence Systems
Vol. **12(2)**, 2019, *pp.* 1029–1046
DOI: https://doi.org/10.2991/ijcis.d.190823.001; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

Revisiting the Role of Hesitant Multiplicative Preference Relations in Group Decision Making With Novel Consistency Improving and Consensus Reaching Processes

Rui Wang[1,2], Bin Shuai[1,2,*], Zhen-Song Chen[3,*], Kwai-Sang Chin[4], Jiang-Hong Zhu[1,2]

[1]*School of Transportation and Logistics, Southwest Jiaotong University, Chengdu Sichuan, 610031, China*
[2]*National United Engineering Laboratory of Intergrated and Intelligent Transportation, Southwest Jiaotong University, Chengdu Sichuan, 610031, China*
[3]*School of Civil Engineering, Wuhan University, Wuhan, 430072, China*
[4]*Department of Systems Engineering and Engineering Management, City University of Hong Kong, Kowloon Tong, Hong Kong, 999077, China*

ARTICLE INFO

Article History
Received 03 Jul 2019
Accepted 20 Aug 2019

Keywords
Group decision making
Hesitant multiplicative preference relation
Consistency improving process
Consensus reaching process
Mathematical programming model

ABSTRACT

In recent years, hesitant multiplicative preference relation (HMPR) has been a powerful means to represent the evaluation information of decision makers during the pairwise comparison concerning alternatives. As the important parts of group decision making (GDM) issues with HMPRs, the consistency improving and consensus reaching processes have been researched by many scholars; however, the existing approaches present several limitations, including defining the consistency index depend on the other HMPR instead of the original HMPR, improving the consistency and consensus levels of HMPRs independently, and that the high computational complexity of the existing iterative algorithms. To overcome these drawbacks, this paper proposes a mathematical programming model to improve the consistency and consensus levels of HMPRs, simultaneously. First, a consistency index according to multiplicative consistency is put forward to compute the consistency degree of normalized HMPR (NHMPR) after the normalization procedure. Second, a programming model by minimizing the difference between the original NHMPR and the revised NHMPR is developed to obtain the acceptably consistent NHMPR; then, a consistency-based method is constructed to solve the decision making problems with an HMPR. Third, considering the GDM issues, a consistency- and consensus-based programming model is established to obtain the acceptably consistent and consensus NHMPRs, in which the original evaluation information can remain as much as possible. Fourth, the normalized hesitant multiplicative weighted geometric operator is introduced to fuse the revised NHMPRs and an algorithm for GDM is proposed with novel consistency improving and consensus reaching processes. Finally, two numerical examples are applied to show the practicality and advantages of the proposed approaches.

1. INTRODUCTION

Group decision making (GDM) issues, in which decision makers determine the best choice from several alternatives, are attracting a substantial amount of attention in many practical fields, such as engineering, science, and technology [1,2]. Generally, decision makers express their evaluation information on the pre-determined alternatives concerning a collection of criteria; subsequently, the individual information can be aggregated to determine the ranking result [3,4]. According to the literature [5], the pairwise comparison approaches are more accurate than the non-pairwise comparison approaches; thus, preference relation, which is the research object of this study, can be constructed by the pairwise comparison between alternatives during the evaluation process. Traditionally, fuzzy preference relations (FPRs) [6] and multiplicative preference relations (MPRs) [7] are the most common used preference relations, in which their elements are numbers belonging to [0.1, 0 the crisp.9] and [1/9, 9], respectively. Furthermore, with the increasing complexity of GDM issues in practice, many generalized preference relations [8–14] were investigated to express the fuzziness of evaluation information; however, they fail to deal with some situations, i.e., the preference information of decision makers may be expressed by several possible crisp numbers. To break this limitation, Torra [15] put forward the hesitant fuzzy set (HFS) theory; later, many scholars have focused on the generalized form of HFS [16–19]. Accordingly, Zhu and Xu [20] developed the hesitant fuzzy preference relations (HFPRs), in which their elements are the possible values belonging to 0.1~0.9 scale; nevertheless, Saaty's 1~9 scale, which is distributed non-uniformly and asymmetrically, is more reasonable than the 0.1~0.9 scale in some practical situations [21,22]. Later, the hesitant multiplicative preference relations (HMPRs) were developed with the combination of HFS and MPRs [23,24]. Since then, the consistency measures, consensus models, aggregation operators, and GDM approaches based on HMPRs were researched by many scholars [25–33]; this paper also focuses on the GDM approaches with HMPRs, in which the consistency and consensus issues can be solved simultaneously.

*Corresponding authors. Emails: shuaibin@home.swjtu.edu.cn; zschen@whu.edu.cn

With respect to the aggregation tools of HMPRs, many hesitant multiplicative aggregation operators have been proposed [23,31,34]. Besides, He *et al.* [32] developed three kinds of error analysis based ranking approaches to deal with the GDM issues with HMPRs. Later, combined with data envelopment analysis theory, Lin and Wang [29] proposed the self-weight and cross weight prioritization methods with HMPRs. However, the aforementioned literature ignored the consistency issue of HMPRs, which is an important factor in measuring the rationality of ranking results based on preference relations. Lin *et al.* [26] constructed the consistent HMPRs according to the original HMPRs, then a least square deviation model was developed to determine the ranking of alternatives. Zhang and Wu [27] normalized the HMPRs and divided them into several reduced MPRs, then the consistency improving method proposed by Xu and Wei [35] was introduced to obtain the acceptably consistent HMPRs. On the other hand, the ranking of alternatives with a relatively high consensus level is desirable [36,37]; hence, the consensus models based on HMPRs have been a hot topic in recent years. Zhang and Wu [24] developed the group consensus index to propose an iteration-based algorithm to increase the consensus degrees of HMPRs. Later, Lin *et al.* [25] reduced the HMPRs to MPRs according to a regression method, and an iterative consensus reaching approach with MPRs was proposed to solve the consensus issue in GDM problems.

Generally, concerning a GDM problem with HMPRs, the best alternative can be determined through three processes, namely, consistency improving process, consensus reaching process, and selection process [38–40]. To ensure the logicality and rationality of the preference information [41,42], the consistency degrees of individual HMPRs are improved to reach the requirement during the consistency improving process. The consensus reaching process aims to revise the individual non-consensus preference information for improving the consensus degree among decision makers [36,37, 43–45]. Once the acceptably consistent and consensus HMPRs are obtained by the aforementioned two steps, the next issue is how to obtain the ranking result during the selection process. At present, many scholars have proposed GDM approaches based on HMPRs that consider the consistency and consensus issues [24,25]; nevertheless, these GDM approaches still present several limitations. (1) The existing consistency indexes, which are utilized to check the consistency level of individual HMPR, were mostly defined depend on the consistent HMPRs derived from the original HMPRs instead of the original HMPRs themselves. The step of constructing the consistent HMPRs greatly increases the computational complexity of consistency improving process. (2) Both the consistency and consensus improving approaches were developed by constructing the specific iterative algorithms; the acceptably consistent and consensus HMPRs were obtained after many iterative rounds, respectively. These processes are time-consuming and require a large number of arithmetic operations. In addition, the original preference information may not be effectively preserved. (3) The existing GDM approaches with HMPRs regarded the consistency improving and consensus reaching processes as two independent steps, and the consistency and consensus levels were improved by two independent iteration-based models. These methods will result in a huge workload if the original HMPRs of decision makers are unacceptably consistent and consensus, simultaneously.

To overcome the aforementioned drawbacks, this paper proposes a novel consistency- and consensus-based algorithm. The main contributions of this study can be summarized as follows. (1) Since the normalized HMPR (NHMPR) is obtained after the normalization procedure, a new consistency index is developed according to the original NHMPR itself instead of other NHMPR, which can be used to check the consistency degree of the original NHMPR appropriately. (2) A consistency-based programming model for minimizing the distance between the original NHMPR and acceptably consistent NHMPR is developed; afterwards, an algorithm based on the proposed consistency improving model is put forward to deal with the decision making issues with an HMPR. (3) A consensus measure is put forward to construct a comprehensive programming model with the constraints of the consistency and consensus requirements, in which the consistency and consensus degrees of the original NHMPRs can be improved, simultaneously. (4) An approach for GDM with HMPRs is established based on the proposed comprehensive model and normalized hesitant multiplicative weighted geometric (NHMWG) operator. To achieve this, the rest of this paper is presented as in the following. The related concepts of HMPRs are introduced in Section 2. Section 3 defines the consistency index of HMPR and develops the decision making method with an HMPR. A GDM approach according to the proposed comprehensive model is constructed in Section 4. Section 5 applies two numerical examples to illustrate the practicality and advantages of the proposed approaches. Some conclusions of this study are summarized in Section 6.

2. PRELIMINARIES

In this section, the definitions of hesitant multiplicative set (HMS), HMPR, and NHMPR are recalled. Furthermore, the distance measure between NHMPRs is developed in detail, which will be used during the decision making process.

2.1. HMS

In practice, determining the exact value of membership degree is sometimes very difficult. Therefore, according to the HFS [15] and 1~9 scale [7], Xia and Xu [23] defined the HMS and proposed the score function of the hesitant multiplicative element (HME).

Definition 1. [23] Let X be a non-empty set, an HMS A on X is given by $h_A(x)$ that will return a finite subset of $[1/9, 9]$ when being applied to X, which are defined as

$$A = \{\langle x, h_A(x)\rangle \,|x \in X\}, \tag{1}$$

where $h_A(x)$ is a set of several values in $[1/9, 9]$, indicating the possible membership values of $x \in X$ to the set A. For convenience, we call the $h = h_A(x)$ an HME and l_h is the number of the elements in HME h.

Definition 2. [23] Let h be an HME, then the score function of h is given by

$$s(h) = \sum\nolimits_{\gamma \in h} \gamma / l_h. \tag{2}$$

Subsequently, the comparison method between two HMEs h_1 and h_2 is presented as

1. if $s(h_1) < s(h_2)$, then $h_1 < h_2$;
2. if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

It is noteworthy that the numbers of elements in HMEs given by decision makers will most likely be diverse without a prior stipulation; hence, the computational complexity of aggregating HMEs will become extremely high. To make sure that the different HMEs have the same number of elements, Zhang and Wu [24] put forward the normalization equation of HMEs as follows.

Definition 3. [24] Let h be an HME and $\delta\,(0 \le \delta \le 1)$ be a preference parameter determined by decision makers, where h^+ and h^- are the maximum and minimum elements in HME h, respectively. Then, the added element in normalized HME (NHME) $\overline{h}$ with δ is determined as

$$\overline{h} = \left(h^+\right)^{\delta} \times \left(h^-\right)^{(1-\delta)}. \tag{3}$$

Based on the NHMEs obtained by the equation in Definition 3, we can construct the distance measure between different NHMEs.

Definition 4. Let h_1 and h_2 be two HMEs, and their NHMEs with δ are $\overline{h}_1$ and $\overline{h}_2$, respectively, where the number of the elements in them is l, $\overline{h}_1^{\sigma(s)}$ and $\overline{h}_2^{\sigma(s)}$ are the sth element in NHMEs $\overline{h}_1$ and $\overline{h}_2$, respectively. Then, the distance between NHMEs $\overline{h}_1$ and $\overline{h}_2$ can be defined by

$$d\left(\overline{h}_1, \overline{h}_2\right) = \frac{1}{2l}\sum_{s=1}^{l}\left|\log_9^{\overline{h}_1^{\sigma(s)}} - \log_9^{\overline{h}_2^{\sigma(s)}}\right|. \tag{4}$$

It can be easily proven that the distance measure has the following properties.

Theorem 1. *The distance $d\left(\overline{h}_1, \overline{h}_2\right)$ between NHMEs $\overline{h}_1$ and $\overline{h}_2$ satisfies the properties as follows:*

1. $0 \le d\left(\overline{h}_1, \overline{h}_2\right) \le 1$;
2. $d\left(\overline{h}_1, \overline{h}_2\right) = 0$ if and only if $\overline{h}_1 = \overline{h}_2$;
3. $d\left(\overline{h}_1, \overline{h}_2\right) = d\left(\overline{h}_2, \overline{h}_1\right)$.

2.2. Hesitant Multiplicative Preference Relations

As the FPR [6] and MPR [7] are the most common used preference relations in literature, Xia and Xu [23] extended the HMS to develop the HMPR for expressing the preference information. Later, Zhang and Wu [24] proposed the concept of NHMPRs.

Definition 5. [23] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, an HMPR $\boldsymbol{H}$ on X is expressed by $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n} \subset X \times X\,(i, j = 1, 2, \ldots, n)$, in which the HME $h_{ij} = \left\{h_{ij}^s | s = 1, 2, \ldots, l_{h_{ij}}\right\}$ indicates the hesitant degree to which x_i is preferred to x_j and satisfies the conditions as

$$h_{ij}^{\rho(s)} \times h_{ji}^{\rho\left(l_{h_{ij}}-s+1\right)} = 1, h_{ii} = \{1\}, l_{h_{ij}} = l_{h_{ji}}, \tag{5}$$

where $h_{ij}^{\rho(s)}$ is the sth largest element in h_{ij}.

Definition 6. [24,31] Let $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ be an HMPR and $\delta\,(0 \le \delta \le 1)$ be an optimized parameter; then, an NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ with δ, where δ is utilized to add elements to $h_{ij}\,(i < j)$ and $1 - \delta$ is utilized to add elements to $h_{ji}\,(i < j)$, and the NHMEs in NHMPR $\overline{\boldsymbol{H}}$ satisfy the conditions as below:

$$\begin{cases} l_{\overline{h}_{ij}} = \max\left\{l_{h_{ij}} | i, j = 1, 2, \ldots, n\right\} = l; \\ \overline{h}_{ij}^{\sigma(s)} \times \overline{h}_{ji}^{\sigma(s)} = 1, \overline{h}_{ii} = \{1\}; \\ \overline{h}_{ij}^{\sigma(s)} \le \overline{h}_{ij}^{\sigma(s+1)}, \overline{h}_{ji}^{\sigma(s+1)} \le \overline{h}_{ji}^{\sigma(s)}, i < j. \end{cases} \tag{6}$$

where $\overline{h}_{ij}^{\sigma(s)}$ and are the sth element in NHMEs $\overline{h}_{ij}$ and $\overline{h}_{ji}$, respectively.

In the following sections, the decision making approach with MPRs in the literature [46] will be introduced to determine the priority vector of alternatives; then, we should transform the NHMPR into a corresponding MPR to obtain the ranking result combined with the improved score function of NHME [24].

Definition 7. [24] Let $\overline{h}$ be the NHME with δ derived from an HME h and $l_{\overline{h}}$ be the number of elements in $\overline{h}$, then the improved score function of $\overline{h}$ is given by

$$\bar{s}\left(\overline{h}\right) = \left(\prod_{s=1}^{l_{\overline{h}}} \overline{h}^{\sigma(s)}\right)^{1/l_{\overline{h}}}, \tag{7}$$

where $\overline{h}^{\sigma(s)}$ is the sth element in NHME $\overline{h}$.

Furthermore, the following theorem can be obtained.

Theorem 2. *Let $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ be an HMPR and its NHMPR be $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ with δ, and l be the number of elements in $\overline{h}_{ij}$, then the matrix $\boldsymbol{S}\left(\overline{\boldsymbol{H}}\right) = \left(s_{ij}\right)_{n\times n}$ is called the improved score matrix of NHMPR $\overline{\boldsymbol{H}}$, where*

$$s_{ij} = \left(\prod_{s=1}^{l} \overline{h}_{ij}^{\sigma(s)}\right)^{1/l}. \tag{8}$$

In addition, the improved score matrix $\boldsymbol{S}\left(\overline{\boldsymbol{H}}\right)$ is an MPR.

