Journal of Statistical Theory and Applications
Vol. **18(4)**, December 2019, *pp.* 375–386
DOI: https://doi.org/10.2991/jsta.d.191031.001; ISSN 1538-7887
https://www.atlantis-press.com/journals/jsta

Bayesian Analysis of Inverse Gaussian Stochastic Conditional Duration Model

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ARTICLE INFO

Article History

Received 20 Sep 2017
Accepted 21 Oct 2019

Keywords

Stochastic conditional duration
Bayesian analysis
Markov Chain Monte Carlo
Inverse Gaussian distribution
Slice sampler

2000 Mathematics Subject Classification: 62M10, 91B70, 62F15

ABSTRACT

This paper discusses Bayesian analysis of stochastic conditional duration model when the innovations follow inverse Gaussian distribution. Estimation is carried out by the methods of Markov Chain Monte Carlo. Applications of the model and methods are illustrated through simulation and data analysis.

1. INTRODUCTION

The empirical analysis of durations between the occurrences of certain financial events is important in understanding the market behavior. To describe the evolution of such durations, Bauwens and Veredas [1] proposed a class of models called the stochastic conditional duration (SCD) models. One can refer Engle and Russell [2], Pacurar [3] for details on such duration models. The SCD model assumes that the conditional mean of durations between events is generated by a latent stochastic process. The likelihood based inference for such models requires evaluation of multiple integral with respect to latent variables. In view of these difficulties Bauwens and Veredas [1] estimated the parameters by quasi-maximum likelihood (ML) by expressing the model into a linear state space form and then applying Kalman filter method. Feng, Jiang and Song [4] proposed an extension to Bauwens and Veredas's [1] SCD model in order to capture the asymmetric behavior or leverage effect in the duration process. They adopted the Markov Chain Maximum Likelihood (MCML) approach proposed by Durbin and Koopman [5]. Strickland, Forbes and Martin [6] proposed a Bayesian Markov Chain Monte Carlo (MCMC) method in which the sampling scheme employed is a hybrid of the Gibbs and Metropolis-Hastings (MH) algorithm, with the latent vector sampled in blocks. Knight and Ning [7] discussed the empirical characteristic function (ECF) and the generalized method of moments estimation. Bauwens and Galli [8] developed a ML estimation based on the efficient importance sampling (EIS) method to estimate the parameters. Xu, Knight and Wirjanto [9] extended the SCD model of Bauwens and Veredas [1] by imposing mixtures of bivariate normal distributions on the innovations of the observation and latent equations of the duration process. Men, Kolkiewicz and Wirjanto [10] introduced a correlation between the error process and the innovations of the duration process and adopted Monte Carlo methods for estimation. Ramanathan, Mishra and Abraham [11] introduced a new procedure for estimation, filtering and smoothing in SCD models, based on estimating functions. Majority of the literature on SCD models assume that the errors follow either a Weibull, Gamma or exponential distribution. Balakrishna and Rahul [12] proposed a SCD model having inverse Gaussian (IG) distribution for innovations. In the present paper, we consider the Bayesian analysis of SCD model with IG error random variables. We propose Bayesian MCMC methods to estimate the parameters of the model. Here we follow the algorithm mentioned in Men, Kolkiewicz and Wirjanto [13], Edwards and Sokal [14] and Neal [15]. Since it is difficult to obtain the analytical conditional densities of observed data, the auxiliary particle filter (APF) in Pitt and Shephard [16] is employed to evaluate the likelihood function.

A brief review of IG SCD model is given in Section 2. The Bayesian estimation methodology and MCMC algorithm are discussed in Section 3. Simulation studies are carried out in Section 4. Section 5 presents the application of proposed method to real life data sets.

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2. THE SCD MODEL

Let τ_i be the time of occurrence of an event (or transaction) of interest with $\tau_0 = 0$ and let $X_i = \tau_i - \tau_{i-1}$, $i = 1, 2, ..., n$ be the i^{th} trade duration, which is defined as the waiting time between two consecutive transactions of an underlying asset from time $i-1$ to i.

The SCD model is defined by

$$\begin{aligned} X_i &= e^{\psi_i}\epsilon_i \\ \psi_i &= \phi\psi_{i-1} + \eta_i, \qquad i = 1, 2, \dots, n \\ \psi_0 &\sim N\left(0, \frac{\sigma^2}{1-\phi^2}\right), \end{aligned} \tag{1}$$

where $|\phi| < 1$ to ensure the stationarity of the process. Further, η_i follows independent and identically distributed (iid) $N\left(0, \sigma^2\right)$, so that $\{\psi_i\}$ follows a Gaussian AR(1) sequence and $\{\epsilon_i\}$ is an iid sequence on the positive support with common pdf $f(\epsilon_i)$ and η_j is independent of ϵ_i $\forall\ i, j$. Note that the model depends on the unobservable ψ_i, called the latent variable. Most of the SCD models available in the literature assume that the innovations follow iid exponential, gamma or Weibull distributions. One can refer Bauwens and Veredas [1], Strickland, Forbes and Martin [6], Knight and Ning [7], Durbin and Koopman [5] and so on for the details on such models. Xu, Knight and Wirjanto [9] proposed a SCD model by assuming a bivariate mixture of normal distribution for (ϵ_i, η_i). Balakrishna and Rahul [12] assumed an IG distribution for ϵ_i and estimated the model parameters by EIS method. In this paper we propose a Bayesian method for this SCD model with IG innovations. We consider the unit mean IG distribution for ϵ_i in the model (1).

