Journal of Statistical Theory and Applications
Vol. **18(4)**, December 2019, *pp.* 361–366
DOI: https://doi.org/10.2991/jsta.d.191112.002; ISSN 1538-7887
https://www.atlantis-press.com/journals/jsta

An Application of Hermite Distribution in Sensitive Surveys

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ARTICLE INFO

Article History

Received 07 Apr 2016
Accepted 10 Sep 2017

Keywords

Hermite distribution
Randomized response techniques
Sensitive surveys

2000 Mathematics Subject Classification: 62D05

ABSTRACT

In this article, we proposed an efficient estimator for estimating population proportion of individuals possessing sensitive attribute in a finite dichotomous population. We used the Hermite distribution to randomize the responses in the randomization design of Kuk [1]. The relative efficiency results depicted that the proposed technique is relatively better than those of Kuk [1], Singh and Grewal [2] and Hussain *et al.* [3] and Hussain *et al.* [4].

1. INTRODUCTION

A random variable Z is said to have a Hermite distribution if its probability mass function (pmf) is given by

$$P_Z = Pr\left(Z = z\right) = e^{-(a+b)} \sum_{i=1}^{\left[\frac{z}{2}\right]} \frac{a^{z-2i} b^i}{(z-2i)!\, i!}, \quad z = 0, 1, 2, ..., \infty. \tag{1}$$

where a and b are the two parameters taking positive numbers. Furthermore, the mean and variance of Z are given by $\mu_Z = a + 2b$, and $\sigma_Z^2 = a + 4b$, respectively. The distribution with pmf (1) is denoted by $Her\left(a, b\right)$.

For estimation of population proportion π of the sensitive group, Warner [5] introduced the randomized response technique (RRT) in order to reduce non-response and misreporting in sensitive surveys. Several modifications of his RRT and new RRTs have been suggested. For better understanding of RRTs we refer to Blair *et al.* [6] and the references therein.

Kuk [1] modified Warner [5] model and argued that respondents feel insecure to answer a sensitive question, even when it is generated by a randomization device. He suggested using two decks each containing cards of two different colors, say C1 and C2. A respondent belonging to sensitive (non-sensitive) group is directed to use first (second) deck with proportion θ_1 (θ_2) of C1 (C2) cards. The i^{th} respondent, using either the first or second deck, is asked to randomly draw a card and report the color of the card drawn without disclosing the deck he/she have used. Let $X_i\left(Y_i\right)$ be the color of the card drawn from the first (second) deck. According to this design $X_i\left(Y_i\right)$ follows Bernoulli distribution with parameter θ_1 (θ_2). The reported response, Z_i, can be written as

$$Z_i = \alpha_i X_i + \left(1 - \alpha_i\right) Y_i, \tag{2}$$

here α_i is an indicator variable which taking value 1 if the respondent possesses sensitive attribute and 0, otherwise. Evidently, $E\left(\alpha_i\right) = \pi$.

Recently, Singh and Grewal [2] argued that the respondent have to report $X_i\left(Y_i\right)$ the number of cards he/she have drawn for getting first C1 card from first (second) deck according to their status. Here $X_i\left(Y_i\right)$ follows geometric distribution, $G\left(\theta_1\right)\left(G\left(\theta_2\right)\right)$. By changing the distribution of X_i and Y_i Singh and Grewal [2] improved the efficiency of estimator of π. Following Singh and Grewal [2] work, Hussain *et al.* [3] and Hussain *et al.* [4], respectively, used negative binomial and geometric distribution of order k, as randomization device [in their notation X and Y are distributed as $\left(NB\left(r, \theta_1\right)\text{ and } NB\left(r, \theta_2\right)\right)$ and $\left(G_{k_1}\left(\theta_1\right)\text{ and } G_{k_2}\left(\theta_2\right)\right)$, respectively] and provided efficient RRTs.

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We intend to use the Hermite distribution, in Kuk [1] set up. The rest of the article is arranged as follows: In Section 2, we present briefly the proposed survey method and give an unbiased estimator of π and its variance. Relative efficiency comparisons are made in Section 3. Section 4 is about discussion of the results and giving a conclusive statement.