Proof. *Since the matrix $\boldsymbol{S}\left(\overline{\boldsymbol{H}}\right) = \left(s_{ij}\right)_{n\times n}$ is the improved score matrix of NHMPR $\overline{\boldsymbol{H}}$, we have*

$$s_{ij} = \left(\prod_{s=1}^{l} \overline{h}_{ij}^{\sigma(s)}\right)^{1/l} (i < j), s_{ji} = \left(\prod_{s=1}^{l} \overline{h}_{ji}^{\sigma(s)}\right)^{1/l} (i < j),$$

$$s_{ii} = \left(\prod_{s=1}^{l} \overline{h}_{ii}^{\sigma(s)}\right)^{1/l}.$$

Due to $\overline{h}_{ij}^{\sigma(s)}, \overline{h}_{ji}^{\sigma(s)} \in [1/9, 9]$ *and* $\overline{h}_{ii}^{\sigma(s)} = 1$, *we can obtain* $s_{ij}, s_{ji} \in [1/9, 9]$ *and* $s_{ii} = 1$. *And* $\overline{\boldsymbol{H}}$ *is an NHMPR, i.e.,* $\overline{h}_{ij}^{\sigma(s)} \times \overline{h}_{ji}^{\sigma(s)} = 1$, *then*

$$s_{ij} \times s_{ji} = \left(\prod_{s=1}^{l} \overline{h}_{ij}^{\sigma(s)}\right)^{1/l} \times \left(\prod_{s=1}^{l} \overline{h}_{ji}^{\sigma(s)}\right)^{1/l}$$

$$= \left(\prod_{s=1}^{l} \left(\overline{h}_{ij}^{\sigma(s)} \times \overline{h}_{ji}^{\sigma(s)}\right)\right)^{1/l} = 1.$$

Accordingly, the improved score matrix $\boldsymbol{S}\left(\overline{\boldsymbol{H}}\right)$ *is proven as an MPR.*

Then, we can define the distance measure as in the following to compute the distance between different NHMPRs.

Definition 8. Let $\overline{\boldsymbol{H}}^1 = \left(\overline{h}_{ij}^1\right)_{n\times n}$ and $\overline{\boldsymbol{H}}^2 = \left(\overline{h}_{ij}^2\right)_{n\times n}$ be two NHMPRs with δ derived from HMPRs $\boldsymbol{H}^1$ and $\boldsymbol{H}^2$, respectively, and l be the number of elements in $\overline{h}_{ij}^1$ and $\overline{h}_{ij}^2$, the distance between them is given by

$$d\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2\right) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{l} \frac{1}{l} \left| \log_9^{\overline{h}_{ij}^{1,\sigma(s)}} - \log_9^{\overline{h}_{ij}^{2,\sigma(s)}} \right|. \quad (9)$$

Theorem 3. *The distance* $d\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2\right)$ *can be simplified as*

$$d\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2\right) = \frac{1}{ln\,(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{l} \left| \log_9^{\overline{h}_{ij}^{1,\sigma(s)}} - \log_9^{\overline{h}_{ij}^{2,\sigma(s)}} \right|. \quad (10)$$

2.3. Consensus Reaching Process

In actual decision making, decision makers may vary in research areas and experiences; then, the individual preference information that deviates far from the group opinions may be given inevitably. However, the ranking of alternatives, which has a relatively high consensus degree or is accepted by all decision makers, is more desirable than that with a low consensus level [43,44,47]. Thus, the consensus reaching process has been a hot research topic, and the classical consensus models under a fuzzy context, in social network GDM, and in large-scale GDM were reviewed by Palomares *et al.* [48], Dong *et al.* [49], and Labella *et al.* [50], respectively.

Accordingly, the general procedure of a classical consensus model can be concluded in the following, and its visualization process is shown in Fig. 1.

1. Collecting information: Decision makers express their evaluation information using different kinds of preference relations.
2. Detection of consensus level: The consensus measures between the preference information are calculated to evaluate the consensus level.
3. Consensus control: The current consensus level is utilized to check whether the acceptable consensus is achieved. If the conditions are satisfied, proceed to the selection process; otherwise, proceed to the feedback mechanism.

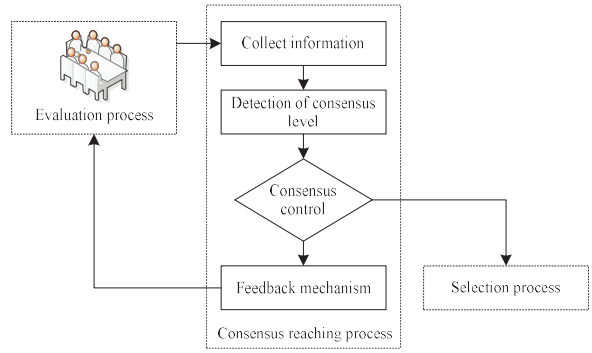

Figure 1 Flowchart of general consensus reaching process.

4. Feedback mechanism: The non-consensus evaluation information can be identified and revised according to several identification and direction rules, respectively.

3. CONSISTENCY FOR HMPRs

Different kinds of consistency of HFPRs, HMPRs, and the extended hesitant fuzzy linguistic preference relations have been investigated systematically by Rodríguez *et al.* [51]; accordingly, inspired by the literature [51], a consistency index is developed to represent the consistency degrees of HMPRs. Furthermore, a mathematical programming model is put forward to obtain the acceptably consistent NHMPR; subsequently, the approach for decision making with an HMPR is proposed.

3.1. A Consistency Index of HMPRs

To investigate the consistency measure of HMPRs, Zhang and Wu [24] constructed the multiplicative consistent HMPR, and we can transform the multiplicative consistency into the following definition.

Definition 9. [24] Let $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ be an HMPR and δ be an optimized parameter; then, an NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ with δ can be determined, if

$$\overline{h}_{ij}^{\sigma(s)} \overline{h}_{jk}^{\sigma(s)} \overline{h}_{ki}^{\sigma(s)} = \overline{h}_{ik}^{\sigma(s)} \overline{h}_{kj}^{\sigma(s)} \overline{h}_{ji}^{\sigma(s)}, \quad i, j, k = 1, 2, \ldots, n; i \neq j \neq k, \quad (11)$$

where $\overline{h}_{ij}^{\sigma(s)}$ $(s = 1, 2, \ldots l)$ is the *sth* element in HME $\overline{h}_{ij}$ and l is the number of elements in $\overline{h}_{ij}$, then the NHMPR $\overline{\boldsymbol{H}}$ is called a multiplicative consistent HMPR.

Theorem 4. *Given an HMPR* $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ *and its NHMPR* $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ *with* δ, *the following two statements are equivalent:*

1. $\overline{h}_{ij}^{\sigma(s)} \overline{h}_{jk}^{\sigma(s)} \overline{h}_{ki}^{\sigma(s)} = \overline{h}_{ik}^{\sigma(s)} \overline{h}_{kj}^{\sigma(s)} \overline{h}_{ji}^{\sigma(s)}$, for all $i, j, k = 1, 2, \ldots, n; i \neq j \neq k$.
2. $\overline{h}_{ij}^{\sigma(s)} \overline{h}_{jk}^{\sigma(s)} \overline{h}_{ki}^{\sigma(s)} = \overline{h}_{ik}^{\sigma(s)} \overline{h}_{kj}^{\sigma(s)} \overline{h}_{ji}^{\sigma(s)}$, for all $i < j < k$.

The proof of Theorem 4 can be easily implemented with reference to the literature [39]. According to Theorem 4, we can obtain the following forms of multiplicative consistency of HMPRs.

Definition 10. Given an HMPR $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ and its NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ with δ, then the NHMPR $\overline{\boldsymbol{H}}$ is multiplicative consistent if and only if

$$\overline{h}_{ij}^{\sigma(s)}\overline{h}_{jk}^{\sigma(s)}\overline{h}_{ki}^{\sigma(s)} = \overline{h}_{ik}^{\sigma(s)}\overline{h}_{kj}^{\sigma(s)}\overline{h}_{ji}^{\sigma(s)}, \quad i<j<k, \tag{12}$$

where $\overline{h}_{ij}^{\sigma(s)}$ $(s = 1, 2, \dots l)$ is the *sth* element in HME $\overline{h}_{ij}$ and l is the number of elements in $\overline{h}_{ij}$.

Definition 11. Given an HMPR $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ and its NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ with δ, then the NHMPR $\overline{\boldsymbol{H}}$ is multiplicative consistent if and only if

$$\log_9^{\overline{h}_{ij}^{\sigma(s)}} + \log_9^{\overline{h}_{jk}^{\sigma(s)}} + \log_9^{\overline{h}_{ki}^{\sigma(s)}} = \log_9^{\overline{h}_{ik}^{\sigma(s)}} + \log_9^{\overline{h}_{kj}^{\sigma(s)}} + \log_9^{\overline{h}_{ji}^{\sigma(s)}}, \quad i<j<k, \tag{13}$$

where $\overline{h}_{ij}^{\sigma(s)}$ $(s = 1, 2, \dots l)$ is the *sth* element in HME $\overline{h}_{ij}$ and l is the number of elements in $\overline{h}_{ij}$.

Afterwards, the consistency index of HMPRs can be defined as follows.

Definition 12. Given an HMPR $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ and its NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ with δ, a consistency index can be defined by

$$CI\left(\overline{\boldsymbol{H}}\right) = \frac{1}{6\cdot C_n^3\cdot l}\sum_{i<j<k}\sum_{s=1}^{l}\left|\left(\log_9^{\overline{h}_{ij}^{\sigma(s)}} + \log_9^{\overline{h}_{jk}^{\sigma(s)}} + \log_9^{\overline{h}_{ki}^{\sigma(s)}}\right) - \left(\log_9^{\overline{h}_{ik}^{\sigma(s)}} + \log_9^{\overline{h}_{kj}^{\sigma(s)}} + \log_9^{\overline{h}_{ji}^{\sigma(s)}}\right)\right|, \tag{14}$$

where $\overline{h}_{ij}^{\sigma(s)}$ $(s = 1, 2, \dots l)$ is the *sth* element in HME $\overline{h}_{ij}$ and l is the number of elements in $\overline{h}_{ij}$.

Obviously, we can utilize the consistency index $CI\left(\overline{\boldsymbol{H}}\right)$ to check the consistency degree of NHMPRs; the larger the value of $CI\left(\overline{\boldsymbol{H}}\right)$, the weaker the consistency degree of NHMPR $\overline{\boldsymbol{H}}$. Hence, we can compare the consistency index $CI\left(\overline{\boldsymbol{H}}\right)$ with a predefined consistency threshold $CI_0 \in (0, 1)$ to check whether the NHMPR $\overline{\boldsymbol{H}}$ is an acceptably consistent HMPR; if $CI\left(\overline{\boldsymbol{H}}\right) \le CI_0$, then the NHMPR $\overline{\boldsymbol{H}}$ is the acceptably consistent HMPR, and vice versa.

3.2. Improve the Consistency Degree of HMPR

For an HMPR $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n}$ and its NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ with δ, where $\overline{h}_{ij} = \left\{\overline{h}_{ij}^{s} | s = 1, 2, \dots, l\right\}$ $(i, j = 1, 2, \dots, n)$. To obtain a revised NHMPR $\hat{\boldsymbol{H}} = \left(\hat{h}_{ij}\right)_{n\times n}$ that reaches the consistency threshold CI_0, in which the preference information in the original NHMPR $\overline{\boldsymbol{H}}$ remains as much as possible; then, we establish the following mathematical programming model by minimizing the deviation between the revised NHMPR $\hat{\boldsymbol{H}}$ and NHMPR $\overline{\boldsymbol{H}}$ with the constraint condition of the acceptable multiplicative consistency.

$$\min\ d\left(\overline{\boldsymbol{H}}, \hat{\boldsymbol{H}}\right)$$

$$s.t.\begin{cases} CI\left(\hat{\boldsymbol{H}}\right) \le CI_0 \\ \hat{\boldsymbol{H}}\ is\ an\ NHMPR \end{cases} \tag{M-1}$$

Combined with Eqs. (10) and (14) and the concept of NHMPR in Definition 6, we can transform model M-1 into the following form.

$$\min \frac{1}{ln(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{l}\left|\log_9^{\overline{h}_{ij}^{\sigma(s)}} - \log_9^{\hat{h}_{ij}^{\sigma(s)}}\right|$$

$$s.t.\begin{cases} \frac{1}{6\cdot C_n^3\cdot l}\sum_{i<j<k}\sum_{s=1}^{l}\left|\left(\log_9^{\hat{h}_{ij}^{\sigma(s)}} + \log_9^{\hat{h}_{jk}^{\sigma(s)}} + \log_9^{\hat{h}_{ki}^{\sigma(s)}}\right) - \left(\log_9^{\hat{h}_{ik}^{\sigma(s)}} + \log_9^{\hat{h}_{kj}^{\sigma(s)}} + \log_9^{\hat{h}_{ji}^{\sigma(s)}}\right)\right| \le CI_0 \\ \hat{h}_{ij}^{\sigma(s)} - \hat{h}_{ij}^{\sigma(s+1)} \le 0,\ \ i<j,\ \ s = 1, 2, \dots, l. \\ \hat{h}_{ji}^{\sigma(s+1)} - \hat{h}_{ji}^{\sigma(s)} \le 0,\ \ i<j,\ \ s = 1, 2, \dots, l. \\ \frac{1}{9} \le \hat{h}_{ij}^{\sigma(s)}, \hat{h}_{ji}^{\sigma(s)} \le 9,\ \ i<j,\ \ s = 1, 2, \dots, l. \\ \hat{h}_{ij}^{\sigma(s)} \times \hat{h}_{ji}^{\sigma(s)} = 1,\ \ i<j,\ \ s = 1, 2, \dots, l. \end{cases} \tag{M-2}$$

Denote $r_{ij}^{\sigma(s)} = 1 + \log_9^{\overline{h}_{ij}^{\sigma(s)}}$ and $\hat{r}_{ij}^{\sigma(s)} = 1 + \log_9^{\hat{h}_{ij}^{\sigma(s)}}$. As $\frac{1}{9} \le \hat{h}_{ij}^{\sigma(s)} \le 9$, we have $0 \le \hat{r}_{ij}^{\sigma(s)} \le 2$. Let $\hat{\varepsilon}_{ij}^{\sigma(s)} = r_{ij}^{\sigma(s)} - \hat{r}_{ij}^{\sigma(s)}$, $\hat{\varepsilon}_{ij}^{\sigma(s)+} = \left(\left|\hat{\varepsilon}_{ij}^{\sigma(s)}\right| + \hat{\varepsilon}_{ij}^{\sigma(s)}\right)/2$, and $\hat{\varepsilon}_{ij}^{\sigma(s)-} = \left(\left|\hat{\varepsilon}_{ij}^{\sigma(s)}\right| - \hat{\varepsilon}_{ij}^{\sigma(s)}\right)/2$, then $\left|\hat{\varepsilon}_{ij}^{\sigma(s)}\right| = \hat{\varepsilon}_{ij}^{\sigma(s)+} + \hat{\varepsilon}_{ij}^{\sigma(s)-}$. Therefore, model M-2 can be rewritten as in the following.