A random variable X is said to have an IG distribution with parameters μ and λ, and is denoted by $IG(\mu, \lambda)$ if its probability density function (pdf) is of the form

$$f(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda (x-\mu)^2}{2\mu^2 x}\right), \quad x > 0,\ \mu > 0,\ \lambda > 0. \tag{2}$$

If we restrict ϵ_i to follow a unit mean IG distribution then its pdf is of the form

$$f_\epsilon(\epsilon_i) = \sqrt{\frac{\lambda}{2\pi\epsilon_i^3}} \exp\left(-\frac{\lambda (\epsilon_i - 1)^2}{2\epsilon_i}\right), \quad \epsilon_i > 0. \tag{3}$$

In order to develop an estimation procedure for the model let $X = (x_1, x_2, x_3, ..., x_{n-1}, x_n)'$ be a vector of observations from the model Eq. (1) and $\psi = (\psi_1, \psi_2, ..., \psi_n)'$ be a vector of associated latent variables, where $'$ denotes the transpose of a vector. If we denote the joint density function of (X, ψ) by $f\left(X, \psi|\theta\right)$, then the likelihood function of the parameter $\theta = (\phi, \sigma, \lambda)'$ based on the observations is

$$L(\theta; X) = \int f\left(X, \psi|\theta\right) d\psi, \tag{4}$$

which is an n-fold integral with respect to ψ. We need to evaluate this integral to obtain the likelihood function. Toward that end some numerical methods such as MCMC are to be used. In the next section, we propose a Bayesian method for estimating the parameters.

3. BAYESIAN ESTIMATION

We consider the problem of estimation for the SCD model when the innovations of the durations follow an IG distribution in a Bayesian framework. By Bayes theorem, the joint conditional distribution of ψ and θ given the observations is

$$f\left(\theta, \psi|X\right) \propto f\left(X|\psi, \theta\right) f\left(\psi|\theta\right) f(\theta), \tag{5}$$

where $f\left(X|\psi, \theta\right)$ is the density of X given (θ, ψ), $f\left(\psi|\theta\right)$ is the density of ψ and $f(\theta)$ is the prior density of θ. Bayesian inference on (θ, ψ) is based upon the posterior distribution $f\left(\theta, \psi|X\right)$. From this joint posterior, the marginal $f\left(\theta|X\right)$ can be used to make inferences about the parameters of SCD model with IG innovations and the marginal $f\left(\psi|X\right)$ provides the inference about the latent variables. We assume that the prior distributions of ϕ, σ, λ are mutually independent. We take prior distribution of ϕ as a normal truncated in the interval $(-1, 1)$ which results in flat density over the supporting region. For σ^2 we take an inverse gamma prior distribution as in Pitt and Shephard [16], which is a conjugate prior. For the parameter λ, we use a truncated Cauchy prior distribution with density

$$f(\lambda) \propto \frac{1}{1+\lambda^2}, \qquad \lambda > 0, \tag{6}$$

which was also considered by Bauwens and Lubrano [17].

To construct an appropriate Markov Chain sampler, the joint posterior should be expressed as proportional to various conditional distributions. So in Eq. (5), the joint posterior, $f(\psi|\theta)$ is further decomposed into a set of conditionals $f(\psi_i|\psi_{-i}, \theta, X)$, where ψ_{-i} means all terms of ψ except ψ_i. Here, since ψ is a vector of latent variables, it is difficult to sample from these univariate conditional distributions. So we follow the MCMC algorithm proposed by Men, Kolkiewicz and Wirjanto [13] for estimation.

While developing MCMC algorithm, the latent variables are augmented with the vector of parameters and then the estimation is carried out. We start drawing samples from the conditional posterior distribution of the latent variable ψ and the parameters θ. The sampling algorithm required to draw the samples from each conditional posterior and the associated steps are detailed below.

The parameters $\theta = (\phi, \sigma, \lambda)'$ are initialized with values $(0.5, 0.5, 1)'$ and then the initial value of ψ is generated from latent AR(1) model by using these values of θ.

Step 1. Sample $\psi = (\psi_1, \psi_2, ..., \psi_n)$ by drawing ψ_i from $f(\psi_i|\psi_{-i}, X, \theta)$ for $i = 1, 2, 3, \dots, n$. By employing the markovian structure of the model in (1), $f(\psi_i|\psi_{-i}, X, \theta)$ can be expressed as $f(\psi_i|\psi_{i-1}, \psi_{i+1}, X, \theta)$. A single-move MH algorithm is used to sample ψ_i. The conditional distribution of latent random variables ψ_i, given the other parameters in the model have been previously obtained is given by

$$\begin{aligned} f(\psi_1|X, \psi_2, \theta) &\propto f(x_1|\psi_1) f(\psi_1|\theta) f(\psi_2|\psi_1, x_1, \theta), \\ f(\psi_i|X, \psi_{i-1}, \psi_{i+1}, \theta) &\propto f(x_i|\psi_i) f(\psi_i|\psi_{i-1}, x_{i-1}, \theta) f(\psi_{i+1}|\psi_i, x_i, \theta) \quad i = 2, \dots, n-1, \\ f(\psi_n|X, \psi_{n-1}, \theta) &\propto f(x_n|\psi_n) f(\psi_n|\psi_{n-1}, x_{n-1}, \theta), \end{aligned}$$