2. PROPOSED RRT

Suppose, in a sensitive survey, we provide two decks of cards to the respondents. Random number, $X_1, X_2, X_3, ..., X_T$, generated from $Her(a_1, b_1)$, are written on cards placed in deck 1, and random numbers $Y_1, Y_2, Y_3, ..., Y_T$, generated from $Her(a_2, b_2)$, are written on cards placed in deck 2. Each respondent in the sample is directed to use one of the two provided decks depending on his/her own status on sensitive attribute A. If a respondent possesses (does not possesses) sensitive attribute A, he/she is asked to draw a card from deck 1 (deck 2) and report the number X_i (Y_i) written on the drawn card. Evidently, X_i and Y_i follows $Her(a_1, b_1)$, and $Her(a_2, b_2)$, respectively. The expected randomized response from the i^{th} respondent may be written as

$$E(Z_i) = \pi\mu_X + (1-\pi)\mu_Y. \tag{3}$$

On solving (3) for π and estimating $E(Z_i)$ by the sample mean, $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$, of reported responses, we get

$$\hat{\pi}_{(P)} = \frac{\bar{z} - \mu_Y}{\mu_X - \mu_Y}, \tag{4}$$

where $\mu_X = a_1 + 2b_1$ and $\mu_Y = a_2 + 2b_2$.

By (4), the variance of $\hat{\pi}_{(P)}$ is given by

$$Var\left(\hat{\pi}_{(P)}\right) = \frac{Var\left(\bar{z}\right)}{\left(\mu_X - \mu_Y\right)^2} = \frac{\pi(1-\pi)}{n} + \frac{\sigma_X^2\pi + \sigma_Y^2(1-\pi)}{n\left(\mu_X - \mu_Y\right)^2}, \tag{5}$$

where $\sigma_X^2 = a_1 + 4b_1$, and $\sigma_Y^2 = a_2 + 4b_2$.

3. EFFICIENCY COMPARISONS

It is difficult to conclude from analytical comparison, here, so numerical comparisons are made between the proposed RRT and those proposed by Kuk [1], Singh and Grewal [2], Hussain *et al.* [3] and Hussain *et al.* [4]. Let $\hat{\pi}_{(Kuk)}$, $\hat{\pi}_{(SG)}$, $\hat{\pi}_{(H_1)}$ and $\hat{\pi}_{(H_2)}$ denote the estimators proposed by Kuk [1], Singh and Grewal [2], Hussain *et al.* [3] and Hussain *et al.* [4], respectively. The estimators $\hat{\pi}_{(SG)}$, $\hat{\pi}_{(H_1)}$ and $\hat{\pi}_{(H_2)}$ and their variances can be readily obtained by (4) and (5) by setting different parameters as mentioned earlier. The estimator $\hat{\pi}_{(Kuk)}$ may also be obtained from (4) but its variance is given by

$$Var\left(\hat{\pi}_{(Kuk)}\right) = \frac{\theta_{Kuk}(1-\theta_{Kuk})}{nk\left(\theta_1 - \theta_2\right)^2} + \frac{\pi(1-\pi)}{n}\left(1 - \frac{1}{k}\right), \tag{6}$$

where $\theta_{Kuk} = \pi\theta_1 + (1-\pi)\theta_2$, and k is the number of repetitions.

Now, we define the relative efficiency of the proposed estimator $\hat{\pi}_{(P)}$ with respect to $\hat{\pi}_{(Kuk)}$, $\hat{\pi}_{(SG)}$, $\hat{\pi}_{(H_1)}$ and $\hat{\pi}_{(H_2)}$ as

$$RE_J = \frac{Var\left(\hat{\pi}_{(J)}\right)}{Var\left(\hat{\pi}_{(P)}\right)}, \; J = Kuk,\ SG,\ H_1,\ H_2.$$

To know the extent of relative efficiency we have computed the RE_J for different values of the design parameters, and the results are reported in the Tables 1–5 given below.

4. DISCUSSIONS OF RESULTS

From Tables 1–5, it is observed that the proposed estimator performs better than all the considered estimators over the whole range of π. It is also worth mentioning that the proposed RRT is simple to apply as respondents have to just draw a card and report the number written on the card, while in the RRTs proposed by Singh and Grewal [2], Hussain *et al.* [3] and Hussain *et al.* [4] respondents might have consumed a lot of time for observing a specific type of card and sequences of specific cards. Therefore, it is concluded that the proposed strategy of randomizing the response using Hermite distribution performs well without incurring any additional sampling and administrative cost.