$$\min \frac{1}{ln(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{l}\left(\hat{\varepsilon}_{ij}^{\sigma(s)+} + \hat{\varepsilon}_{ij}^{\sigma(s)-}\right)$$

$$s.t.\begin{cases} r_{ij}^{\sigma(s)} - \hat{r}_{ij}^{\sigma(s)} + \hat{\varepsilon}_{ij}^{\sigma(s)-} - \hat{\varepsilon}_{ij}^{\sigma(s)+} = 0, \\ i<j,\ \ s = 1, 2, \dots, l. \\ \sum_{1\le i<j<k\le n}\sum_{s=1}^{n}\left|\left(\hat{r}_{ij}^{\sigma(s)} + \hat{r}_{jk}^{\sigma(s)} + \hat{t}_{ki}^{\sigma(s)}\right) - \left(\hat{r}_{ik}^{\sigma(s)} + \hat{r}_{kj}^{\sigma(s)} + \hat{r}_{ji}^{\sigma(s)}\right)\right| \le 6\cdot l\cdot C_n^3\cdot CI_0 \\ 9^{\hat{r}_{ij}^{\sigma(s)}-1} - 9^{\hat{r}_{ij}^{\sigma(s+1)}-1} \le 0,\ i<j,\ s = 1, 2, \dots, l. \\ 9^{\hat{r}_{ji}^{\sigma(s+1)}-1} - 9^{\hat{r}_{ji}^{\sigma(s)}-1} \le 0,\ i<j,\ s = 1, 2, \dots, l. \\ 0 \le \hat{r}_{ij}^{\sigma(s)}, \hat{r}_{ji}^{\sigma(s)} \le 2,\ \ i<j,\ \ s = 1, 2, \dots, l. \\ 9^{\hat{r}_{ij}^{\sigma(s)}-1} \times 9^{\hat{r}_{ji}^{\sigma(s)}-1} = 1,\ i<j,\ s = 1, 2, \dots, l. \\ \hat{\varepsilon}_{ij}^{\sigma(s)-}, \hat{\varepsilon}_{ij}^{\sigma(s)+} \ge 0,\ \ i<j,\ \ s = 1, 2, \dots, l. \end{cases} \tag{M-3}$$

Subsequently, set $\hat{\delta}_{ijk}^{\sigma(s)} = \left(\hat{r}_{ij}^{\sigma(s)} + \hat{r}_{jk}^{\sigma(s)} + \hat{t}_{ki}^{\sigma(s)}\right) - \left(\hat{r}_{ik}^{\sigma(s)} + \hat{r}_{kj}^{\sigma(s)} + \hat{r}_{ji}^{\sigma(s)}\right)$, $\hat{\delta}_{ijk}^{\sigma(s)+} = \left(\left|\hat{\delta}_{ijk}^{\sigma(s)}\right| + \hat{\delta}_{ijk}^{\sigma(s)}\right)/2$, and $\hat{\delta}_{ijk}^{\sigma(s)-} = \left(\left|\hat{\delta}_{ijk}^{\sigma(s)}\right| - \hat{\delta}_{ijk}^{\sigma(s)}\right)/2$, then $\left|\hat{\delta}_{ijk}^{\sigma(s)}\right| = \hat{\delta}_{ijk}^{\sigma(s)+} + \hat{\delta}_{ijk}^{\sigma(s)-}$; the following form can be utilized to replace model M-3.

$$\min \frac{1}{ln(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{l} \left(\hat{\varepsilon}_{ij}^{\sigma(s)+} + \hat{\varepsilon}_{ij}^{\sigma(s)-}\right)$$

$$s.t. \begin{cases} r_{ij}^{\sigma(s)} - \hat{r}_{ij}^{\sigma(s)} + \hat{\varepsilon}_{ij}^{\sigma(s)-} - \hat{\varepsilon}_{ij}^{\sigma(s)+} = 0, \\ i < j, \quad s = 1, 2, \dots, l. \\ \sum_{1 \le i < j < k \le n} \sum_{s=1}^{n} \left(\hat{\delta}_{ijk}^{\sigma(s)+} + \hat{\delta}_{ijk}^{\sigma(s)-}\right) \le 6 \cdot l \cdot C_n^3 \cdot CI_0 \\ \left(\hat{r}_{ij}^{\sigma(s)} + \hat{r}_{jk}^{\sigma(s)} + \hat{t}_{ki}^{\sigma(s)}\right) - \left(\hat{r}_{ik}^{\sigma(s)} + \hat{r}_{kj}^{\sigma(s)} + \hat{r}_{ji}^{\sigma(s)}\right) + \\ \hat{\delta}_{ijk}^{\sigma(s)-} - \hat{\delta}_{ijk}^{\sigma(s)+} = 0, \ i < j < k, \quad s = 1, 2, \dots, l. \\ \hat{r}_{ij}^{\sigma(s)} - \hat{r}_{ij}^{\sigma(s+1)} \le 0, \ i < j, \quad s = 1, 2, \dots, l. \\ \hat{r}_{ji}^{\sigma(s+1)} - \hat{r}_{ji}^{\sigma(s)} \le 0, \ i < j, \quad s = 1, 2, \dots, l. \\ 0 \le \hat{r}_{ij}^{\sigma(s)}, \hat{r}_{ji}^{\sigma(s)} \le 2, \quad i < j, \quad s = 1, 2, \dots, l. \\ \hat{r}_{ij}^{\sigma(s)} + \hat{r}_{ji}^{\sigma(s)} = 2, \quad i < j, \quad s = 1, 2, \dots, l. \\ \hat{\varepsilon}_{ij}^{\sigma(s)-}, \hat{\varepsilon}_{ij}^{\sigma(s)+} \ge 0, \ i < j, \quad s = 1, 2, \dots, l. \\ \hat{\delta}_{ijk}^{\sigma(s)+}, \hat{\delta}_{ijk}^{\sigma(s)-} \ge 0, \quad i < j < k \end{cases} \tag{M-4}$$

Combined with the optimal solutions of model M-4, i.e., $\hat{r}_{ij}^{\sigma(s)*}$, and $\hat{r}_{ij}^{\sigma(s)*} = 1 + \log_9^{\hat{h}_{ij}^{\sigma(s)*}}$, the acceptably consistent NHMPR $\hat{\boldsymbol{H}}$ can be obtained based on Definition 6 as in the following.

$$\hat{h}_{ij} = \begin{cases} \left\{9^{\hat{r}_{ij}^{\sigma(s)*}-1} | s = 1, 2, \dots, l\right\}, & if\, k \ne j; \\ \{i\} & if\, i = j. \end{cases} \tag{15}$$

Remark 1.

In practice, if we aim at the completely consistent NHMPR to implement the subsequent decision making, i.e., $CI\left(\hat{\boldsymbol{H}}\right) = 0$. Then, let $CI_0 = 0$ in model M-4, we have $\hat{\delta}_{ijk}^{\sigma(s)+} = \hat{\delta}_{ijk}^{\sigma(s)-} = 0\ (i < j < k)$; the model M-4 can be transformed into

$$\min \frac{1}{ln(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{l} \left(\hat{\varepsilon}_{ij}^{\sigma(s)+} + \hat{\varepsilon}_{ij}^{\sigma(s)-}\right)$$

$$s.t. \begin{cases} r_{ij}^{\sigma(s)} - \hat{r}_{ij}^{\sigma(s)} + \hat{\varepsilon}_{ij}^{\sigma(s)-} - \hat{\varepsilon}_{ij}^{\sigma(s)+} = 0, \\ i < j, \quad s = 1, 2, \dots, l. \\ \left(\hat{r}_{ij}^{\sigma(s)} + \hat{r}_{jk}^{\sigma(s)} + \hat{t}_{ki}^{\sigma(s)}\right) - \left(\hat{r}_{ik}^{\sigma(s)} + \hat{r}_{kj}^{\sigma(s)} + \hat{r}_{ji}^{\sigma(s)}\right) = 0, \\ i < j < k, \quad s = 1, 2, \dots, l. \\ \hat{r}_{ij}^{\sigma(s)} - \hat{r}_{ij}^{\sigma(s+1)} \le 0, \ i < j, \quad s = 1, 2, \dots, l. \\ \hat{r}_{ji}^{\sigma(s+1)} - \hat{r}_{ji}^{\sigma(s)} \le 0, \ i < j, \quad s = 1, 2, \dots, l. \\ 0 \le \hat{r}_{ij}^{\sigma(s)}, \hat{r}_{ji}^{\sigma(s)} \le 2, \quad i < j, \quad s = 1, 2, \dots, l. \\ \hat{r}_{ij}^{\sigma(s)} + \hat{r}_{ji}^{\sigma(s)} = 2, \quad i < j, \quad s = 1, 2, \dots, l. \\ \hat{\varepsilon}_{ij}^{\sigma(s)-}, \hat{\varepsilon}_{ij}^{\sigma(s)+} \ge 0, \ i < j, \quad s = 1, 2, \dots, l. \end{cases} \tag{M-5}$$

Thus, according to the solutions of model M-5, the completely multiplicative consistent NHMPR can be obtained from the original NHMPR $\overline{\boldsymbol{H}}$.

3.3. Decision Making Approach With an HMPR

With respect to a decision making issue, suppose that decision maker evaluates several alternatives $A_i\ (i = 1, 2, \dots, n)$ by using the HMPR; then an HMPR $\boldsymbol{H} = \left(h_{ij}\right)_{n \times n}$ can be constructed by pairwise comparison. In this subsection, a decision making approach with an HMPR combined with the proposed consistency-based programming model, i.e., Algorithm 1, is proposed as in the following, and the flowchart of Algorithm 1 is presented in Fig. 2.

Algorithm 1:

Step 1: Utilize an optimized parameter δ to obtain the NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n \times n}$ from the HMPR $\boldsymbol{H}$.

Step 2: Compute the consistency index of the NHMPR $\overline{\boldsymbol{H}}$ by Eq. (14). If $CI\left(\overline{\boldsymbol{H}}\right) \le CI_0$, then proceed to Step 5; otherwise, proceed to the next step.

Step 3: Obtain the acceptably consistent HMPR $\hat{\boldsymbol{H}} = \left(\hat{h}_{ij}\right)_{n \times n}$ according to model M-4.

Step 4: Construct the improved score matrix $\boldsymbol{S}\left(\hat{\boldsymbol{H}}\right) = \left(\hat{s}_{ij}\right)_{n \times n}$ combined with Definition 7, where $\hat{s}_{ij}$ is the improved score value of NHME $\hat{h}_{ij}$.

Step 5: Utilize the method developed by Crawford and Williams [46] to compute the priority weights of alternatives as

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} \hat{s}_{ij}}}{\sum_{i=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \hat{s}_{ij}}}. \tag{16}$$

Step 6: Determine the ranking of alternatives based on their priority weights.

4. A NEW CONSENSUATED GDM APPROACH WITH HMPRs

Suppose that a team of decision makers $D_p\ (p = 1, 2, \dots, m)$ evaluates a collection of alternatives $A_i\ (i = 1, 2, \dots, n)$ using HMPRs, and $\lambda = (\lambda_1, \lambda_2, \dots \lambda_m)^T$ is the weight vector of decision makers, which satisfies the conditions of $0 \le \lambda_p \le 1$ and $\sum_{p=1}^{m} \lambda_p = 1$. Thus, let $\boldsymbol{H^p} = \left(h_{ij}^p\right)_{n \times n}$ be the individual HMPRs, where $h_{ij}^p = \left\{h_{ij}^{p,s} | s = 1, 2, \dots, l_{h_{ij}^p}\right\}$ represents the possible preference values of decision maker D_p; subsequently, we can transform the individual HMPRs $\boldsymbol{H^p}$ into the NHMPRs $\overline{\boldsymbol{H}}^{\boldsymbol{p}} = \left(\overline{h}_{ij}^p\right)_{n \times n}$ combined with an optimized parameter δ, where $\overline{h}_{ij}^p = \left\{\overline{h}_{ij}^{p,\sigma(s)} | s = 1, 2, \dots, l\right\}$ and l is the unified number of elements in $\overline{h}_{ij}^p$.

In general, consensus reaching process is a critical step during the normal GDM procedure, in which the consensus level between

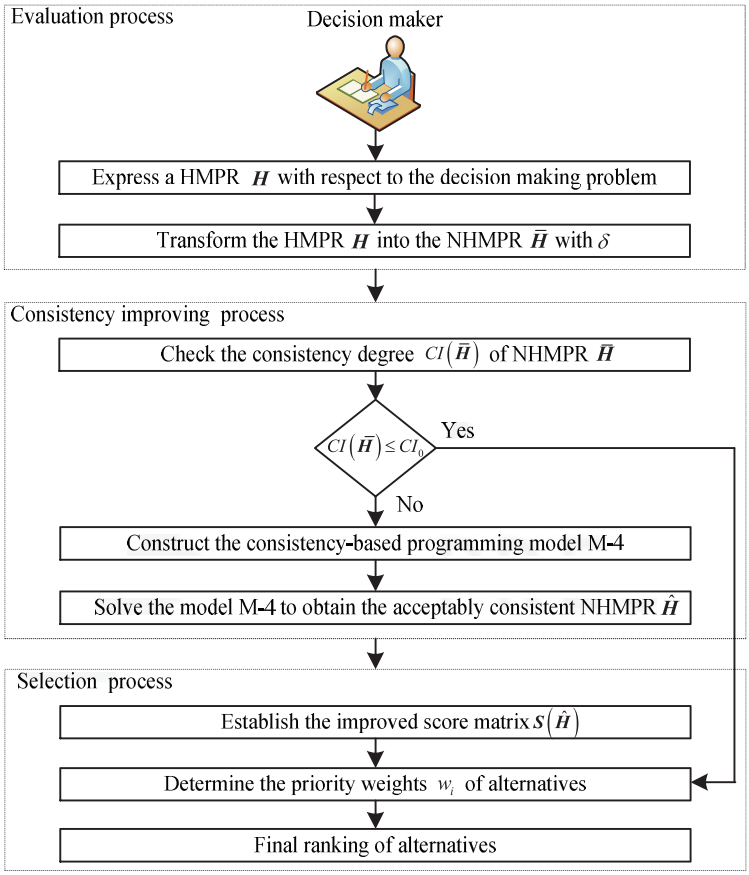

Figure 2 | Flowchart of Algorithm 1.

decision makers will affect the accuracy of the ranking result [52]. Therefore, this section develops a consensus measure of NHMPRs to check whether the consensus level among decision makers has satisfied the predefined requirement. Subsequently, a consistency- and consensus-based programming model is proposed to improve the consistency degrees of individual NHMPRs and consensus level among decision makers, simultaneously; afterwards, the revised NHMPRs can be obtained by solving the model. Finally, combined with the NHMWG operator, the best alternative can be determined.

4.1. Consensus Measure of NHMPRs

To define the consensus measure of NHMPRs, we first develop the similarity and proximity measures between the NHMPRs of different decision makers as in the following.