where $f(x_i|\psi_i)$, $i = 1, 2, \dots, n$ are the conditional density functions of the durations, $f(\psi_1|\theta)$ is the density of the latent state ψ_1, $f(\psi_i|\psi_{i-1}, x_{i-1}, \theta)$ is the conditional density of ψ_i given ψ_{i-1} and $f(\psi_{i+1}|\psi_i, x_i, \theta)$ is the conditional density of ψ_{i+1} given ψ_i. Since x_n is the last observation, the posterior distribution of ψ_n depends only on x_n, x_{n-1} and ψ_{n-1}. The conditional distribution of ψ_1 and ψ_n are given in Appendix A. If the conditional distribution does not possess a simple form as in the present case then it is not possible to draw the samples directly. In such cases one of the options is to consider an accept/reject method. Following Men, Kolkiewicz and Wirjanto [13] we use a single-move Metropolis Hastings algorithm to sample the latent states, where the proposal distribution is simulated by slice sampler method. The slice sampler method proposed by Edwards and Sokal [14] and Neal [15] is a method to sample random variables which do not have simple pdfs or their pdfs are known up to a normalizing constant. As the slice sampler adapts to the analytical structure of the underlying density, it is more efficient. Also it ensures faster convergence to the underlying distribution. So, to generate random variates from the conditional distribution we employ the method of slice sampler. While analyzing a data, we estimate these unobservable latent variables ψ, usingAPF method proposed by Pitt and Shephard [16]. In the following discussion, we obtain specific forms of the required samplers.

For $i = 2, 3, \dots, n-1$,

$$(\psi_i|\psi_{i-1}) \sim N(\phi\psi_{i-1}, \sigma^2) \text{ and } (\psi_{i+1}|\psi_i) \sim N(\phi\psi_i, \sigma^2).$$

Substituting for the corresponding normal densities and on simplifying the square terms we get

$$\begin{aligned} f(\psi_i|\psi_{-i}, \theta) &\propto \exp\left(-\frac{(1+\phi^2)}{2\sigma^2}\left(\psi_i^2 - 2\psi_i\left(\frac{\phi}{\phi^2+1}(\psi_{i-1}+\psi_{i+1})\right)\right)\right) \\ &\propto N(\omega_1, \sigma_1^2), \end{aligned}$$

$$\text{where } \omega_1 = \frac{\phi}{1+\phi^2}(\psi_{i-1}+\psi_{i+1}) \text{ and } \sigma_1^2 = \frac{\sigma^2}{1+\phi^2}.$$

Hence the conditional posterior distribution of ψ_i can be represented as

$$f(\psi_i|X, \psi_{i-1}, \psi_{i+1}, \theta) \propto f(x_i|\psi_i) f(\psi_i|\psi_{i-1}, \theta) f(\psi_{i+1}|\psi_i, \theta) \tag{7}$$

$$\begin{aligned} &\propto \sqrt{\frac{\lambda e^{\psi_i}}{x_i^3}} \exp\left(-\frac{\lambda(x_i - e^{\psi_i})^2}{2e^{\psi_i}x_i}\right) \exp\left(-\frac{(\psi_i - \phi\psi_{i-1})^2}{2\sigma^2}\right) \\ &\qquad \exp\left(-\frac{(\psi_{i+1} - \phi\psi_i)^2}{2\sigma^2}\right) \\ &\propto (\lambda\exp(\psi))^{\frac{1}{2}} \exp\left(-\frac{\lambda(x_i - e^{\psi_i})^2}{2e^{\psi_i}x_i}\right) \exp\left(-\frac{(\psi_i - a_i)^2}{2b}\right), \end{aligned} \tag{8}$$

where $a_i = \dfrac{\phi(\psi_i + \psi_{i+1})}{1+\phi^2}$ and $b = \dfrac{\sigma^2}{1+\phi^2}$.

The posterior distribution in (8) is proportional to a product of three positive functions and data cannot be simulated directly from them. So we propose a method of slice sampler whose algorithm is given below.

Slice sampler algorithm for ψ_i. Let us rewrite the conditional distribution in (8) as

$$g(\psi_i) \propto \exp\left(-\frac{\lambda(x_i - e^{\psi_i})^2}{2e^{\psi_i}x_i}\right)\exp\left(-\frac{(\psi_i - a_{1i})^2}{2b}\right), \tag{9}$$

where $a_{1i} = a_i + \frac{b}{2}$ and follow the algorithm given below.

To start the slice sampling procedure, the sampled value of ψ_i from the last MCMC step is set as the initial value.

i. Initialize $\psi_i^{(0)}$. Set $t = 0$.

ii. Draw a random observation u_1 uniformly from the interval $\left(0, \exp\left(-\frac{\lambda\left(x_i - e^{\psi_i^{(t)}}\right)^2}{2e^{\psi_i^{(t)}}x_i}\right)\right)$. Then we define an interval for ψ_i through the inequality $u_1 \leq \exp\left(-\frac{\lambda\left(x_i - e^{\psi_i}\right)^2}{2e^{\psi_i}x_i}\right)$, which is equivalent to

$$\psi_i \geq log\left(\frac{x_i\left(1 + \frac{1}{\epsilon_i^2}\right)}{2\left(1 - \frac{(log(u_1))}{\lambda}\right)}\right). \tag{10}$$

iii. Similarly draw a random observation u_2 uniformly from the interval $\left(0, \exp\left(-\frac{\left(\psi_i - a_{1i}^{(t)}\right)^2}{2b}\right)\right)$, where $a_{1i}^{(t)}$ is calculated from Eq. (9). We define an interval for ψ_i through the inequality $u_2 \leq \exp\left(-\frac{(\psi_i - a_{1i})^2}{2b}\right)$, which is equivalent to

$$a_{1i}^{(t)} - \sqrt{-2\ b\ log(u_2)} \leq \psi_i \leq a_{1i}^{(t)} + \sqrt{-2\ b\ log(u_2)}. \tag{11}$$

iv. Draw $\psi_i^{(t+1)}$ uniformly from the interval $\left(max\left\{log\left(\frac{x_i\left(1+\frac{1}{\epsilon_i^2}\right)}{2\left(1-\frac{(log(u_1))}{\lambda}\right)}\right), a_{1i}^{(t)} - \sqrt{-2\ b\ log(u_2)}\right\}, a_{1i}^{(t)} + \sqrt{-2\ b\ log(u_2)}\right)$ determined by the inequalities (10) and (11).