Table 1 Relative efficiency of $\hat{\pi}_{(P)}$ for $r = k = 3,\ k_1 = 4,\ k_2 = 2,\ a_1 = 11,\ b_1 = 12,\ a_2 = 1$ and $b_2 = 2$.

θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$	θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$
			$\pi = 0.1$						$\pi = 0.2$		
0.1	0.2	50.116	11.084	4.263	1.805	0.1	0.2	27.497	8.393	3.387	1.993
	0.3	17.432	4.263	1.989	1.802		0.3	9.184	3.893	1.887	1.99
	0.4	9.695	2.937	1.547	1.801		0.4	4.92	2.945	1.571	1.989
	0.5	6.418	2.451	1.386	1.800		0.5	3.133	2.574	1.447	1.988
	0.6	4.642	2.217	1.307	1.800		0.6	2.172	2.385	1.384	1.988
	0.7	3.537	2.084	1.263	1.800		0.7	1.577	2.273	1.347	1.988
	0.8	2.786	2.001	1.235	1.800		0.8	1.174	2.2	1.322	1.988
	0.9	2.244	1.945	1.217	1.800		0.9	0.885	2.148	1.305	1.988
0.2	0.3	66.537	31.547	11.084	1.830	0.2	0.3	37.436	21.202	7.656	2.022
	0.4	20.116	9.000	3.568	1.814		0.4	10.933	7.067	2.945	2.004
	0.5	10.256	4.853	2.186	1.807		0.5	5.37	4.319	2.029	1.996
	0.6	6.379	3.411	1.705	1.803		0.6	3.202	3.313	1.693	1.991
	0.7	4.389	2.747	1.484	1.801		0.7	2.098	2.827	1.531	1.988
	0.8	3.204	2.389	1.365	1.799		0.8	1.444	2.552	1.44	1.986
	0.9	2.425	2.175	1.293	1.798		0.9	1.016	2.38	1.382	1.985
0.3	0.4	76.642	57.505	19.737	1.882	0.3	0.4	43.693	37.104	12.957	2.072
	0.5	21.221	14.589	5.432	1.839		0.5	11.761	10.684	4.15	2.029
	0.6	10.116	6.916	2.874	1.818		0.6	5.411	5.742	2.503	2.006
	0.7	5.945	4.322	2.009	1.805		0.7	3.041	3.996	1.921	1.993
	0.8	3.884	3.164	1.623	1.797		0.8	1.877	3.18	1.649	1.984
	0.9	2.695	2.558	1.421	1.792		0.9	1.209	2.733	1.5	1.978
0.4	0.5	80.432	83.274	28.326	1.958	0.4	0.5	46.27	52.785	18.184	2.135
	0.6	20.747	19.611	7.105	1.871		0.6	11.669	13.914	5.227	2.054
	0.7	9.274	8.495	3.4	1.825		0.7	5.043	6.847	2.871	2.01
	0.8	5.116	4.832	2.179	1.798		0.8	2.65	4.417	2.061	1.983
	0.9	3.126	3.24	1.648	1.781		0.9	1.509	3.313	1.693	1.965
0.5	0.6	77.905	103.168	34.958	2.045	0.5	0.6	45.166	64.933	22.233	2.191
	0.7	18.695	22.642	8.116	1.895		0.7	10.656	15.929	5.899	2.063
	0.8	7.73	8.958	3.554	1.816		0.8	4.266	7.264	3.01	1.992
	0.9	3.892	4.583	2.096	1.77		0.9	2.029	4.369	2.045	1.947
0.6	0.7	69.063	111.505	37.737	2.108	0.6	0.7	40.38	70.233	24	2.207
	0.8	15.063	22.263	7.989	1.883		0.8	8.724	15.902	5.89	2.029
	0.9	5.484	7.674	3.126	1.765		0.9	3.08	6.626	2.798	1.929
0.7	0.8	53.905	102.6	34.768	2.083	0.7	0.8	31.914	65.374	22.38	2.136
	0.9	9.853	17.053	6.253	1.784		0.9	5.871	13.003	4.923	1.913
0.8	0.9	32.432	70.768	24.158	1.868	0.8	0.9	19.767	47.043	16.27	1.907

Table 2 Relative efficiency of $\hat{\pi}_{(P)}$ for $r = k = 3,\ k_1 = 4,\ k_2 = 2,\ a_1 = 11,\ b_1 = 12,\ a_2 = 1$ and $b_2 = 2$.

θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$	θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$
			$\pi = 0.3$						$\pi = 0.4$		
0.1	0.2	20.324	7.817	3.197	2.161	0.1	0.2	16.971	7.935	3.233	2.358
	0.3	6.873	3.972	1.915	2.157		0.3	5.767	4.243	2.002	2.354
	0.4	3.798	3.113	1.629	2.156		0.4	3.257	3.38	1.714	2.352
	0.5	2.525	2.762	1.512	2.156		0.5	2.232	3.017	1.593	2.352
	0.6	1.845	2.577	1.451	2.155		0.6	1.69	2.821	1.528	2.351
	0.7	1.427	2.465	1.413	2.155		0.7	1.359	2.7	1.488	2.351
	0.8	1.146	2.389	1.388	2.155		0.8	1.138	2.618	1.46	2.351
	0.9	0.945	2.336	1.37	2.155		0.9	0.98	2.559	1.441	2.351
0.2	0.3	28.211	18.296	6.69	2.193	0.2	0.3	24.073	17.633	6.465	2.395
	0.4	8.352	6.718	2.831	2.173		0.4	7.176	6.906	2.89	2.372
	0.5	4.236	4.362	2.045	2.164		0.5	3.72	4.637	2.133	2.362
	0.6	2.648	3.465	1.746	2.159		0.6	2.4	3.747	1.837	2.356
	0.7	1.845	3.017	1.597	2.156		0.7	1.739	3.291	1.685	2.352
	0.8	1.372	2.756	1.51	2.153		0.8	1.352	3.02	1.595	2.349
	0.9	1.066	2.588	1.454	2.152		0.9	1.103	2.843	1.535	2.348
0.3	0.4	33.282	31.056	10.944	2.248	0.3	0.4	28.727	29.241	10.335	2.456
	0.5	9.127	9.761	3.845	2.201		0.5	7.971	9.79	3.851	2.403
	0.6	4.362	5.62	2.465	2.176		0.6	3.91	5.878	2.547	2.374

(*continued*)

Table 2 Relative efficiency of $\hat{\pi}_{(P)}$ for $r = k = 3,\ k_1 = 4,\ k_2 = 2,\ a_1 = 11,\ b_1 = 12,\ a_2 = 1$ and $b_2 = 2$. *(Continued)*

θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$	θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$
		$\pi = 0.3$						$\pi = 0.4$			
	0.7	2.595	4.104	1.96	2.16		0.7	2.415	4.404	2.056	2.357
	0.8	1.732	3.372	1.715	2.15		0.8	1.69	3.673	1.812	2.346
	0.9	1.239	2.958	1.577	2.143		0.9	1.278	3.251	1.671	2.338
0.4	0.5	35.535	43.563	15.113	2.31	0.4	0.5	30.931	40.555	14.106	2.523
	0.6	9.197	12.465	4.746	2.226		0.6	8.155	12.343	4.702	2.43
	0.7	4.174	6.606	2.793	2.178		0.7	3.829	6.857	2.873	2.376
	0.8	2.366	4.521	2.099	2.148		0.8	2.278	4.849	2.204	2.343
	0.9	1.507	3.541	1.772	2.128		0.9	1.543	3.879	1.881	2.32
0.5	0.6	34.972	53.282	18.352	2.356	0.5	0.6	30.686	49.371	17.045	2.566
	0.7	8.563	14.197	5.324	2.228		0.7	7.727	14.014	5.259	2.428
	0.8	3.673	7.038	2.937	2.154		0.8	3.475	7.331	3.031	2.347
	0.9	1.961	4.558	2.111	2.107		0.9	1.987	4.945	2.236	2.294
0.6	0.7	31.592	57.676	19.817	2.353	0.6	0.7	27.992	53.486	18.416	2.55
	0.8	7.225	14.324	5.366	2.181		0.8	6.686	14.253	5.339	2.369
	0.9	2.859	6.634	2.803	2.08		0.9	2.849	7.053	2.939	2.26
0.7	0.8	25.394	54.211	18.662	2.256	0.7	0.8	22.849	50.694	17.486	2.427
	0.9	5.183	12.211	4.662	2.048		0.9	5.033	12.508	4.757	2.214
0.8	0.9	16.38	40.352	14.042	2.000	0.8	0.9	15.257	38.792	13.518	2.136