Definition 13. Let $\overline{H}^p$ and $\overline{H}^q$ be two NHMPRs, then the similarity measure between them is given by

$$SM\left(\overline{H}^p, \overline{H}^q\right) = 1 - d\left(\overline{H}^p, \overline{H}^q\right), \tag{17}$$

where $d\left(\overline{H}^p, \overline{H}^q\right)$ is the distance measure defined in Definition 8. Substituting Eq. (10) into the equation above; then, the similarity measure can be converted into

$$SM\left(\overline{H}^p, \overline{H}^q\right) = 1 - \frac{1}{ln\,(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{l} \left| \log_9 \overline{h}_{ij}^{p,\sigma(s)} - \log_9 \overline{h}_{ij}^{q,\sigma(s)} \right|. \tag{18}$$

Furthermore, the proximity measure between decision maker D_p and other decision makers is given by

$$PM\left(\overline{H}^q\right) = \frac{1}{m-1} \sum_{p=1, p\neq q}^{m} SM\left(\overline{H}^p, \overline{H}^q\right). \tag{19}$$

Similarly, substituting the similarity measures into the equation above, we can obtain the proximity measure as

$$PM\left(\overline{H}^q\right) = 1 - \frac{1}{ln\,(m-1)\,(n-1)} \sum_{p=1, p\neq q}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{l} \left| \log_9 \overline{h}_{ij}^{p,\sigma(s)} - \log_9 \overline{h}_{ij}^{q,\sigma(s)} \right|. \tag{20}$$

Subsequently, the consensus measure of NHMPRs is presented as below.

Definition 14. Let $\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m$ be the NHMPRs derived from the HMPRs of decision makers, then the consensus measure can be given by

$$CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m\right) = \frac{1}{m}\sum_{q=1}^{m} PM\left(\overline{\boldsymbol{H}}^q\right) = \frac{1}{m(m-1)}\sum_{q=1}^{m}\sum_{p=1,p\neq q}^{m} SM\left(\overline{\boldsymbol{H}}^p, \overline{\boldsymbol{H}}^q\right). \tag{21}$$

Because of $SM\left(\overline{\boldsymbol{H}}^p, \overline{\boldsymbol{H}}^q\right) = SM\left(\overline{\boldsymbol{H}}^q, \overline{\boldsymbol{H}}^p\right)$, Eq. (21) can be simplified to

$$CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m\right) = \frac{2}{m(m-1)}\sum_{q=1}^{m-1}\sum_{p=q+1}^{m} SM\left(\overline{\boldsymbol{H}}^p, \overline{\boldsymbol{H}}^q\right) \tag{22}$$

Then, substituting the similarity measures into Eq. (22), we can obtain the consensus measure as

$$CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m\right) = 1 - \frac{2}{lmn(m-1)(n-1)}\sum_{q=1}^{m-1}\sum_{p=q+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{l}\left|\log_9^{\overline{h}_{ij}^{p,\sigma(s)}} - \log_9^{\overline{h}_{ij}^{q,\sigma(s)}}\right|. \tag{23}$$

Remark 2.

The consensus measure $CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m\right)$ can be utilized to check the agreement degree of decision makers in practice; the larger the $CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m\right)$, the higher the consensus degree among decision makers. We can check the consensus degree by a comparison between the $CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m\right)$ and a predefined consensus threshold $CM_0 \in (0,1)$; if $CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \ldots, \overline{\boldsymbol{H}}^m\right) \geq CM_0$, then the acceptable consensus level has been reached between NHMPRs $\overline{\boldsymbol{H}}^p$, which can be utilized to obtain the ranking of alternatives during the selection process.

4.2. Improve Consistency and Consensus Degrees of HMPRs

For the GDM problem with HMPRs, a collection of NHMPRs $\overline{\boldsymbol{H}}^p$ can be obtained from the HMPRs given by decision makers. Before aggregating them to determine the ranking of alternatives, two important issues should be confirmed: (1) whether the individual NHMPRs $\overline{\boldsymbol{H}}^p$ have achieved the consistency threshold, and (2) whether a relatively high consensus level has been reached between decision makers. This paper aims to solve these two problems simultaneously, which significantly improve the efficiency of GDM process; then, the GDM issue can be solved combined with the NHMWG operator. The flowchart of the novel GDM process with HMPRs is presented in Fig. 3.

Consequently, a mathematical programming model is developed to improve the consistency and consensus degrees of NHMPRs, while the original preference information can remain as much as possible.

$$\min \frac{1}{m}\sum_{p=1}^{m} d\left(\overline{\mathbf{H}}^p, \hat{\boldsymbol{H}}^p\right)$$

$$s.t.\begin{cases} CI\left(\hat{\boldsymbol{H}}^p\right) \leq CI_0, p = 1,2,\ldots,m. \\ CM\left(\hat{\boldsymbol{H}}^1, \hat{\boldsymbol{H}}^2, \ldots, \hat{\boldsymbol{H}}^m\right) \geq CM_0 \\ \hat{\boldsymbol{H}}^p \text{ is an NHMPR}, p = 1,2,\ldots,m. \end{cases} \tag{M-6}$$

Combined with Eqs. (10), (14), and (23), model M-6 can be transformed into

$$\min \frac{1}{lmn(n-1)}\sum_{p=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{l}\left|\log_9^{\overline{h}_{ij}^{p,\sigma(s)}} - \log_9^{\hat{h}_{ij}^{p,\sigma(s)}}\right|$$

$$s.t.\begin{cases} \sum_{1\leq i<j<k\leq n}\sum_{s=1}^{l}\left|\left(\log_9^{\hat{h}_{ij}^{p,\sigma(s)}} + \log_9^{\hat{h}_{jk}^{p,\sigma(s)}} + \log_9^{\hat{h}_{ki}^{p,\sigma(s)}}\right) - \left(\log_9^{\hat{h}_{ik}^{p,\sigma(s)}} + \log_9^{\hat{h}_{kj}^{p,\sigma(s)}} + \log_9^{\hat{h}_{ji}^{p,\sigma(s)}}\right)\right| \leq 6\cdot CI_0\cdot C_n^3\cdot l, \\ p = 1,2,\ldots,m. \\ 1 - \frac{2}{lmn(m-1)(n-1)}\sum_{q=1}^{m-1}\sum_{p=q+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{l}\left|\log_9^{\overline{h}_{ij}^{p,\sigma(s)}} - \log_9^{\overline{h}_{ij}^{q,\sigma(s)}}\right| \geq CM_0 \\ \hat{h}_{ij}^{p,\sigma(s)} - \hat{h}_{ij}^{p,\sigma(s+1)} \leq 0,\ \ i<j,\ \ s = 1,2,\ldots,l,\ \ p = 1,2,\ldots,m. \\ \hat{h}_{ji}^{p,\sigma(s+1)} - \hat{h}_{ji}^{p,\sigma(s)} \leq 0,\ \ i<j,\ \ s = 1,2,\ldots,l,\ \ p = 1,2,\ldots,m. \\ \frac{1}{9} \leq \hat{h}_{ij}^{p,\sigma(s)}, \hat{h}_{ji}^{p,\sigma(s)} \leq 9,\ \ i<j,\ \ s = 1,2,\ldots,l,\ \ p = 1,2,\ldots,m. \\ \hat{h}_{ij}^{p,\sigma(s)} \times \hat{h}_{ji}^{p,\sigma(s)} = 1,\ \ i<j,\ \ s = 1,2,\ldots,l,\ \ p = 1,2,\ldots,m. \end{cases} \tag{M-7}$$

Let $t_{ij}^{p,\sigma(s)} = 1 + \log_9^{\overline{h}_{ij}^{p,\sigma(s)}}$ and $\hat{t}_{ij}^{p,\sigma(s)} = 1 + \log_9^{\hat{h}_{ij}^{p,\sigma(s)}}$, we can rewrite model M-7 into the following form.

$$\min \frac{1}{lmn(n-1)}\sum_{p=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{l}\left|t_{ij}^{p,\sigma(s)} - \hat{t}_{ij}^{p,\sigma(s)}\right|$$

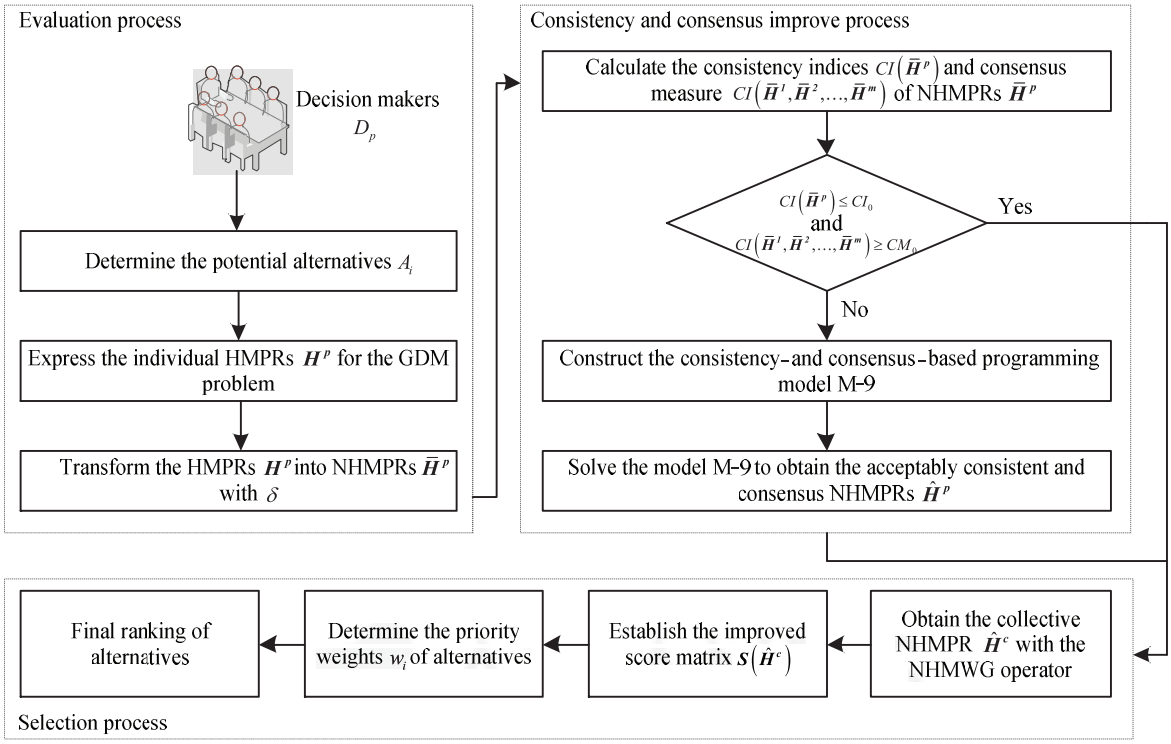

Figure 3 Flowchart of the novel group decision making (GDM) process with hesitant multiplicative preference relations (HMPRs).

$$
s.t.\begin{cases}
\sum\limits_{1\le i<j<k\le n}\sum\limits_{s=1}^{l}\left|\left(\hat{t}_{ij}^{p,\sigma(s)}+\hat{t}_{jk}^{p,\sigma(s)}+\hat{t}_{ki}^{p,\sigma(s)}\right)-\right.\\
\left.\left(\hat{t}_{ik}^{p,\sigma(s)}+\hat{t}_{kj}^{p,\sigma(s)}+\hat{t}_{ji}^{p,\sigma(s)}\right)\right|\le 6\cdot CI_0\cdot C_n^3\cdot l, p=1,2,\ldots,m.\\
1-\dfrac{2}{lmn(m-1)(n-1)}\sum\limits_{q=1}^{m-1}\sum\limits_{p=q+1}^{m}\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n}\sum\limits_{s=1}^{l}\left|\hat{t}_{ij}^{p,\sigma(s)}-\hat{t}_{ij}^{q,\sigma(s)}\right|\\
\ge CM_0\\
9^{\hat{t}_{ij}^{p,\sigma(s)}-1}-9^{\hat{t}_{ij}^{p,\sigma(s+1)}-1}\le 0,\ i<j,\ s=1,2,\ldots,l,\ p=1,2,\ldots,m.\\
9^{\hat{t}_{ji}^{p,\sigma(s+1)}-1}-9^{\hat{t}_{ji}^{p,\sigma(s)}-1}\le 0,\ i<j,\ s=1,2,\ldots,l,\ p=1,2,\ldots,m.\\
0\le \hat{t}_{ij}^{p,\sigma(s)},\hat{t}_{ji}^{p,\sigma(s)}\le 2,\ \ i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m.\\
9^{\hat{t}_{ij}^{p,\sigma(s)}-1}\times 9^{\hat{t}_{ji}^{p,\sigma(s)}-1}=1,\ i<j,\ s=1,2,\ldots,l,\ p=1,2,\ldots,m.
\end{cases}
\tag{M-8}
$$

Denote $\hat{\eta}_{ij}^{p,\sigma(s)}=t_{ij}^{p,\sigma(s)}-\hat{t}_{ij}^{p,\sigma(s)}$, $\hat{\xi}_{ijk}^{p,\sigma(s)}=\left(\hat{t}_{ij}^{p,\sigma(s)}+\hat{t}_{jk}^{p,\sigma(s)}+\hat{t}_{ki}^{p,\sigma(s)}\right)-\left(\hat{t}_{ik}^{p,\sigma(s)}+\hat{t}_{kj}^{p,\sigma(s)}+\hat{t}_{ji}^{p,\sigma(s)}\right)$, $\hat{\varphi}_{ij}^{pq,\sigma(s)}=\hat{t}_{ij}^{p,\sigma(s)}-\hat{t}_{ij}^{q,\sigma(s)}$, $\hat{\eta}_{ij}^{p,\sigma(s)+}=\left(\left|\hat{\eta}_{ij}^{p,\sigma(s)}\right|+\hat{\eta}_{ij}^{p,\sigma(s)}\right)/2$, $\hat{\eta}_{ij}^{p,\sigma(s)-}=\left(\left|\hat{\eta}_{ij}^{p,\sigma(s)}\right|-\hat{\eta}_{ij}^{p,\sigma(s)}\right)/2$, $\hat{\xi}_{ijk}^{p,\sigma(s)+}=\left(\left|\hat{\xi}_{ijk}^{p,\sigma(s)}\right|+\hat{\xi}_{ijk}^{p,\sigma(s)}\right)/2$, $\hat{\xi}_{ijk}^{p,\sigma(s)-}=\left(\left|\hat{\xi}_{ijk}^{p,\sigma(s)}\right|-\hat{\xi}_{ijk}^{p,\sigma(s)}\right)/2$, $\hat{\varphi}_{ij}^{pq,\sigma(s)+}=\left(\left|\hat{\varphi}_{ij}^{pq,\sigma(s)}\right|+\hat{\varphi}_{ij}^{pq,\sigma(s)}\right)/2$ and $\hat{\varphi}_{ij}^{pq,\sigma(s)-}=\left(\left|\hat{\varphi}_{ij}^{pq,\sigma(s)}\right|-\hat{\varphi}_{ij}^{pq,\sigma(s)}\right)/2$, then $\left|\hat{\eta}_{ij}^{p,\sigma(s)}\right|=\hat{\eta}_{ij}^{p,\sigma(s)+}+\hat{\eta}_{ij}^{p,\sigma(s)-}$, $\left|\hat{\xi}_{ijk}^{p,\sigma(s)}\right|=\hat{\xi}_{ijk}^{p,\sigma(s)+}+\hat{\xi}_{ijk}^{p,\sigma(s)-}$ and $\left|\hat{\varphi}_{ij}^{pq,\sigma(s)}\right|=\hat{\varphi}_{ij}^{pq,\sigma(s)+}+\hat{\varphi}_{ij}^{pq,\sigma(s)-}$. Subsequently, we can transform model M-8 into

$$
\min\frac{1}{lmn(n-1)}\sum_{p=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{l}\left(\hat{\eta}_{ij}^{p,\sigma(s)+}+\hat{\eta}_{ij}^{p,\sigma(s)-}\right)
$$