v. Stop, if a stopping criterion is met; otherwise, set $t = t + 1$ and repeat from ii.

Step 2. Sample ϕ. The prior distribution of ϕ is assumed to follow a univariate normal distribution truncated in the interval (-1,1). Given a truncated normal prior distribution, $\phi \sim N\left(\alpha_\phi, \beta_\phi^2\right)$ and the other parameters in the model have been previously sampled, the conditional distribution of ϕ is

$$\begin{aligned} f\left(\phi|X, \sigma^2, \lambda\right) &\propto f\left(\psi|\theta, X\right) f(\phi). \\ &= f\left(\psi_1|\theta, X\right)\prod_{i=2}^{n} f\left(\psi_i|\psi_{i-1}, x_i, \theta\right)\exp\left(-\frac{(\phi - \alpha_\phi)^2}{2\beta_\phi^2}\right). \\ &\propto N\left(\frac{d}{c}, \frac{1}{c}\right)\sqrt{1-\phi^2}, \end{aligned} \tag{12}$$

where $c = \frac{\sum_{i=2}^{n}\psi_i^2}{\sigma^2} + \frac{1}{\beta_\phi^2}$, $d = \frac{\sum_{i=2}^{n}\psi_i\psi_{i-1}}{\sigma^2} + \frac{\alpha_\phi}{\beta_\phi^2}$. The conditional distribution of ϕ is given in Appendix A. It is proportional to the product of a normal distribution and a positive function. Hence we can use the slice sampling method to sample the parameter ϕ.

Step 3. Sample σ^2. We sample σ^2 by taking an inverse gamma prior distribution, i.e., $\sigma^2 \propto$ inverse Gamma $(\alpha_{\sigma^2}, \beta_{\sigma^2})$, where α_{σ^2} and β_{σ^2} are hyperparameters. As the prior for σ^2 is a conjugate prior the sampling is carried out by simulating from the inverse Gamma distribution with the corresponding parameters obtained. The conditional distribution of σ^2 is presented in Appendix A.

Step 4. Sample λ. The conditional distribution of λ is

$$\begin{aligned} f\left(\lambda|X,\psi,\phi,\sigma^2\right) &= f\left(X|\psi,\lambda\right) f(\lambda). \\ &= f(\lambda)\prod_{i=1}^{n}\sqrt{\frac{\lambda e^{\psi_i}}{x_i^3}}\exp\left(-\frac{\lambda(x_i-e^{\psi_i})^2}{2e^{\psi_i}x_i}\right), \end{aligned} \tag{13}$$

where $f(\lambda)$ is a prior density of λ, given by (6). Here the samples cannot be simulated directly from (13). So we use a random walk MH algorithm with standard normal distribution as the proposal distribution to sample λ. The acceptance probability is computed using (13).

The following is a summary of the procedure used to sample $(\theta', \psi')'$.

- Sample ψ' using the single-move MH algorithm with proposal distribution simulated by the method of slice sampling which is briefly explained in Step 1.
- Sample ϕ from (12) following the explanations in Step 2.
- Sample σ^2 directly from the inverse Gamma density using Step 3.
- Sample λ using a random walk MH algorithm with the acceptance probability computed through (13) given in Step 4.

In the next section we demonstrate the applications of the above methods through a simulation study.

4. SIMULATION STUDY

A simulation study is carried out to assess the performance of the Bayes estimators, described in the previous section. We generate 5000 observations from the model (1). Then the MCMC algorithm discussed in Section 3 is run and first 25000 iterations were discarded as burn-in from 100000 iterations. The parameters are estimated and the simulation results are tabulated in Table 1. The plots of histograms of posterior samples of ϕ, σ^2, λ are shown in Figure 1 (a), 1 (b) and 1 (c) respectively. The trace plots are shown in Figure 2 (a), 2 (b) and 2 (c) respectively.

To illustrate the application of Bayesian estimation method we analyze two sets of data. We perform model diagnosis based on the residuals to explain model adequacy. The residuals of SCD model with IG innovations are defined as $\hat{\epsilon}_i = x_i/e^{\hat{\psi}_i}$, where $\hat{\psi}_i$ is the estimator of ψ_i. The estimators of the parameters ϕ, σ and λ can be obtained by the MCMC algorithm discussed in Section 3. To obtain the estimates of the unobservable component ψ_i, we employ an APF proposed by Pitt and Shephard [16]. This method is described in the following subsection:

4.1. Particle Filter

Particle filters are a class of simulation-based filters that recursively approximate the filtering distribution using a collection of particles with some probability masses. The particles are samples of unknown states from the state space, and the particle weights are probability mass

Table 1 True and estimated parameters of the IG-SCD model.