Table 3 Relative efficiency of $\hat{\pi}_{(P)}$ for $r = k = 3,\ k_1 = 4,\ k_2 = 2,\ a_1 = 11,\ b_1 = 12,\ a_2 = 1$ and $b_2 = 2$.

θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$	θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$
		$\pi = 0.5$						$\pi = 0.6$			
0.1	0.2	15.324	8.514	3.417	2.615	0.1	0.2	14.753	9.6	3.765	2.975
	0.3	5.189	4.691	2.143	2.61		0.3	4.929	5.382	2.359	2.969
	0.4	2.969	3.764	1.834	2.608		0.4	2.831	4.329	2.008	2.967
	0.5	2.076	3.366	1.701	2.607		0.5	2.003	3.869	1.854	2.966
	0.6	1.61	3.148	1.629	2.607		0.6	1.576	3.614	1.769	2.965
	0.7	1.328	3.012	1.583	2.606		0.7	1.322	3.453	1.716	2.965
	0.8	1.141	2.918	1.552	2.606		0.8	1.154	3.342	1.679	2.964
	0.9	1.008	2.851	1.529	2.606		0.9	1.036	3.261	1.652	2.964
0.2	0.3	22.274	18.243	6.66	2.658	0.2	0.3	22.047	20.047	7.247	3.027
	0.4	6.637	7.471	3.069	2.631		0.4	6.518	8.471	3.388	2.995
	0.5	3.484	5.116	2.284	2.619		0.5	3.433	5.867	2.52	2.98
	0.6	2.293	4.17	1.969	2.612		0.6	2.282	4.8	2.165	2.971
	0.7	1.703	3.676	1.805	2.607		0.7	1.718	4.235	1.976	2.966
	0.8	1.36	3.378	1.705	2.604		0.8	1.393	3.89	1.861	2.962
	0.9	1.141	3.181	1.639	2.602		0.9	1.188	3.659	1.784	2.96
0.3	0.4	26.907	29.71	10.483	2.729	0.3	0.4	26.988	32.188	11.294	3.113
	0.5	7.506	10.425	4.054	2.667		0.5	7.518	11.7	4.465	3.038
	0.6	3.741	6.429	2.722	2.634		0.6	3.773	7.341	3.012	2.998
	0.7	2.366	4.887	2.208	2.613		0.7	2.415	5.625	2.44	2.973
	0.8	1.703	4.107	1.948	2.6		0.8	1.765	4.744	2.146	2.957
	0.9	1.328	3.649	1.795	2.59		0.9	1.4	4.218	1.971	2.945
0.4	0.5	29.224	40.83	14.189	2.803	0.4	0.5	29.576	43.906	15.2	3.2
	0.6	7.795	13.031	4.923	2.697		0.6	7.929	14.541	5.412	3.073
	0.7	3.741	7.471	3.069	2.635		0.7	3.851	8.518	3.404	2.999
	0.8	2.293	5.386	2.375	2.596		0.8	2.4	6.212	2.635	2.952
	0.9	1.61	4.358	2.032	2.569		0.9	1.718	5.054	2.249	2.919
0.5	0.6	29.224	49.517	17.085	2.847	0.5	0.6	29.812	53.082	18.259	3.248
	0.7	7.506	14.768	5.502	2.692		0.7	7.753	16.465	6.053	3.064
	0.8	3.484	8.012	3.25	2.599		0.8	3.668	9.161	3.618	2.954
	0.9	2.076	5.538	2.425	2.538		0.9	2.238	6.428	2.707	2.881
0.6	0.7	26.907	53.687	18.475	2.82	0.6	0.7	27.694	57.6	19.765	3.207
	0.8	6.637	15.116	5.618	2.618		0.8	6.988	16.941	6.212	2.972
	0.9	2.969	7.819	3.185	2.496		0.9	3.224	9.035	3.576	2.828
0.7	0.8	22.274	51.255	17.664	2.668	0.7	0.8	23.224	55.341	19.012	3.017
	0.9	5.189	13.552	5.097	2.436		0.9	5.635	15.441	5.712	2.75
0.8	0.9	15.324	40.135	13.958	2.327	0.8	0.9	16.4	44.188	15.294	2.604