$$
s.t.\begin{cases}
\sum\limits_{1\le i<j<k\le n}\sum\limits_{s=1}^{l}\left(\hat{\xi}_{ijk}^{p,\sigma(s)+}+\hat{\xi}_{ijk}^{p,\sigma(s)-}\right)\le 6\cdot CI_0\cdot C_n^3\cdot l,\\
p=1,2,\ldots,m.\\
1-\dfrac{2}{lmn(m-1)(n-1)}\sum\limits_{q=1}^{m-1}\sum\limits_{p=q+1}^{m}\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n}\sum\limits_{s=1}^{l}\left(\hat{\varphi}_{ij}^{pq,\sigma(s)+}\right.\\
\left.+\ \hat{\varphi}_{ij}^{pq,\sigma(s)-}\right)\ge CM_0\\
t_{ij}^{p,\sigma(s)}-\hat{t}_{ij}^{p,\sigma(s)}-\hat{\eta}_{ij}^{p,\sigma(s)+}+\hat{\eta}_{ij}^{p,\sigma(s)-}=0,\\
i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m.\\
\left(\hat{t}_{ij}^{p,\sigma(s)}+\hat{t}_{jk}^{p,\sigma(s)}+\hat{t}_{ki}^{p,\sigma(s)}\right)-\left(\hat{t}_{ik}^{p,\sigma(s)}+\hat{t}_{kj}^{p,\sigma(s)}+\hat{t}_{ji}^{p,\sigma(s)}\right)-\\
\hat{\xi}_{ijk}^{p,\sigma(s)+}+\hat{\xi}_{ijk}^{p,\sigma(s)-}=0,\ i<j<k,\ \ s=1,2,\ldots,l,\\
p=1,2,\ldots,m.\\
\hat{t}_{ij}^{p,\sigma(s)}-\hat{t}_{ij}^{q,\sigma(s)}-\hat{\varphi}_{ij}^{pq,\sigma(s)+}+\hat{\varphi}_{ij}^{pq,\sigma(s)-}=0,\\
i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m-1,\ \ p<q.\\
\hat{t}_{ij}^{p,\sigma(s)}-\hat{t}_{ij}^{p,\sigma(s+1)}\le 0,\ i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m.\\
\hat{t}_{ji}^{p,\sigma(s+1)}-\hat{t}_{ji}^{p,\sigma(s)}\le 0,\ i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m.\\
0\le \hat{t}_{ij}^{p,\sigma(s)},\hat{t}_{ji}^{p,\sigma(s)}\le 2,\ \ i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m.\\
\hat{t}_{ij}^{p,\sigma(s)}+\hat{t}_{ji}^{p,\sigma(s)}=2,\ \ i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m.\\
\hat{\eta}_{ij}^{p,\sigma(s)+},\hat{\eta}_{ij}^{p,\sigma(s)-}\ge 0,\ \ i<j,\ \ s=1,2,\ldots,l,\ \ p=1,2,\ldots,m.\\
\hat{\xi}_{ijk}^{p,\sigma(s)+},\hat{\xi}_{ijk}^{p,\sigma(s)-}\ge 0,\ i<j<k,\ \ s=1,2,\ldots,l,\\
p=1,2,\ldots,m.\\
\hat{\varphi}_{ij}^{pq,\sigma(s)+},\hat{\varphi}_{ij}^{pq,\sigma(s)-}\ge 0,\ i<j,\ \ s=1,2,\ldots,l,\\
p=1,2,\ldots,m-1,\ \ p<q.
\end{cases}
\tag{M-9}
$$

Then, we can obtain the optimal solutions as $\hat{t}_{ij}^{p,\sigma(s)*}$ by solving model M-9. From $\hat{t}_{ij}^{p,\sigma(s)*} = 1 + \log_9^{\hat{h}_{ij}^{p,\sigma(s)*}}$, we can complement the elements of $\hat{\boldsymbol{H}}^{\boldsymbol{p}}$ to obtain the acceptably consistent and consensus NHMPR of decision maker D_p as

$$\hat{h}_{ij}^{p} = \begin{cases} \left\{9^{\hat{t}_{ij}^{p,\sigma(s)*}-1} | s = 1, 2, \dots, l\right\}, \text{ if } i \neq j; \\ \{1\}, \text{ if } i = j. \end{cases} \tag{24}$$

Then, we can aggregate the acceptably consistent and consensus NHMPRs to determine the best alternative during the selection process.

4.3. Aggregate the Individual NHMPRs

The aggregation tools play a critical role during the selection process, Zhang and Wu [24] defined the concept of NHMPRs and proposed the HMWG operator to aggregate them; hence, we can transform the HMWG operator into the following form, i.e., the NHMWG operator, to complement the selection process in this paper.

Definition 15. [24] Let $h_j\left(j = 1, 2, \dots, n\right)$ be a collection of HMEs, then the NHMEs $\overline{h}_j$ can be obtained with an optimized parameter δ, the NHMWG operator is a mapping $\boldsymbol{H}^{\boldsymbol{n}} \to \boldsymbol{H}$ as

$$NHMWG\left(\overline{h}_1, \overline{h}_2, \dots, \overline{h}_n\right) = \bigotimes_{j=1}^{n} \overline{h}_j^{w_j} = \tag{25}$$

$$\bigcup_{s=1,2,\dots,l} \left\{\prod_{j=1}^{n} \left(\overline{h}_j^{\sigma(s)}\right)^{w_j}\right\},$$

where $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ is the weight vector of NHMEs $\overline{h}_j$, in which $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$, and l is the unified number of elements in $\overline{h}_j$.

Theorem 5. *Suppose that $\hat{\boldsymbol{H}}^{\boldsymbol{p}}\left(p = 1, 2, \dots, m\right)$ be a collection of individual acceptably consistent NHMPRs, then the collective NHMPR $\hat{\boldsymbol{H}}^{\boldsymbol{c}} = \left(\hat{h}_{ij}^{c}\right)_{n\times n}$ obtained by the NHMWG operator is also acceptably consistent and satisfies the condition as*

$$CI\left(\hat{\boldsymbol{H}}^{\boldsymbol{c}}\right) \leq \max_p \left\{CI\left(\hat{\boldsymbol{H}}^{\boldsymbol{p}}\right)\right\}. \tag{26}$$

Proof. According to the consistency index of NHMPR, we have

$$CI\left(\hat{\boldsymbol{H}}^{\boldsymbol{c}}\right) = \frac{1}{6 \cdot C_n^3 \cdot l} \sum_{i<j<k} \sum_{s=1}^{l} \left| \left(\log_9^{\hat{h}_{ij}^{c,\sigma(s)}} + \log_9^{\hat{h}_{jk}^{c,\sigma(s)}} + \log_9^{\hat{h}_{ki}^{c,\sigma(s)}}\right) - \left(\log_9^{\hat{h}_{ik}^{c,\sigma(s)}} + \log_9^{\hat{h}_{kj}^{c,\sigma(s)}} + \log_9^{\hat{h}_{ji}^{c,\sigma(s)}}\right) \right|.$$

Because of the NHMPRs $\hat{\boldsymbol{H}}^{\boldsymbol{p}}$ are aggregated by the NHMWG operator, then

$$\begin{aligned} CI\left(\hat{\boldsymbol{H}}^{\boldsymbol{c}}\right) &= \frac{1}{6 \cdot C_n^3 \cdot l} \sum_{i<j<k} \sum_{s=1}^{l} \left| \left(\log_9^{\left(\prod_{p=1}^{m}\left(\hat{h}_{ij}^{p,\sigma(s)}\right)^{\lambda_p}\right)} + \log_9^{\left(\prod_{p=1}^{m}\left(\hat{h}_{jk}^{p,\sigma(s)}\right)^{\lambda_p}\right)} + \log_9^{\left(\prod_{p=1}^{m}\left(\hat{h}_{ki}^{p,\sigma(s)}\right)^{\lambda_p}\right)}\right) \right. \\ &\quad \left. - \left(\log_9^{\left(\prod_{p=1}^{m}\left(\hat{h}_{ik}^{p,\sigma(s)}\right)^{\lambda_p}\right)} + \log_9^{\left(\prod_{p=1}^{m}\left(\hat{h}_{kj}^{p,\sigma(s)}\right)^{\lambda_p}\right)} + \log_9^{\left(\prod_{p=1}^{m}\left(\hat{h}_{ji}^{p,\sigma(s)}\right)^{\lambda_p}\right)}\right) \right| \\ &= \frac{1}{6 \cdot C_n^3 \cdot l} \sum_{i<j<k} \sum_{s=1}^{l} \sum_{p=1}^{m} \lambda_p \left| \left(\log_9^{\hat{h}_{ij}^{p,\sigma(s)}} + \log_9^{\hat{h}_{jk}^{p,\sigma(s)}} + \log_9^{\hat{h}_{ki}^{p,\sigma(s)}}\right) \right. \\ &\quad \left. - \left(\log_9^{\hat{h}_{ik}^{p,\sigma(s)}} + \log_9^{\hat{h}_{kj}^{p,\sigma(s)}} + \log_9^{\hat{h}_{ji}^{p,\sigma(s)}}\right) \right| \\ &\leq \frac{1}{6 \cdot C_n^3 \cdot l} \sum_{i<j<k} \sum_{s=1}^{l} \left(\max_p \left\{ \left| \left(\log_9^{\hat{h}_{ij}^{p,\sigma(s)}} + \log_9^{\hat{h}_{jk}^{p,\sigma(s)}} + \log_9^{\hat{h}_{ki}^{p,\sigma(s)}}\right) \right. \right.\right. \\ &\quad \left.\left.\left. - \left(\log_9^{\hat{h}_{ik}^{p,\sigma(s)}} + \log_9^{\hat{h}_{kj}^{p,\sigma(s)}} + \log_9^{\hat{h}_{ji}^{p,\sigma(s)}}\right) \right| \right\}\right) \\ &= \max_p \left\{CI\left(\hat{\boldsymbol{H}}^{\boldsymbol{p}}\right)\right\}. \end{aligned}$$

Thus, we can obtain that $CI\left(\hat{\boldsymbol{H}}^{\boldsymbol{c}}\right) \leq \max_p \left\{CI\left(\hat{\boldsymbol{H}}^{\boldsymbol{p}}\right)\right\}$.

Remark 3.

Theorem 5 shows that the consistency index of collective NHMPR aggregated by the NHMWG operator is smaller than the maximum consistency index of the individual NHMPRs. In other words, an acceptably consistent MHMPR $\hat{\boldsymbol{H}}^{\boldsymbol{c}}$ can be determined by aggregating the acceptably consistent NHMPRs $\hat{\boldsymbol{H}}^{\boldsymbol{p}}$ with the NHMWG operator.

4.4. A Novel Approach for GDM With HMPRs

Combined with the aforementioned consistency and consensus analysis, a novel approach for GDM, i.e., Algorithm 2, is developed as in the following.

5. NUMERICAL EXAMPLES

Two numerical examples implemented in the literature [27,24] are introduced to show the feasibility of the Algorithms 1 and 2 proposed in this paper, respectively. Later, several existing decision making approaches are applied to obtain the rankings of the same numerical examples; thus, the advantages of the proposed approaches are presented with the comparison and discussion analysis.

Algorithm 2:

Step 1: Decision makers express their HMPRs $\boldsymbol{H^p}$; then, after the normalization procedure, the corresponding NHMPRs $\overline{\boldsymbol{H}}^{\boldsymbol{p}}$ can be derived with an optimized parameter δ, where $\overline{h}_{ij}^{p} = \left\{\overline{h}_{ij}^{p,\sigma(s)} | s = 1, 2, \ldots, l\right\}$ and l is the unified number of elements in $\overline{h}_{ij}^{p}$.

Step 2: Set the consistency threshold CI_0 and consensus threshold CM_0, respectively. Then, check the consistency and consensus degrees of NHMPRs $\overline{\boldsymbol{H}}^{\boldsymbol{p}}$. If $CI\left(\overline{\boldsymbol{H}}^{\boldsymbol{p}}\right) \leq CI_0$ and $CI\left(\overline{\boldsymbol{H}}^{\boldsymbol{1}}, \overline{\boldsymbol{H}}^{\boldsymbol{2}}, \ldots, \overline{\boldsymbol{H}}^{\boldsymbol{m}}\right) \geq CM_0$, then let $\hat{\boldsymbol{H}}^{\boldsymbol{p}} = \overline{\boldsymbol{H}}^{\boldsymbol{p}}$ and proceed to Step 5; otherwise, proceed to the next step.

Step 3: Construct the consistency- and consensus-based mathematical programming model M-9.

Step 4: Solve model M-9 and acquire the acceptably consistent and consensus NHMPRs $\hat{\boldsymbol{H}}^{\boldsymbol{p}} = \left(\hat{h}_{ij}^{p}\right)_{n\times n}$ with Eq. (24).

Step 5: According to the weights of decision makers, the collective NHMPR $\hat{\boldsymbol{H}}^{\boldsymbol{c}} = \left(\hat{h}_{ij}^{c}\right)_{n\times n}$ can be determined by the NHMWG operator.

Step 6: Construct the improved score matrix $\boldsymbol{S}\left(\hat{\boldsymbol{H}}^{\boldsymbol{c}}\right) = \left(\hat{s}_{ij}^{c}\right)_{n\times n}$ combined with Definition 7, where $\hat{s}_{ij}^{c}$ is the improved score value of $\hat{h}_{ij}^{c}$.

Step 7: Utilize the method developed by Crawford and Williams [46] to compute the priority weights of alternatives as

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} \hat{s}_{ij}^{c}}}{\sum_{i=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \hat{s}_{ij}^{c}}}. \tag{27}$$

Step 8: Determine the ranking of alternatives based on their priority weights.

5.1. Implementation

Example 1. Suppose that an enterprise plans to build the enterprise resource planning system; then, four potential enterprise resource planning systems in the market are determined as the alternatives, i.e., $A = \{A_1, A_2, A_3, A_4\}$. Decision makers in the enterprise evaluate the four alternatives using HMPR; then, the HMPR $\boldsymbol{H} = \left(h_{ij}\right)_{n\times n} \left(i, j = 1, 2, 3, 4; n = 4\right)$ is obtained as [27]:

$$H = \begin{pmatrix} \{1\} & \left\{\frac{1}{7}, \frac{1}{6}, \frac{1}{5}\right\} & \{1, 3\} & \left\{\frac{1}{8}, \frac{2}{7}\right\} \\ \{5, 6, 7\} & \{1\} & \{1, 2\} & \{7, 9\} \\ \left\{\frac{1}{3}, 1\right\} & \left\{\frac{1}{2}, 1\right\} & \{1\} & \left\{\frac{1}{4}\right\} \\ \left\{\frac{7}{2}, 8\right\} & \left\{\frac{1}{9}, \frac{1}{7}\right\} & \{4\} & \{1\} \end{pmatrix}.$$

Subsequently, the best alternative can be obtained using the proposed decision making method, i.e., Algorithm 1, which is presented as follows.

Step 1: Set the optimized parameter $\delta = 1$ and utilize δ to obtain the NHMPR $\overline{\boldsymbol{H}} = \left(\overline{h}_{ij}\right)_{n\times n}$ as

$$\overline{\boldsymbol{H}} = \begin{pmatrix} \{1\} & \left\{\frac{1}{7}, \frac{1}{6}, \frac{1}{5}\right\} & \{1, 3, 3\} & \left\{\frac{1}{8}, \frac{2}{7}, \frac{2}{7}\right\} \\ \{7, 6, 5\} & \{1\} & \{1, 2, 2\} & \{7, 9, 9\} \\ \left\{1, \frac{1}{3}, \frac{1}{3}\right\} & \left\{1, \frac{1}{2}, \frac{1}{2}\right\} & \{1\} & \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\} \\ \left\{8, \frac{7}{2}, \frac{7}{2}\right\} & \left\{\frac{1}{7}, \frac{1}{9}, \frac{1}{9}\right\} & \{4, 4, 4\} & \{1\} \end{pmatrix}.$$

Step 2: Set the consistency threshold $CI_0 = 0.1$. Combined with Eq. (14), we can check the consistency degree of NHMPR $\overline{\boldsymbol{H}}$ as $CI\left(\overline{\boldsymbol{H}}\right) = 0.2967$; obviously, $CI\left(\overline{\boldsymbol{H}}\right) > CI_0$, then proceed to the next step.