	ϕ	σ	λ
True.	0.75	1.5	1
Est.	0.763302	1.529561	0.999191
mse.	0.000039	0.000094	0.0000039
HPD CI(95%)	(0.74633, 0.78840)	(1.47808, 1.57561)	(0.9557, 1.1477)
True.	0.85	1.5	1
Est.	0.84699	1.55057	0.999242
mse.	0.0000322	0.000095	0.0000041
HPD CI(95%)	(0.82633, 0.86840)	(1.49808, 1.58561)	(0.9657, 1.1413)
Highest probability density	Confidence interval (HPD CI)		

computed by Bayes theory. The basic idea is the recursive computation of relevant probability distribution and approximation of probability distribution.

By successive conditional decomposition, the likelihood of the IG-SCD model is

$$f\left(X|\theta\right) = f\left(x_1|\theta\right)\prod_{i=2}^{n} f\left(x_i|\mathcal{F}_{i-1},\theta\right), \tag{14}$$

where $\mathcal{F}_i = \sigma\left(x_1, x_2, \dots, x_i\right)$, the sigma field generated by $(x_1, x_2, \dots, x_i)$ is the information known at time i. The conditional density of x_{i+1} given θ and $\mathcal{F}_i$ is given by

$$\begin{aligned} f\left(x_{i+1}|\mathcal{F}_i,\theta\right) &= \int f\left(x_{i+1}|\psi_{i+1},\theta\right) dF\left(\psi_{i+1}|\mathcal{F}_i,\theta\right) \\ &= \int f\left(x_{i+1}|\psi_{i+1},\theta\right) f\left(\psi_{i+1}|\psi_i,\theta\right) dF\left(\psi_i|\mathcal{F}_i,\theta\right). \end{aligned} \tag{15}$$

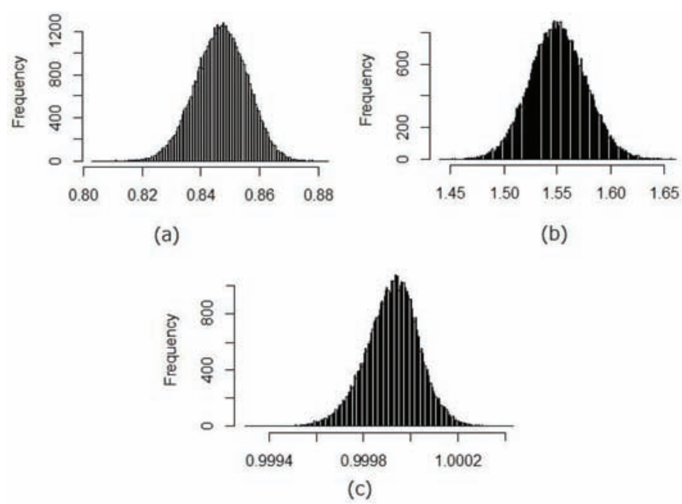

Figure 1 | Histogram of the posterior samples using simulated data.

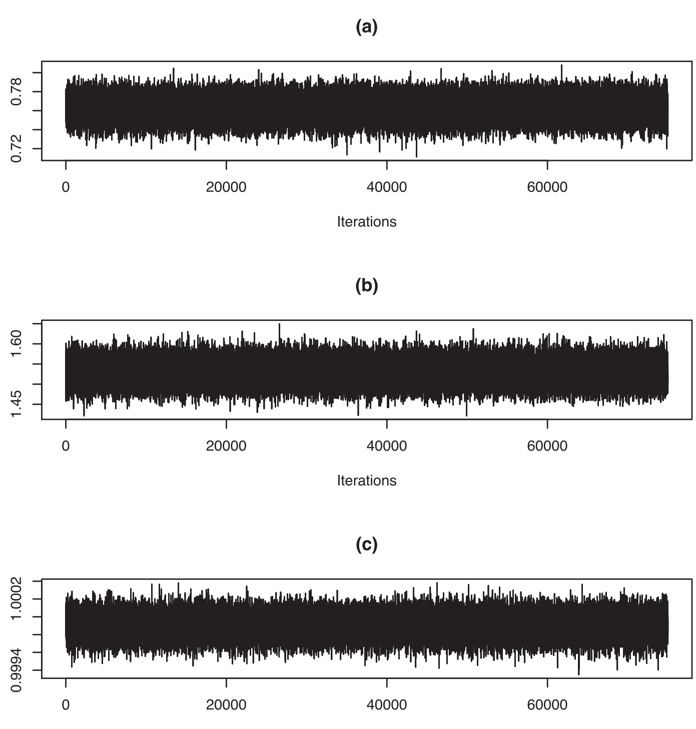

Figure 2 | Trace plots of the posterior samples using simulated data.

The difficulty in obtaining an analytical form of the above integral leads to the utilization of APF. Suppose that we have a particle sample $\{\psi_i^{(t)}, t = 1, 2, \dots, N\}$ of ψ_i from the filtered distribution $(\psi_i|\mathcal{F}_i, \theta)$ with weights $\{\pi_t, t = 1, 2, \dots, N\}$ such that $\sum_{t=1}^{N} \pi_t = 1$. From this sample, the one-step ahead predictive density of ψ_{i+1} is

$$f\left(\psi_{i+1}|\mathcal{F}_i, \theta\right) \approx \sum_{t=1}^{N} \pi_t f\left(\psi_{i+1}|\psi_i^{(t)}, \theta\right) \tag{16}$$

The one step ahead prediction distribution of ψ_{i+1} can then be sampled and the conditional density (15) can be evaluated numerically by

$$f\left(x_{i+1}|\mathcal{F}_i, \theta\right) \approx \sum_{t=1}^{N} \pi_t f\left(x_{i+1}|\psi_{i+1}^{(t)}, \theta\right), \tag{17}$$

where $\psi_{i+1}^{(t)}$ are particles from the prediction distribution of $\left(\psi_{i+1}|\mathcal{F}_i, \theta\right)$. The predictive density of ψ_{i+1} should be known for the approximation (16) to be feasible. From latent AR(1) process, we have ψ_{i+1} has a conditional normal distribution $\psi_{i+1} \sim N\left(\phi\psi_i, \sigma^2\right)$.