Table 4 Relative efficiency of $\hat{\pi}_{(p)}$ for $r = k = 3,\ k_1 = 4,\ k_2 = 2,\ a_1 = 11,\ b_1 = 12,\ a_2 = 1$ and $b_2 = 2$.

θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$	θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$
		$\pi = 0.7$						$\pi = 0.8$			
0.1	0.2	15.18	11.472	4.365	3.528	0.1	0.2	17.067	14.922	5.472	4.495
	0.3	4.944	6.489	2.704	3.52		0.3	5.332	8.464	3.319	4.484
	0.4	2.82	5.215	2.279	3.518		0.4	2.974	6.777	2.756	4.48
	0.5	1.999	4.65	2.091	3.516		0.5	2.087	6.021	2.505	4.478
	0.6	1.584	4.334	1.985	3.516		0.6	1.648	5.596	2.363	4.477
	0.7	1.339	4.133	1.918	3.515		0.7	1.394	5.324	2.272	4.477
	0.8	1.18	3.994	1.872	3.515		0.8	1.231	5.135	2.209	4.476
	0.9	1.069	3.893	1.838	3.515		0.9	1.12	4.997	2.163	4.476
0.2	0.3	23.421	23.524	8.382	3.595	0.2	0.3	27.326	30.218	10.57	4.587
	0.4	6.811	10.159	3.927	3.554		0.4	7.741	13.244	4.912	4.53
	0.5	3.564	7.077	2.9	3.534		0.5	3.976	9.244	3.579	4.503
	0.6	2.369	5.794	2.472	3.523		0.6	2.611	7.554	3.016	4.488
	0.7	1.79	5.106	2.243	3.516		0.7	1.959	6.64	2.711	4.479
	0.8	1.461	4.682	2.102	3.512		0.8	1.592	6.073	2.522	4.472
	0.9	1.254	4.396	2.006	3.509		0.9	1.365	5.687	2.393	4.468
0.3	0.4	29.086	37.352	12.991	3.703	0.3	0.4	34.477	47.565	16.352	4.735
	0.5	8.034	13.944	5.189	3.608		0.5	9.373	18.117	6.536	4.605
	0.6	4.021	8.845	3.489	3.557		0.6	4.632	11.565	4.352	4.535
	0.7	2.578	6.803	2.808	3.526		0.7	2.942	8.901	3.464	4.492
	0.8	1.893	5.74	2.454	3.505		0.8	2.145	7.498	2.997	4.463
	0.9	1.511	5.099	2.24	3.49		0.9	1.705	6.645	2.712	4.443
0.4	0.5	32.176	50.639	17.421	3.812	0.4	0.5	38.518	64.166	21.886	4.881
	0.6	8.614	17.266	6.296	3.651		0.6	10.228	22.383	7.959	4.663
	0.7	4.193	10.262	3.961	3.558		0.7	4.943	13.43	4.974	4.535
	0.8	2.627	7.532	3.052	3.498		0.8	3.078	9.886	3.793	4.453
	0.9	1.893	6.142	2.588	3.457		0.9	2.207	8.058	3.183	4.396
0.5	0.6	32.691	61.069	20.897	3.867	0.5	0.6	39.451	77.223	26.238	4.953
	0.7	8.549	19.545	7.056	3.638		0.7	10.306	25.345	8.946	4.642
	0.8	4.079	11.069	4.23	3.499		0.8	4.908	14.528	5.34	4.453
	0.9	2.514	7.836	3.153	3.407		0.9	3.019	10.335	3.942	4.327
0.6	0.7	30.631	66.322	22.648	3.808	0.6	0.7	37.275	83.938	28.477	4.864
	0.8	7.841	20.202	7.275	3.518		0.8	9.606	26.301	9.264	4.474
	0.9	3.678	11.009	4.21	3.339		0.9	4.528	14.549	5.347	4.231
0.7	0.8	25.996	64.082	21.901	3.56	0.7	0.8	31.99	81.513	27.668	4.517
	0.9	6.489	18.657	6.76	3.234		0.9	8.13	24.552	8.681	4.082
0.8	0.9	18.785	52.03	17.884	3.036	0.8	0.9	23.596	67.15	22.881	3.796