Step 3: Develop the consistency-based model, i.e., model M-4, according to the NHMPR $\overline{\boldsymbol{H}}$; afterwards, the acceptably consistent NHMPR $\hat{\boldsymbol{H}} = \left(\hat{h}_{ij}\right)_{n\times n}$ can be determined by solving model M-4 as

$$\hat{\boldsymbol{H}} = \begin{pmatrix} \{1\} & \{0.1429, 0.1667, 0.2000\} & \{0.5000, 1.4497, 1.8000\} & \{0.1250, 0.2857, 0.4500\} \\ \{6.9997, 6.0005, 5.0001\} & \{1\} & \{3.4996, 3.4996, 9.0000\} & \{0.8749, 9.0000, 9.0000\} \\ \{2.0002, 0.6898, 0.5558\} & \{0.2858, 0.2858, 0.1111\} & \{1\} & \{0.2500, 0.2500, 0.2500\} \\ \{8.0001, 3.5003, 2.2221\} & \{1.1429, 0.1111, 0.1111\} & \{3.9997, 3.9997, 3.9997\} & \{1\} \end{pmatrix}.$$

In addition, the visible approach [24], i.e., the "Figure of area," is introduced to compare the consistency levels of the original NHMPR $\overline{\boldsymbol{H}}$ and the acceptably consistent NHMPR $\hat{\boldsymbol{H}}$ as shown in Fig. 4. Then, we can see that the acceptably consistent NHMPR $\hat{\boldsymbol{H}}$ perform more regularly concerning the areas of different colors than that of the original NHMPR $\overline{\boldsymbol{H}}$, which indicates that the acceptably consistent NHMPR $\hat{\boldsymbol{H}}$ obtained by the proposed approach has a higher consistency level.

Step 4: Develop the improved score matrix $\boldsymbol{S}\left(\hat{\boldsymbol{H}}\right) = \left(\hat{s}_{ij}\right)_{n\times n}$ with Eq. (7) as

$$\boldsymbol{S}\left(\hat{\boldsymbol{H}}\right) = \begin{pmatrix} 1 & 0.1682 & 1.0927 & 0.2524 \\ 5.9440 & 1 & 4.7946 & 4.1383 \\ 0.9152 & 0.2086 & 1 & 0.2500 \\ 3.9627 & 0.2416 & 3.9997 & 1 \end{pmatrix}.$$

Step 5: Utilize the method developed by Crawford and Williams [46] to compute the priority weights of alternatives as

$$w_1 = 0.0825, w_2 = 0.5858, w_3 = 0.0831, w_4 = 0.2487.$$

Step 6: Based on the priority weights of alternatives, we can obtain the ranking as $A_2 \succ A_4 \succ A_3 \succ A_1$.

Example 2. Suppose that an investment enterprise plans to select a manufacturing company for investing; four companies have entered the vision of investors, which include a car company (A_1), a food company (A_2), a computer company (A_3), and an arms company (A_4) [24]. Four decision makers D_p $\left(p = 1, 2, 3, 4\right)$ from the investment enterprise evaluate the future developments of the aforementioned four alternatives using HMPRs, in which the weights of

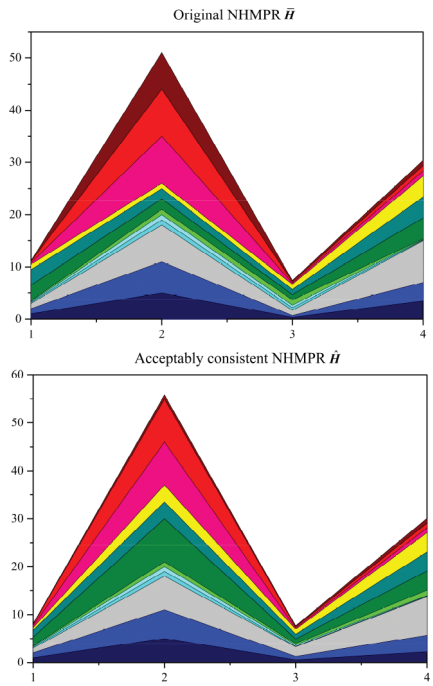

Figure 4 | Areas of $\overline{\boldsymbol{H}}$ and $\hat{\boldsymbol{H}}$.

decision makers are assumed as $\lambda = (0.1, 0.5, 0.3, 0.1)^T$ according to the literature [24]. Then, the individual HMPRs $\boldsymbol{H}^p = \left(h_{ij}^p\right)_{n\times n}$ $(i, j = 1, 2, 3, 4; n = 4)$ of each decision maker are presented as in the following.

$$\boldsymbol{H}^1 = \begin{pmatrix} \{1\} & \{3\} & \{5,7\} & \{3\} \\ \left\{\frac{1}{3}\right\} & \{1\} & \left\{\frac{1}{9},\frac{1}{7}\right\} & \{5\} \\ \left\{\frac{1}{5},\frac{1}{7}\right\} & \{9,7\} & \{1\} & \left\{\frac{1}{7},\frac{1}{5}\right\} \\ \left\{\frac{1}{3}\right\} & \left\{\frac{1}{5}\right\} & \{7,5\} & \{1\} \end{pmatrix},$$

$$\boldsymbol{H}^2 = \begin{pmatrix} \{1\} & \{3,5\} & \left\{\frac{1}{5},\frac{1}{3}\right\} & \{5\} \\ \left\{\frac{1}{3},\frac{1}{5}\right\} & \{1\} & \left\{\frac{1}{3},1\right\} & \{1,3,5\} \\ \{5,3\} & \{3,1\} & \{1\} & \{5,7,9\} \\ \left\{\frac{1}{5}\right\} & \left\{1,\frac{1}{3},\frac{1}{5}\right\} & \left\{\frac{1}{5},\frac{1}{7},\frac{1}{9}\right\} & \{1\} \end{pmatrix}$$

$$\boldsymbol{H}^3 = \begin{pmatrix} \{1\} & \left\{\frac{1}{5},\frac{1}{3}\right\} & \left\{\frac{1}{7}\right\} & \{7,9\} \\ \{5,3\} & \{1\} & \{1,3,5\} & \left\{\frac{1}{3},1\right\} \\ \{7\} & \left\{1,\frac{1}{3},\frac{1}{5}\right\} & \{1\} & \{5,7,9\} \\ \left\{\frac{1}{7},\frac{1}{9}\right\} & \{3,1\} & \left\{\frac{1}{5},\frac{1}{7},\frac{1}{9}\right\} & \{1\} \end{pmatrix},$$

$$\boldsymbol{H}^4 = \begin{pmatrix} \{1\} & \{3,5,7\} & \left\{\frac{1}{5},\frac{1}{3}\right\} & \left\{\frac{1}{7},\frac{1}{5}\right\} \\ \left\{\frac{1}{3},\frac{1}{5},\frac{1}{7}\right\} & \{1\} & \{3\} & \{5,7,9\} \\ \{5,3\} & \left\{\frac{1}{3}\right\} & \{1\} & \{5,7\} \\ \{7,5\} & \left\{\frac{1}{5},\frac{1}{7},\frac{1}{9}\right\} & \left\{\frac{1}{5},\frac{1}{7}\right\} & \{1\} \end{pmatrix}.$$

Subsequently, we can utilize the proposed GDM method, i.e., Algorithm 2, to determine the best manufacturing company for the investment enterprise, which is presented as below.

Step 1: Set the optimized parameter $\delta = 1$. Then, the corresponding NHMPRs $\overline{H}^p = \left(\overline{h}_{ij}^p\right)_{n\times n}$ can be obtained after the normalization procedure, where $\overline{h}_{ij}^p = \left\{\overline{h}_{ij}^{p,\sigma(s)} | s = 1, 2, 3\right\}$.

Step 2: Set the consistency threshold $CI_0 = 0.1$ and consensus threshold $CM_0 = 0.9$. Combined with Eqs. (14) and (23), we can obtain the consistency degrees of NHMPRs $\overline{\boldsymbol{H}}^p$ as

$$CI\left(\overline{\boldsymbol{H}}^1\right) = 0.4066, CI\left(\overline{\boldsymbol{H}}^2\right) = 0.1906,$$

$$CI\left(\overline{\boldsymbol{H}}^3\right) = 0.4042, CI\left(\overline{\boldsymbol{H}}^4\right) = 0.4707,$$

and the consensus level of NHMPRs $\overline{\boldsymbol{H}}^p$ as $CM\left(\overline{\boldsymbol{H}}^1, \overline{\boldsymbol{H}}^2, \overline{\boldsymbol{H}}^3, \overline{\boldsymbol{H}}^4\right) = 0.6099$. Obviously, both the consistency and consensus degrees of NHMPRs $\overline{\boldsymbol{H}}^p$ have not met the predefined requirements; then, proceed to the next step.

Step 3: Construct the consistency- and consensus-based mathematical programming model, i.e., model M-9, according to NHMPRs $\overline{\boldsymbol{H}}^p$.

Step 4: Solve model M-9 and acquire the acceptably consistent and consensus NHMPR $\hat{\boldsymbol{H}}^p$ with Eq. (24) as

$$\hat{\boldsymbol{H}}^1 = \begin{pmatrix} \{1\} & \{2.8909, 2.8909, 2.8909\} & \{0.6697, 2.0651, 2.0651\} & \{3.0000, 3.0000, 3.0000\} \\ \{0.3459, 0.3459, 0.3459\} & \{1\} & \{0.1655, 0.7143, 0.7143\} & \{1.6667, 5.0001, 5.0001\} \\ \{1.4933, 0.4842, 0.4842\} & \{6.0431, 1.3999, 1.3999\} & \{1\} & \{4.4799, 4.4799, 4.4799\} \\ \{0.3333, 0.3333, 0.3333\} & \{0.6000, 0.2000, 0.2000\} & \{0.2232, 0.2232, 0.2232\} & \{1\} \end{pmatrix},$$

$$\hat{\boldsymbol{H}}^2 = \begin{pmatrix} \{1\} & \{2.8909, 2.8909, 2.8909\} & \{0.4286, 0.4286, 0.4286\} & \{4.8182, 5.0001, 5.0001\} \\ \{0.3459, 0.3459, 0.3459\} & \{1\} & \{0.3333, 0.7143, 0.7143\} & \{1.6667, 3.0000, 5.0001\} \\ \{2.3332, 2.3332, 2.3332\} & \{3.0000, 1.3999, 1.3999\} & \{1\} & \{5.0001, 6.9997, 9.0000\} \\ \{0.2075, 0.2000, 0.2000\} & \{0.6000, 0.3333, 0.2000\} & \{0.2000, 0.1429, 0.1111\} & \{1\} \end{pmatrix},$$

$$\hat{H}^3 = \begin{pmatrix} \{1\} & \{0.4519, 0.4519, 0.4519\} & \{0.1506, 0.3228, 0.3228\} & \{4.8182, 5.0001, 5.0001\} \\ \{2.2127, 2.2127, 2.2127\} & \{1\} & \{0.3333, 0.7143, 0.7143\} & \{1.6667, 3.0000, 5.0001\} \\ \{6.6381, 3.0975, 3.0975\} & \{3.0000, 1.3999, 1.3999\} & \{1\} & \{5.0001, 6.9997, 9.0000\} \\ \{0.2075, 0.2000, 0.2000\} & \{0.6000, 0.3333, 0.2000\} & \{0.2000, 0.1429, 0.1111\} & \{1\} \end{pmatrix},$$

$$\hat{H}^4 = \begin{pmatrix} \{1\} & \{2.8909, 2.8909, 2.8909\} & \{0.4286, 0.4286, 0.4286\} & \{2.1430, 3.0000, 3.0000\} \\ \{0.3459, 0.3459, 0.3459\} & \{1\} & \{0.3333, 0.7143, 0.7143\} & \{1.6667, 5.0001, 5.0001\} \\ \{2.3332, 2.3332, 2.3332\} & \{3.0000, 1.3999, 1.3999\} & \{1\} & \{5.0001, 6.9997, 6.9997\} \\ \{0.4666, 0.3333, 0.3333\} & \{0.6000, 0.2000, 0.2000\} & \{0.2000, 0.1429, 0.1429\} & \{1\} \end{pmatrix}.$$

To compare the consistency levels of the original NHMPRs and the acceptably consistent NHMPRs visibly, the "Figure of area" [24] of different NHMPRs are presented in Figs. 5 and 6. Obviously, Figs. 5 and 6 show that the consistency levels of the acceptably consistent NHMPRs determined by the proposed approach have been improved significantly. Moreover, the proposed GDM approach considers the consensus issue between decision makers, the acceptably consistent NHMPRs also have a relatively high consensus degree. Thus, the "Figure of area" of the acceptably consistent NHMPRs are much more similar than that of the original NHMPRs.

Step 5: According to the weights of decision makers, the collective NHMPR $\hat{H}^c$ can be determined by the NHMWG operator as

$$\hat{H}^c = \begin{pmatrix} \{1\} & \{1.6567, 1.6567, 1.6567\} & \{0.3275, 0.4607, 0.4607\} & \{4.2376, 4.5145, 4.5145\} \\ \{0.6036, 0.6036, 0.6036\} & \{1\} & \{0.3108, 0.7143, 0.7143\} & \{1.6667, 3.2227, 5.0001\} \\ \{3.0535, 2.1706, 2.1706\} & \{3.2176, 1.3999, 1.3999\} & \{1\} & \{4.9455, 6.6942, 8.1852\} \\ \{0.2360, 0.2215, 0.2215\} & \{0.6000, 0.3010, 0.2000\} & \{0.2022, 0.1494, 0.1222\} & \{1\} \end{pmatrix}.$$

Step 6: Develop the improved score matrix $S\left(\hat{H}^c\right) = \left(\hat{s}_{ij}^c\right)_{n\times n}$ with Eq. (7) as

$$S\left(\hat{H}^c\right) = \begin{pmatrix} 1 & 1.6567 & 0.4112 & 4.4203 \\ 0.6036 & 1 & 0.5413 & 3.0254 \\ 2.4321 & 1.8475 & 1 & 6.4711 \\ 0.2262 & 0.3305 & 0.1545 & 1 \end{pmatrix}.$$

Step 7: Utilize the method developed by Crawford and Williams [46] to compute the priority weights of alternatives as

$$w_1 = 0.2653, w_2 = 0.2009, w_3 = 0.4678, w_4 = 0.0660.$$

Step 8: Finally, according to the priority weights of alternatives, the ranking is determined as $A_3 \succ A_1 \succ A_2 \succ A_4$; then, the best investment target is A_3, i.e., the computer company.