Given the particle sample from a filtered distribution $\left(\psi_i|\mathcal{F}_i, \theta\right)$ we need to sample $\left(\psi_{i+1}|\mathcal{F}_{i+1}, \theta\right)$. For that, we follow the procedure of Chib, Nardari and Shephard [18] and Men, Kolkiewicz and Wirjanto [13], which is summarized below.

4.1.1. Algorithm for APF

1. (a) Given a sample $\{\psi_i^{(t)}, t = 1, 2, \dots, N\}$ from $\left(\psi_i|\mathcal{F}_i, \theta\right)$, calculate the expectation $\hat{\psi}_{i+1}^{*(t)} = E\left(\psi_{i+1}|\psi_i^{(t)}\right)$ and

$$\pi_t = f\left(x_{i+1}|\hat{\psi}_{i+1}^{*(t)}, \theta\right), \qquad t = 1, 2, \dots, N$$

and sample N times with replacement the integers 1,2,$\cdots$,N with probability $\hat{\pi}_t = \frac{\pi_t}{\sum_{t=1}^{N} \pi_t}$. Let the sampled indexes be $k_1, k_2, \dots, k_N$ and associate these with particles $\{\hat{\psi}_i^{*(k_1)}, \hat{\psi}_i^{*(k_2)}, \dots, \hat{\psi}_i^{*(k_N)}\}$.

2. For each value of k_t from 1(a), sample the values $\left\{\psi_{i+1}^{*(1)}, \dots, \psi_{i+1}^{*(N)}\right\}$ from

$$\psi_{i+1}^{*(t)} = \phi\psi_i^{(k_t)} + \eta_{i+1}, \qquad t = 1, \dots, N$$

where $\eta_{i+1} \sim N(0, 1)$.

3. Calculate the weights of the values $\left\{\psi_{i+1}^{*(1)}, \dots, \psi_{i+1}^{*(N)}\right\}$ as

$$\pi_t^* = \frac{f\left(x_{i+1}|\psi_{i+1}^{*(t)}, \theta\right)}{f\left(x_{i+1}|\hat{\psi}_{i+1}^{*(k_t)}, \theta\right)}, \qquad t = 1, 2, \dots, N$$

and using these weights resample the values $\left\{\psi_{i+1}^{*(1)}, \dots, \psi_{i+1}^{*(N)}\right\}$ N times with replacement to obtain a sample $\left\{\psi_{i+1}^{(1)}, \dots, \psi_{i+1}^{(N)}\right\}$ from the filtered distribution $(\psi_{i+1}|\mathcal{F}_{i+1}, \theta)$. Here we use N=2000.

5. DATA ANALYSIS

We demonstrate the applications of the model, by analysing two sets of data. Data sets are on intraday trades of IBM OHLC bar data downloaded from Algoseek Website and intraday trades of US Brent Crude Oil downloaded from the Website of a Swiss Forex bank. Only trades between 9:30:00 am and 4:00:00 pm are recorded as this is the normal trading hour.

5.1. IBM Trades Data

The model is applied to intraday IBM trades data as on 16^{th} June 2015. Consider the 1 second Trade OHLC Bar data with a sample size of 6708 observations. The trade durations are defined as the time intervals between consecutive trades, measured in seconds. We ignore the zero trade duration and the time plot of the nonzero intraday IBM trade duration series is shown in Figure 3. Now, removing the effect of the diurnal pattern we take the adjusted time duration (Tsay [19], pp. 298–300) to model the intraday pattern. In Table 2 the summary statistics of IBM trades data are given.

The estimation of the parameters is carried out by the MCMC algorithm described in Section 3 and the estimates are provided in Table 3. For model diagnosis we compute residuals, $\hat{\epsilon}_i = x_i/e^{\hat{\psi}_i}$, where $\hat{\psi}_i$ is the estimator of ψ_i which is obtained by APF. If the fitted model is adequate

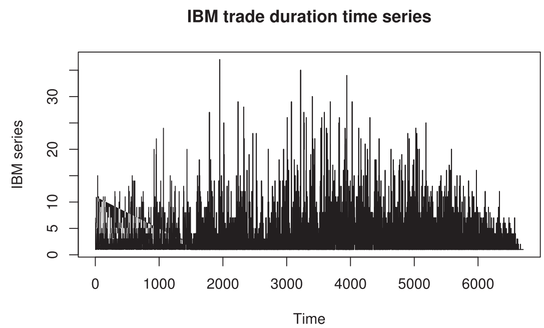

Figure 3 IBM trade duration time series.

Table 2 Descriptive statistics for IBM trades data.

Statistic	IBM Trades Data
Sample size	6708
Minimum	1
Maximum	37
Mean	3.48
Median	2

Table 3 Estimated parameters of the IG-SCD model based on IBM Trades duration data.