Table 5 Relative efficiency of $\hat{\pi}_{(p)}$ for $r = k = 3,\ k_1 = 4,\ k_2 = 2,\ a_1 = 11,\ b_1 = 12,\ a_2 = 1$ and $b_2 = 2$.

θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$	θ_1	θ_2	$RE_{(Kuk)}$	$RE_{(SG)}$	$RE_{(H_1)}$	$RE_{(H_2)}$
		$\pi = 0.9$						$\pi = 0.9$			
0.1	0.2	22.467	22.733	7.978	6.629	0.3	0.7	3.828	13.575	4.925	6.624
	0.3	6.578	12.867	4.689	6.612		0.8	2.733	11.4	4.2	6.578
	0.4	3.504	10.244	3.815	6.606		0.9	2.133	10.067	3.756	6.545
	0.5	2.383	9.058	3.419	6.603	0.4	0.5	54.467	95.933	32.378	7.245
	0.6	1.844	8.387	3.196	6.601		0.6	14.244	34.067	11.756	6.897
	0.7	1.541	7.956	3.052	6.6		0.7	6.763	20.556	7.252	6.693
	0.8	1.351	7.656	2.952	6.599		0.8	4.133	15.133	5.444	6.561
	0.9	1.224	7.435	2.878	6.599		0.9	2.911	12.307	4.502	6.47
0.2	0.3	37.578	45.667	15.622	6.777	0.5	0.6	56.244	115.267	38.822	7.353
	0.4	10.244	20.2	7.133	6.686		0.7	14.578	38.6	13.267	6.861
	0.5	5.084	14.081	5.094	6.643		0.8	6.862	22.304	7.835	6.56
	0.6	3.244	11.467	4.222	6.618		0.9	4.161	15.892	5.697	6.359
	0.7	2.378	10.04	3.747	6.603	0.6	0.7	53.578	125.4	42.2	7.202
	0.8	1.899	9.148	3.449	6.593		0.8	13.8	40.2	13.8	6.588
	0.9	1.605	8.54	3.247	6.586		0.9	6.467	22.467	7.889	6.203
0.3	0.4	48.244	71.4	24.2	7.014	0.7	0.8	46.467	122.333	41.178	6.637
	0.5	12.8	27.6	9.6	6.806		0.9	11.911	37.867	13.022	5.957
	0.6	6.17	17.667	6.289	6.693	0.8	0.9	34.911	102.067	34.422	5.481

CONFLICT OF INTEREST

There are no conflicts of interest among the authors.

AUTHORS' CONTRIBUTIONS

Zawar Hussain and Said Farooq Shah conceived the idea and contributed in calculation of results and writing of the manuscript.

Funding Statement

There is no funding for this work.

REFERENCES

1. A.Y.C. Kuk, Biometrika. 77 (1990), 436–438.
2. S. Singh, I.S. Grewal, Int. J. Contemp. Math. Sci. 8 (2013), 243–248.
3. Z. Hussain, B. Al-Zahrani, J. Shabbir, M. Khan, Commun. Stat. Simul. Comput. 45 (2016), 3584–3596.
4. Z. Hussain, J. Shabbir, Z. Pervez, S.F. Shah, M. Khan, Commun. Stat. Simul. Comput. 46 (2017), 4708–4721.
5. S.L. Warner, J. Am. Stat. Assoc. 60 (1965), 63–69.
6. G. Blair, K. Imai, Y.Y. Zhou, J. Am. Stat. Assoc. 110 (2015), 1304–1319.