5.2. Comparison Analysis

To illustrate the superiority of the proposed approaches, several decision making methods based on HMPRs developed by scholars are introduced to implement the aforementioned numerical

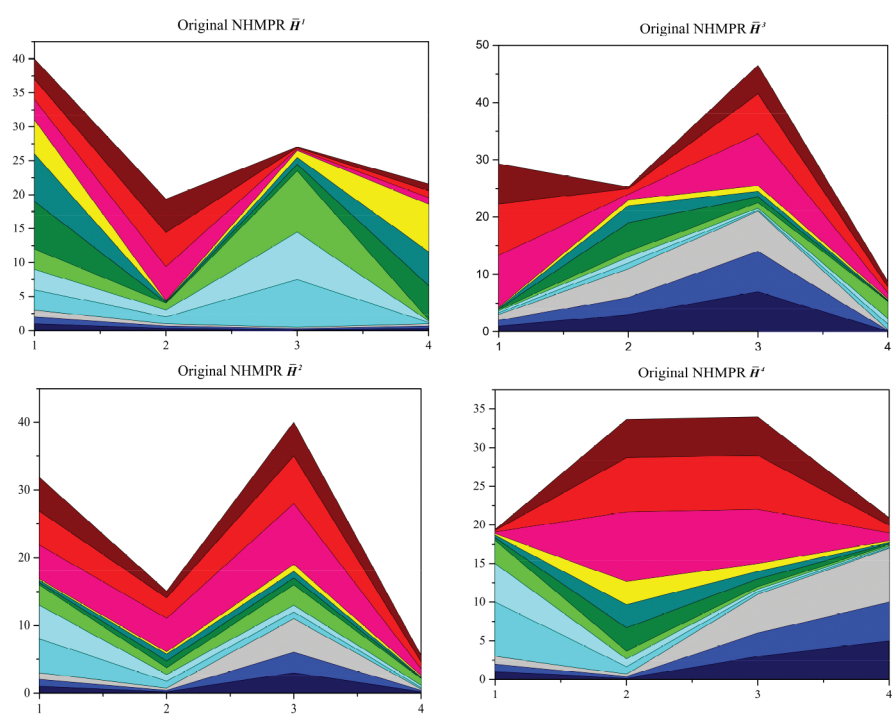

Figure 5 Areas of the original normalized hesitant multiplicative preference relations (NHMPRs).

examples, including Xia and Xu [23], Zhang and Wu [27], Zhang and Wu [24], and Lin *et al.* [25]. Subsequently, we compare the rankings determined in this paper with the results of the methods in the literature above; furthermore, the comparison analysis results of Example 1 are presented in Table 1.

From Table 1, we can see that the ranking of the hesitant multiplicative weighted averaging (HMWA) operator [23] is $A_2 \succ A_4 \succ A_1 \succ A_3$, while the rankings obtained by other decision making approaches are always $A_2 \succ A_4 \succ A_3 \succ A_1$. The inconsistent results can be explained by ignoring the consistency degree of HMPR $\boldsymbol{H}$ in the HWMA operator [23]; thus, the rationality of decision making result cannot be guaranteed. In addition, the HMPR $\boldsymbol{H}$ is aggregated by the HMWA operator without the normalization step; then, the operation process may become extremely complicated, and the elements of the evaluation information may become large-scale. Both Algorithm 1 [27] and Algorithm 2 [24] utilize the iteration-based model to increase the consistency degree of HMPR $\boldsymbol{H}$; however, the complexity of calculation is larger than that of the proposed consistency-based programming model. Furthermore, the effect of consistency optimization also needs to be improved in these two methods; the proposed consistency indexes of the revised HMPR $\hat{\boldsymbol{H}}$ in both Algorithm 1 [27] and Algorithm 2 [24] have not reached the consistency threshold $CI_0 = 0.1$. The distances between the original HMPR $\boldsymbol{H}$ and the revised HMPR $\hat{\boldsymbol{H}}$ in Algorithm 1 [27], Algorithm 2 [24], and the proposed approach are all relatively small, which indicates that all the three methods can maintain the original evaluation information of decision makers effectively. The Algorithm 2 [25] constructs an iteration-based decision making method with an HMPR by transforming the HMPR into an MPR; thus, although the complexity of calculation can be reduced, most of the original information will be ignored.

On the other hand, the existing approaches still present several advantages. The optimized parameter δ is obtained by a programming model in the normalization step of Algorithm 2 [24], which can maintain the original HMPR $\boldsymbol{H}$ as much as possible. The consistency threshold is determined according to the confidence level of decision makers objectively in Algorithm 2 [25]; then, the rationality of the decision making result can be improved.

With respect to Example 2, we make a comparison between the result obtained in this paper and those of the HMWA operator [23], Algorithm 2 [27], Algorithm 4 [24], and Algorithm 4 [25]. Then,

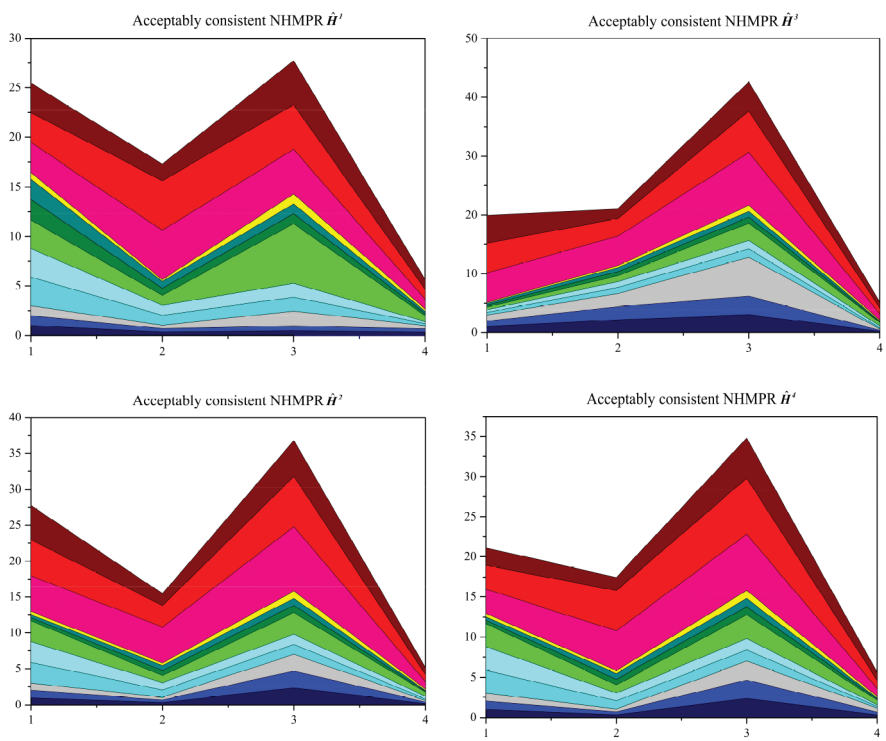

Figure 6 | Areas of the acceptably consistent normalized hesitant multiplicative preference relations (NHMPRs).

Table 1 | Comparison analysis results of Example 1.

Methods	Ranking Indexes				Ranking	$CI(\hat{H})$	$d(\hat{H}, H)$	Improving Consistency
	A_1	A_2	A_3	A_4				
The HMWA operator [23]	0.6861	3.1865	0.6338	1.9210	$A_2 \succ A_4 \succ A_1 \succ A_3$	0.2967	0	No
Algorithm 1 [27]	[0.3404, 0.6185]	[2.7600, 3.2789]	[0.4677, 0.7418]	[1.0911, 1.4346]	$A_2 \succ A_4 \succ A_3 \succ A_1$	0.1454	0.0875	Yes
Algorithm 2 [24]	0.1006	0.5699	0.1048	0.2248	$A_2 \succ A_4 \succ A_3 \succ A_1$	0.1033	0.1092	Yes
Algorithm 2 [25]	0.1336	0.4267	0.1776	0.2621	$A_2 \succ A_4 \succ A_3 \succ A_1$	0.9347	–	Yes
The proposed approach	0.0825	0.5858	0.0831	0.2487	$A_2 \succ A_4 \succ A_3 \succ A_1$	0.1000	0.0984	Yes

the rankings of different GDM methods are presented in Table 2, furthermore, the comparison analysis results are summarized in Table 3. It is noteworthy that the initial weight vector of decision makers is all assumed as $\lambda = (0.1, 0.5, 0.3, 0.1)^T$, and the distance measures in Table 3 indicate the average distance between the original NHMPRs $\overline{\boldsymbol{H}}^p$ and the revised NHMPRs $\hat{\boldsymbol{H}}^p$ determined by each GDM methods. Table 2 shows that the rankings of all the GDM methods remain unchanged as $A_3 \succ A_1 \succ A_2 \succ A_4$; however, we can still analyze the differences between these methods from the data in Table 3.

The GDM approach [23] aggregates the original HMPRs $\boldsymbol{H}^p$ based on the HMWA operator without the normalization step, which is time-consuming especially in the case of a large-scale decision makers. In contrast, the proposed GDM approach normalizes the HMPRs $\boldsymbol{H}^p$ into the NHMPRs $\overline{\boldsymbol{H}}^p$. Furthermore, the HMWA operator [23] ignores the consistency and consensus problems of HMPRs in Example 2, and Table 3 shows that the consistency indexes of HMPRs and consensus measure are all quite different from the threshold values in the HMWA operator [23]. Nevertheless, both the consistency and consensus levels of HMPRs are important factors that influence the rationality of GDM results. In the proposed GDM approach, a consistency- and consensus-based programming model is developed for improving the consistency and consensus levels, simultaneously; thus, the ranking obtained by the proposed approach is more reasonable. Similarly, although the Algorithm 2 [27] develops an iterative model to increase the consistency degrees of NHMPRs $\overline{\boldsymbol{H}}^p$, the consensus degree among decision makers is omitted, i.e., the consensus measure is 0.7634. The NHMPRs $\overline{\boldsymbol{H}}^p$ are divided into several MPRs in Algorithm 2 [27], then the consistency improving method based on MPRs [35] is introduced to revise NHMPRs $\overline{\boldsymbol{H}}^p$. Nevertheless, Table 3 indicates that the consistency indexes of the revised HMPRs determined by Algorithm 2 [27] all exceed the consistency threshold $CI_0 = 0.1$, which means that the consistency improving process of Algorithm 2 [27] is not effective in some situations.

Both Algorithm 4 [24] and Algorithm 4 [25] focus on checking and improving the consistency and consensus levels of the original HMPRs $\boldsymbol{H}^p$, and the consistency indexes and consensus measures in Table 3 also prove that these two consistency and consensus improving methods are highly effective. However, both the Algorithm 4 [24] and Algorithm 4 [25] adopt the mode of improving consistency and consensus levels of HMPRs $\boldsymbol{H}^p$ separately; two different iteration-based models are developed in Algorithm 4 [24] and Algorithm 4 [25] to improve the consistency and consensus levels, respectively. Thus, the consistency and consensus issues are solved after several iterations in Algorithm 4 [24] and Algorithm 4 [25]; in contrast, a programming model is constructed in the proposed GDM approach to improve the consistency and consensus levels of NHMPRs $\overline{\boldsymbol{H}}^p$, simultaneously, which reduces the complexity degree of calculation significantly. Furthermore, from Table 3, we can see that the average distance between the original NHMPRs $\overline{\boldsymbol{H}}^p$ and the revised NHMPRs $\hat{\boldsymbol{H}}^p$ in the proposed GDM approach is smaller than that in Algorithm 4 [24]; the proposed GDM approach can maintain more original evaluation information on the basis of ensuring the consistency and consensus of HMPRs $\boldsymbol{H}^p$. The HMPRs $\boldsymbol{H}^p$ are transformed into MPRs to implement the GDM procedure in Algorithm 4 [25]; although this method can reduce the computational complexity, it also causes the loss of original evaluation information. On the other hand, Algorithm 4 [24] and Algorithm 4 [25] also have their advantages. The optimized parameter δ is determined by minimizing the distances between HMPRs $\boldsymbol{H}^p$ and NHMPRs $\overline{\boldsymbol{H}}^p$ in Algorithm 4 [24], and the weight vector of decision makers is determined according to the consensus levels of decision makers objectively in Algorithm 4 [25]; all these improvements have improved the rationality of ranking results to a certain extent.

According to the aforementioned comparison analysis, the advantages of using the proposed approaches to solve the decision making problems with an HMPR and HMPRs can be summarized as in the following.

Table 2 Rankings of different group decision making methods.

Methods	Ranking Indexes				Ranking
	A_1	A_2	A_3	A_4	
The HMWA operator [23]	1.6885	1.2862	2.5197	0.6101	$A_3 \succ A_1 \succ A_2 \succ A_4$
Algorithm 2 [27]	[1.0456, 1.3712]	[0.7917, 1.2004]	[1.8232, 2.2818]	[0.3337, 0.5286]	$A_3 \succ A_1 \succ A_2 \succ A_4$
Algorithm 4 [24]	1.1760	0.9967	2.1068	0.4050	$A_3 \succ A_1 \succ A_2 \succ A_4$
Algorithm 4 [25]	1.1625	1.0023	1.7060	0.5031	$A_3 \succ A_1 \succ A_2 \succ A_4$
The proposed approach	0.2653	0.2009	0.4678	0.0660	$A_3 \succ A_1 \succ A_2 \succ A_4$

Table 3 Comparison analysis results of Example 2.

Methods	Consistency Indexes				Consensus Measure	Distance Measure	Improving Consensus	Improving Consistency
	$CI\left(\hat{H}^1\right)$	$CI\left(\hat{H}^2\right)$	$CI\left(\hat{H}^3\right)$	$CI\left(\hat{H}^4\right)$				
The HMWA operator [23]	0.4066	0.1906	0.4042	0.4707	0.6099	0	No	No
Algorithm 2 [27]	0.1205	0.1218	0.1324	0.1234	0.7634	0.1443	No	Yes
Algorithm 4 [24]	0.0156	0.0205	0.0240	0.0163	0.9538	0.2422	Yes	Yes
Algorithm 4 [25]	0.8663	0.8829	0.9052	0.8544	0.9583	–	Yes	Yes
The proposed GDM approach	0.1000	0.1000	0.1000	0.1000	0.9000	0.1706	Yes	Yes

1. The consistency index of individual HMPR proposed in this paper is defined by the HMPR itself without constructing any corresponding HMPR. Similarly, the novel consensus measure of HMPRs is developed depending on the HMPRs without aggregating the HMPRs into a collective HMPR. Both the proposed consistency and consensus measures are convenient to be operated, which saves several steps during the decision making process.
2. During the GDM process, instead of dividing the consistency and consensus improving processes into two issues and solving them separately, this paper constructs a consistency- and consensus-based programming model to acquire the acceptably consistent and consensus HMPRs simultaneously. Thus, the computational complexity of consistency and consensus improving processes can be reduced significantly.
3. Both the consistency-based programming model and consistency- and consensus-based programming model in this paper are constructed by minimizing the original evaluation information and the revised evaluation information. Then, the original preference information can remain as much as possible under the situations of the acceptable consistency and consensus levels.
4. According to the proposed programming models, regardless of the consistency and consensus threshold values, the acceptably consistent and consensus HMPRs can be obtained easily by solving the corresponding programming model. In practice, the values of consistency and consensus threshold are always determined by the preference of decision makers or enterprise; subsequently, the proposed approaches are highly flexible in dealing with different situations.