	ϕ	σ	λ
Est.	0.69803	1.2366	1.1566
Std error	0.00143	0.00906	0.00658

Figure 4 ACF plot of residuals.

then the Autocorrelation function (acf) of $\{\hat{\epsilon}_i\}$ will be negligible. The residual acf plot given in Figure 4 indicates that they are uncorrelated. Bauwens and Veredas [1] considered the time series version of Spearman's ρ correlation coefficient and the p-value plots instead of the Ljung-Box statistic. Here we follow a rank portmanteau statistic given in Dufour and Roy [20] to check the lack of autocorrelation of the obtained residuals. The p-value obtained is 0.7. Also the run test confirms the independence of the residuals $\{\hat{\epsilon}_i\}$. In Figure 5 the histogram of the residuals is superimposed by the IG density curve for the IG-SCD model. So it can be concluded that the fitted model is adequate for explaining the dynamics, which generated the data.

5.2. US Brent Crude Oil

The second set of data is the intraday trades data of US Brent Crude Oil downloaded from the Website of a Swiss Forex bank and Marketplace. The intraday trade of the Brent Crude Oil on 20 February 2017 is considered. Taking the normal trading hours and ignoring the zero durations the sample size obtained is 1625 trade durations. The time plot of the nonzero intraday durations and the time plot of the adjusted duration series is shown in Figures 6 and 7. The summary of data is given in Table 4.

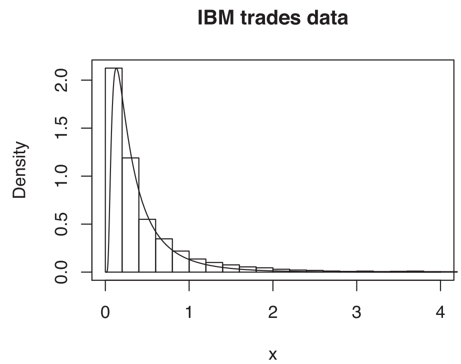

Figure 5 Histogram of residuals superimposed by inverse Gaussian density for IG-SCD model.

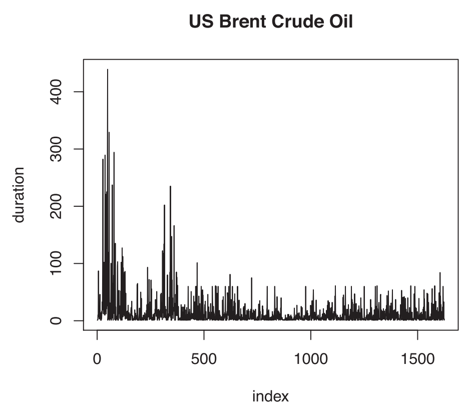

Figure 6 Duration plot of US brent crude oil.

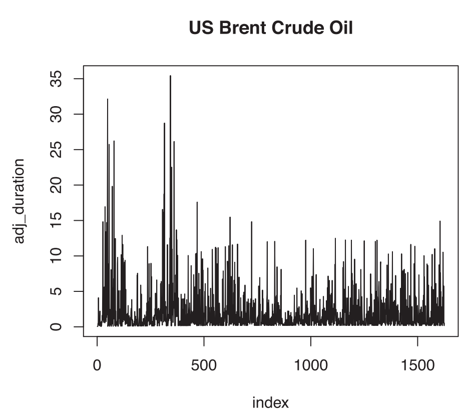

Figure 7 Adjusted duration plot of US brent crude oil.

Table 4 Descriptive statistics for US Brent Crude Oil Trades data.

Statistic	IBM Trades Data
Sample size	1625
Minimum	1
Maximum	439
Mean	14.4
Median	6

The parameters are estimated by the algorithm mentioned in Section 3 and are tabulated in Table 5. From the residual plot given in Figure 8 and the p-value (= 0.8) obtained from rank portmanteau statistic we conclude that the residuals are independent. The superimposition of the histogram of the residuals and the IG density curve in Figure 9, shows a good fit of error distribution.

Table 5 Estimated parameters of the IG-SCD model based on US Brent Crude Oil trades duration data.

	ϕ	σ	λ
Est.	0.77263	1.34885	0.945069
std error.	0.00567	0.00975	0.01018

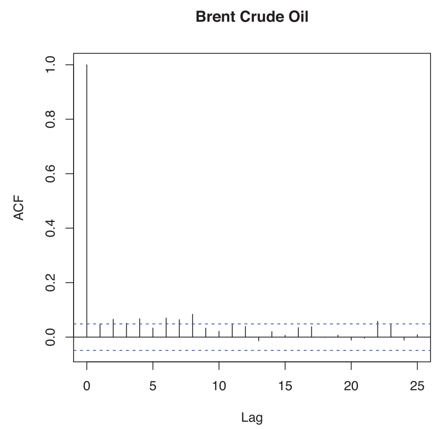

Figure 8 Residual plot of US brent crude oil.

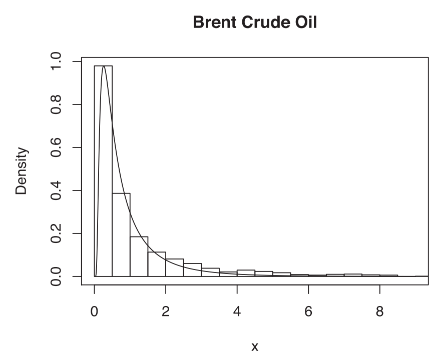

Figure 9 Histogram of residuals superimposed by inverse Gaussian density for inverse Gaussian-stochastic conditional duration (IG-SCD) model.