However, the proposed approaches also present several limitations. During the GDM process, the weights of decision makers play a critical role in determining the ranking of alternatives. Hence, the weights of decision makers should be computed comprehensively by considering the subjective and objective factors, which is an important issue ignored in the proposed GDM approach. Similarly, the determinations of the optimized parameter δ, the consistency threshold CI_0, and the consensus threshold CM_0 are omitted in this paper. Furthermore, with the GDM issues becoming increasingly complex in practice, decision makers may feel difficult in evaluating all the alternatives [53,54]; the proposed approaches cannot deal with the GDM problems with incomplete HMPRs, which will be the focus of the future research. According to the literature [45], the fusion of individual preference relations can be solved by the consistency- and consensus-based mathematical programming model; hence, this is also an aspect that the proposed approach needs to be improved.

6. CONCLUSIONS

To investigate the GDM problems with HMPRs, this paper focuses on the consistency- and consensus-based GDM approach. After the normalization procedure, a revised distance measure between different NHMPRs is utilized to develop a consistency index based on multiplicative consistency of NHMPRs, which is defined on the basis of the original NHMPRs themselves rather than the corresponding consistent NHMPRs. To remain the original evaluation information as much as possible, a goal programming model is constructed to improve the consistency degree by minimizing the distance between the original NHMPR and the acceptably consistent NHMPR; furthermore, we develop a consistency-based approach for decision making with an HMPR according to the proposed programming model. For the GDM problems with HMPRs, the consensus level between decision makers is also an important factor that affects the accuracy of ranking result. A novel consensus measure is put forward to check the consensus level among the individual NHMPRs; thus, a mathematical programming model is developed to improve the consistency and consensus levels of NHMPRs, simultaneously. Subsequently, the acceptably consistent and consensus NHMPRs can be obtained by solving the proposed consistency- and consensus-based programming model, and the NHMWG operator is introduced to aggregate the individual NHMPRs to determine the ranking of alternatives. Two numerical examples are applied to demonstrate the feasibility of the proposed approaches; next, the effectiveness of the proposed methods is illustrated by the comparative analysis.

In future research, with the increasingly complicated of GDM issues in practice, the incomplete evaluation information will inevitably be expressed; thus, we will improve the proposed GDM approach to deal with the GDM problems with incomplete HMPRs. Furthermore, the large-scale GDM problems have received more and more attention in the existing literature [55–59]; then, the GDM approach based on HMPRs, in which a large-scale group of decision makers participates in the evaluation process, is also a future research focus. Further endeavors will also be devoted to the incorporation of the attitudinal dimension into HMPRs [18,60,61].

CONFLICT OF INTEREST

The authors declare no conflict of interest.

AUTHORS' CONTRIBUTIONS

Concept of this study, Rui Wang, Bin Shuai, and Zhen-Song Chen; methodology, Rui Wang and Jiang-Hong Zhu; writing-original draft preparation, Rui Wang; writing-review and editing, Zhen-Song Chen and Bin Shuai.

Funding Statement

This work was supported by the National Natural Science Foundation of China (grant nos. 71801175, 71871171, 71971182, 71373222, and 71231007), the Theme-based Research Projects of the Research Grants Council (grant no. T32-101/15-R), the Fundamental Research Funds for the Central Universities (grant no. 2042018kf0006), the Ger/HKJRS project (grant no. G-CityU103/17), the City University of Hong Kong SRG (grant no. 7004969), and partly by the Doctoral Innovation Fund Program of Southwest Jiaotong University (grant no. D-CX201727).

ACKNOWLEDGMENTS

The authors would like to express their gratitude to the editors and anonymous reviewers for their help in commenting and improving the manuscript.

REFERENCES

[1] A. Mardani, A. Jusoh, E.K. Zavadskas, Fuzzy multiple criteria decision-making techniques and applications – two decades review from 1994 to 2014, Expert Syst. Appl. 42 (2015), 4126–4148.

[2] L. Martínez, D. Ruan, F. Herrera, Computing with words in decision support systems: an overview on models and applications, Int. J. Comput. Intell. Syst. 3 (2010), 382–395.

[3] Z. Zhang, C. Guo, Deriving priority weights from intuitionistic multiplicative preference relations under group decision-making settings, J. Oper. Res. Soc. 68 (2017), 1582–1599.

[4] Z.S. Chen, K.S. Chin, K.L. Tsui, Constructing the geometric Bonferroni mean from the generalized bonferroni mean with several extensions to linguistic 2-tuples for decision-making, Appl. Soft Comput. 78 (2019), 595–613.

[5] R. Ureña, F. Chiclana, J.A. Morente-Molinera, E. Herrera-Viedma, Managing incomplete preference relations in decision making: a review and future trends, Inf. Sci. 302 (2015), 14–32.

[6] S.A. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets Syst. 1 (1978), 155–167.

[7] T.L. Saaty, The analytic hierarchy process, McGraw-Hill, New York, 1980.

[8] H. Herrera-Viedma, J.L. Verdegay, A model of consensus in group decision making under linguistic assessments, Fuzzy Sets Syst. 78 (1996), 73–87.

[9] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, A rational consensus model in group decision making using linguistic assessments, Fuzzy Sets Syst. 88 (1997), 31–49.

[10] Z.S. Xu, Intuitionistic preference relations and their application in group decision making, Inf. Sci. 177 (2007), 2363–2379.

[11] J. Wu, F. Chiclana, H.C. Liao, Isomorphic multiplicative transitivity for intuitionistic and interval-valued fuzzy preference relations and its application in deriving their priority vectors, IEEE Trans. Fuzzy Syst. 26 (2016), 193–202.

[12] M.M. Xia, Z.S. Xu, H.C. Liao, Preference relations based on intuitionistic multiplicative information, IEEE Trans. Fuzzy Syst. 21 (2013), 113–133.

[13] H. Garg, Distance and similarity measures for intuitionistic multiplicative preference relation and its applications, Int. J. Uncertain. Quan. 7 (2017), 117–133.

[14] S.M. Chen, L.W. Lee, Fuzzy multiple criteria hierarchical group decision-making based on interval type-2 fuzzy sets, IEEE Trans. Syst. Man Cybern. Part A Syst. Hum. 40 (2010), 1120–1128.

[15] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010), 529–539.

[16] R.M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, IEEE Trans. Fuzzy Syst. 20 (2011), 109–119.

[17] Z.S. Chen, K.S. Chin, Y.L. Li, Y. Yang, Proportional hesitant fuzzy linguistic term set for multiple criteria group decision making, Inf. Sci. 357 (2016), 61–87.

[18] Z.S. Chen, K.S. Chin, L. Martinez, K.L. Tsui, Customizing semantics for individuals with attitudinal HFLTS possibility distributions, IEEE Trans. Fuzzy Syst. 26 (2018), 3452–3466.

[19] Z.S. Chen, L. Martínez, K.S. Chin, K.L. Tsui, Two-stage aggregation paradigm for HFLTS possibility distributions: a hierarchical clustering perspective, Expert Syst. Appl. 104 (2018), 43–66.

[20] B. Zhu, Z.S. Xu, Regression methods for hesitant fuzzy preference relations, Technol. Econ. Dev. Econ. 19 (2013), S214–S227.

[21] F. Herrera, E. Herrera-Viedma, L. Martinez, A fuzzy linguistic methodology to deal with unbalanced linguistic term sets, IEEE Trans. Fuzzy Syst. 16 (2008), 354–370.

[22] V. Torra, Knowledge-based validation: synthesis of diagnoses through synthesis of relations, Fuzzy Sets Syst. 113 (2000), 267–276.

[23] M.M. Xia, Z.S. Xu, Managing hesitant information in GDM problems under fuzzy and multiplicative preference relations, Int. J. Uncertain. Fuzziness Knowl. Based Syst. 21 (2013), 865–897.

[24] Z. Zhang, C. Wu, A decision support model for group decision making with hesitant multiplicative preference relations, Inf. Sci. 282 (2014), 136–166.

[25] M. Lin, Q. Zhan, Z. Xu, R. Chen, Group decision-making model with hesitant multiplicative preference relations based on regression method and feedback mechanism, IEEE Access. 6 (2018), 61130–61150.

[26] Y. Lin, Y.M. Wang, S.Q. Chen, Multistage decision making based on prioritization of hesitant multiplicative preference relations, J. Intell. Fuzzy Syst. 32 (2017), 691–701.

[27] Z. Zhang, W. Chong, Z. Zhang, W. Chong, Deriving the priority weights from hesitant multiplicative preference relations in group decision making, Appl. Soft Comput. 25 (2014), 107–117.

[28] Q. Mou, Z.S. Xu, H.C. Liao, Z. Hao, Two regression methods for hesitant multiplicative preference relations with different consistencies, Soft Comput. 23 (2019), 7029–7044.

[29] Y. Lin, Y.M. Wang, Prioritization of hesitant multiplicative preference relations based on data envelopment analysis for group decision making, Neural Comput. Appl. 31 (2019), 437–447.

[30] Z. Zhen, C. Guo, Fusion of heterogeneous incomplete hesitant preference relations in group decision making, Int. J. Comput. Intell. Syst. 9 (2016), 245–262.

[31] Z. Zhang, Some hesitant multiplicative aggregation operators and their application in group decision making with hesitant multiplicative preference relations, Int. J. Fuzzy Syst. 18 (2016), 177–197.

[32] Y. He, Z. Xu, J. Gu, An approach to group decision making with hesitant information and its application in credit risk evaluation of enterprises, Appl. Soft Comput. 43 (2016), 159–169.

[33] Y. He, Z. Xu, A consensus reaching model for hesitant information with different preference structures, Knowl. Based Syst. 135 (2017), 99–112.

[34] Z. Zhang, Hesitant triangular multiplicative aggregation operators and their application to multiple attribute group decision making, Neural Comput. Appl. 28 (2017), 195–217.

[35] Z. Xu, C. Wei, A consistency improving method in the analytic hierarchy process, Eur. J. Oper. Res. 116 (1999), 443–449.

[36] F. Chiclana, F. Mata, L. Martínez, E. Herrera-Viedma, S. Alonso, Integration of a consistency control module within a consensus model, Int. J. Uncertain. Fuzziness Knowl. Based Syst. 16 (2008), 35–53.

[37] S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, A web based consensus support system for group decision making problems and incomplete preferences, Inf. Sci. 180 (2010), 4477–4495.

[38] H.C. Liao, Z.S. Xu, X.J. Zeng, J.M. Merigo, Framework of group decision making with intuitionistic fuzzy preference information, IEEE Trans. Fuzzy Syst. 23 (2015), 1211–1227.

[39] G.L. Xu, S.P. Wan, F. Wang, J.Y. Dong, Y.F. Zeng, Mathematical programming methods for consistency and consensus in group decision making with intuitionistic fuzzy preference relations, Knowl. Based Syst. 98 (2016), 30–43.

[40] Z. Zhang, W. Pedrycz, Goal programming approaches to managing consistency and consensus for intuitionistic multiplicative preference relations in group decision-making, IEEE Trans. Fuzzy Syst. 26 (2018), 3261–3275.
[41] C.C. Li, Y.C. Dong, Y.J. Xu, F. Chiclana, E. Herrera-Viedma, F. Herrera, An overview on managing additive consistency of reciprocal preference relations for consistency-driven decision making and fusion: taxonomy and future directions, Inf. Fusion. 52 (2019), 143–156.
[42] X. Liu, Y.J. Xu, R. Montes, Y.C. Dong, F. Herrera, Analysis of self-confidence indices-based additive consistency for fuzzy preference relations with self-confidence and its application in group decision making, Int. J. Intell. Syst. 34 (2019), 920–946.
[43] W. Jian, F. Chiclana, H. Fujita, E. Herrera-Viedma, A visual interaction consensus model for social network group decision making with trust propagation, Knowl. Based Syst. 122 (2017), 39–50.
[44] Y.J. Xu, F. Herrera, Visualizing and rectifying different inconsistencies for fuzzy reciprocal preference relations, Fuzzy Sets Syst. 362 (2019), 85–109.
[45] L.H. Wang, Z.W. Gong, N. Zhang, Consensus modelling on interval-valued fuzzy preference relations with normal distribution, Int. J. Comput. Intell. Syst. 11 (2018), 706–715.
[46] G. Crawford, C. Williams, A note on the analysis of subjective judgment matrices, J. Math. Psychol. 29 (1985), 387–405.
[47] Q.X. Dong, X. Zhou, L. Martínez, A hybrid group decision making framework for achieving agreed solutions based on stable opinions, Inf. Sci. 490 (2019), 227–243.
[48] I. Palomares, F.J. Estrella, L. Martínez, F. Herrera, Consensus under a fuzzy context: taxonomy, analysis framework AFRYCA and experimental case of study, Inf. Fusion. 20 (2014), 252–271.
[49] Y.C. Dong, Q.B. Zha, H.J. Zhang, G. Kou, H. Fujita, F. Chiclana, E. Herrera-Viedma, Consensus reaching in social network group decision making: research paradigms and challenges, Knowl. Based Syst. 162 (2018), 3–13.
[50] Á. Labella, Y. Liu, R. Rodríguez, L. Martínez, Analyzing the performance of classical consensus models in large scale group decision making: a comparative study, Appl. Soft Comput. 67 (2018), 677–690.
[51] R.M. Rodríguez, Y.J. Xu, L. Martínez, F. Herrera, Exploring consistency for hesitant preference relations in decision making: discussing concepts, meaning and taxonomy, J. Mult. Valued Log. Soft Comput. 30 (2018), 129–154.
[52] E. Herrera-Viedma, F.J. Cabrerizo, J. Kacprzyk, W. Pedrycz, A review of soft consensus models in a fuzzy environment, Inf. Fusion. 17 (2014), 4–13.
[53] E.E. Kerre, A.U. Rehman, S. Ashraf, Group decision making with incomplete reciprocal preference relations based on multiplicative consistency, Int. J. Comput. Intell. Syst. 11 (2018), 1030–1040.
[54] Y. Xu, C. Li, X. Wen, Missing values estimation and consensus building for incomplete hesitant fuzzy preference relations with multiplicative consistency, Int. J. Comput. Intell. Syst. 11 (2018), 101–119.
[55] H.J. Zhang, Y.C. Dong, E. Herrera-Viedma, Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions, IEEE Trans. Fuzzy Syst. 26 (2018), 884–898.
[56] Z. Zhang, C. Guo, L. Martínez, Managing multi-granular linguistic distribution assessments in large-scale multi-attribute group decision making, IEEE Trans. Syst. Man Cybern. Syst. 47 (2017), 3063–3076.
[57] Y.C. Dong, S. Zhao, H. Zhang, F. Chiclana, E. Herrera-Viedma, A self-management mechanism for non-cooperative behaviors in large-scale group consensus reaching processes, IEEE Trans. Fuzzy Syst. 26 (2018), 3276–3288.
[58] Z.S. Chen, Y. Yang, X.J. Wang, K.S. Chin, K.L. Tsui, Fostering linguistic decision-making under uncertainty: a proportional interval type-2 hesitant fuzzy TOPSIS approach based on hamacher aggregation operators and andness optimization models, Inf. Sci. 500 (2019), 229–258.
[59] R.M. Rodríguez, Á. Labella, G. De Tré, L. Martínez, A large scale consensus reaching process managing group hesitation, Knowl. Based Syst. 159 (2018), 86–97.
[60] Z.S. Chen, C. Yu, K.S. Chin, L. Martínez, An enhanced ordered weighted averaging operators generation algorithm with applications for multicriteria decision making, Appl. Math. Model. 71 (2019), 467–490.
[61] Z.S. Chen, L. Martínez, J.P. Chang, X.J. Wang, S.H. Xiong, K.S. Chin, Sustainable building material selection: a QFD- and ELECTRE III-embedded hybrid MCGDM approach with consensus building, Eng. Appl. Artif. Intell. 85 (2019), 783–807.