6. CONCLUSION

In this article we considered the Bayesian estimation of SCD model with IG error random variables. The slice sampling within an MCMC estimation methodology and the APF method are utilized to estimate the parameters and the latent states of the model. Simulation study shows that the proposed estimation method is reliable. Two sets of financial data are considered to demonstrate the utility of the model. The non-parametric tests applied to residuals support the assumptions on the errors. However, more rigorous statistical tests are to be developed to perform model diagnosis in the presence of latent variables.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

ACKNOWLEDGMENTS

The authors thank the editor and the reviewers for their careful reading of the article. Sri Ranganath acknowledge Kerala State Council for Science Technology & Environment, for the financial assistance for carrying out this research work. The research of N. Balakrishna is partially supported by the DST SERB under the project No. SR/S4/MS:837/13.

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APPENDIX A

The conditional distribution of parameter ϕ is

$$
\begin{aligned}
f\left(\phi|X,\sigma,\lambda\right) &\propto f\left(\psi|\theta,X\right) f(\phi) \\
&= f\left(\psi_1|\theta,X\right) \prod_{i=2}^{n} f\left(\psi_i|\psi_{i-1},\theta,X\right) \exp\left(-\frac{(\phi-\alpha_\phi)^2}{2\beta_\phi^2}\right) \\
&\propto \exp\left(-\frac{\left(1-\phi^2\right)\psi_1^2}{2\sigma^2}\right) \exp\left(-\frac{\sum_{i=1}^{n}(\psi_i-\phi\psi_{i-1})^2}{2\sigma^2}\right) \exp\left(-\frac{(\phi-\alpha_\phi)^2}{2\beta_\phi^2}\right) (1-\phi^2)^{\frac{1}{2}} \\
&\propto \exp\left(-\frac{1}{2}\left(\phi^2\left\{\frac{\sum_{i=1}^{n}\psi_1^2}{\sigma^2}+\frac{1}{\beta_\phi^2}\right\} - 2\phi\left\{\frac{\sum_{i=1}^{n}\psi_i\psi_{i-1}}{\sigma^2}+\frac{\alpha_\phi}{\beta_\phi^2}\right\}\right)\right)(1-\phi^2)^{\frac{1}{2}} \\
&\propto N\left(\frac{d}{c},\frac{1}{c}\right)\sqrt{1-\phi^2},
\end{aligned}
$$

where $c = \frac{\sum_{i=1}^{n}\psi_1^2}{\sigma^2} + \frac{1}{\beta_\phi^2}$, $d = \frac{\sum_{i=2}^{n}\psi_i\psi_{i-1}}{\sigma^2} + \frac{\alpha_\phi}{\beta_\phi^2}$. The conditional distribution of parameter σ^2 , by taking an Inverse Gamma prior is

$$
\begin{aligned}
f\left(\sigma^2|X,\phi,\lambda\right) &= f\left(\psi|\theta,X\right) f(\alpha_{\sigma^2},\beta_{\sigma^2}) \\
&= f\left(\psi_1|\theta,X\right) \prod_{i=2}^{n} f\left(\psi_i|\psi_{i-1},\theta,X\right) f(\alpha_{\sigma^2},\beta_{\sigma^2}) \\
&\propto \exp\left(-\frac{\psi_i^2\left(1-\phi^2\right)}{2\sigma^2}\right) \times \exp\left(\frac{\sum_{i=2}^{N}(\psi_i-\phi\psi_{i-1})^2}{2\sigma^2}\right) \\
&\qquad \times (\sigma^2)^{-\frac{n}{2}} \times \frac{(\beta_{\sigma^2})^{\alpha_{\sigma^2}}}{\Gamma\left(\alpha_{\sigma^2}\right)} (\sigma^2)^{-\alpha_{\sigma^2}-1} \exp\left(\frac{-\beta_{\sigma^2}}{\sigma^2}\right) \\
&\propto (\sigma^2)^{-\frac{(2\,\alpha_{\sigma^2})-n+2}{2}} \exp\left(-\frac{\psi_i^2\left(1-\phi^2\right) - \sum_{i=2}^{N}(\psi_i-\phi\psi_{i-1})^2 + 2\,\beta_{\sigma^2}}{2\,\sigma^2}\right) \\
&\propto inverseGamma\left(a,b\right).
\end{aligned}
$$

where $a = \alpha_{\sigma^2} + \frac{n}{2}$ and $b = \frac{\left(1-\phi^2\right)(\psi_i)^2}{2} + \frac{1}{2}\sum_{i=2}^{n}(\psi_i-\phi\psi_{i-1})^2 + \beta_{\sigma^2}$.

The conditional distribution of parameter ψ_1 is

$$
\begin{aligned}
f\left(\psi_1|X,\psi_2,\theta\right) &\propto f\left(x_1|\psi_1\right) f\left(\psi_1|\theta\right) f\left(\psi_2|\psi_1,x_1,\theta\right) \\
&= f\left(x_1|\psi_1\right) \exp\left(-\frac{\left(1-\phi^2\right)\psi_1^2}{2\sigma^2}\right) \exp\left(-\frac{(\psi_2-\phi\psi_1)^2}{2\sigma^2}\right).
\end{aligned}
$$

The conditional distribution of parameter ψ_n is

$$
\begin{aligned}
f\left(\psi_n|X,\psi_{n-1},\theta\right) &\propto f\left(x_n|\psi_n\right) f\left(\psi_n|\psi_{n-1},x_{n-1},\theta\right) \\
&\propto f\left(x_n|\psi_n\right) \exp\left(\frac{(\psi_n-\phi\psi_{n-1})^2}{2\sigma^2}\right).
\end{aligned}
$$