International Journal of Computational Intelligence Systems
Vol. 12(2), 2019, pp. 1339–1352
DOI: https://doi.org/10.2991/ijcis.d.191104.001; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

On Consistency and Priority Weights for Uncertain 2-Tuple Linguistic Preference Relations

Peng Wu[1], Jiaming Zhu[1], Ligang Zhou[1,2,*], Huayou Chen[1], Yu Chen[1]

[1] School of Mathematical Sciences, Anhui University, Hefei 230601, Anhui, P. R. China
[2] China Institute of Manufacturing Development, Nanjing University of Information Science and Technology, Nanjing 210044, Jiangsu, P. R. China

ARTICLE INFO

Article History

Received 08 Nov 2018
Accepted 15 Oct 2019

Keywords

Group decision-making
Uncertain 2-tuple linguistic preference relations
Additive consistency
Uncertain 2-tuple linguistic priority weights

ABSTRACT

Consistency and priority weights of preference relations are two important phases of decision-making process since the decision-making solutions are determined by them. Therefore, it is meaningful to investigate consistency and priority weights for preference relations. In this paper, consistency and uncertain 2-tuple linguistic priority weights of uncertain 2-tuple linguistic preference relations (U2TLPRs) are investigated. First, based on the additive consistency, an additive consistency index is developed to measure the additive consistency level of U2TLPRs. Second, a goal programming model is proposed to adjust the unacceptable additive consistent U2TLPR until it satisfies acceptable additive consistency. Furthermore, an optimization model is developed to derive the uncertain 2-tuple linguistic priority weights from an U2TLPR. Meanwhile, in group decision-making (GDM) problems, similarities and confidence degrees of decision makers (DMs) are defined to determine DMs' weights. Subsequently the properties of collective U2TLPR are discussed. Finally, the proposed methods are implemented in two examples including a GDM problem to verify the validity of the proposed methods.

1. INTRODUCTION

In decision-making process, the preference relations, which consist of pairwise comparison information provided by decision makers (DMs), are popular and powerful tools to model DMs preferences regarding decision-making problems. In recent years, various preference relations have been investigated, including multiplicative preference relation [1], fuzzy preference relation [2], interval multiplicative preference relation [3,4], interval fuzzy preference relation [5,6], triangular fuzzy reciprocal preference relation [7], triangular fuzzy additive reciprocal preference relations [8,9], multiplicative trapezoidal fuzzy preference relation [10–12], additive trapezoidal fuzzy preference relation [13], intuitionistic fuzzy preference relation [14] and hesitant fuzzy preference relation [15,16]. These preference relations have been mainly studied from two perspectives, which are as follows:

- Consistency measure: It is a critical step for any kinds of preference relations since consistency directly affects the rationality of final decision-making results.
- Priority weights: The other pivotal phase is to obtain priority weights from preference relations because they are utilized to select the optimal alternative(s).

The aforementioned preference relations utilized numerical values to assess pairs of alternatives. However, in the uncertainty of decision-making environment, DMs prefer to give their assessments using linguistic terms. Nevertheless, the traditional fuzzy linguistic approach does not capture all available information and may lead to the loss of information [17]. Therefore, to overcome the limitations of traditional fuzzy linguistic approach, an useful 2-tuple linguistic representation model was introduced by Herrera and Martínez [17] and the merit of 2-tuple linguistic representation model is that allows one to compute with words (CWWs) without loss of information. Since the 2-tuple linguistic representation model was put forward, research on the measurement theories [18,19], information fusion theories [20,21], preference relation theories [22] and applications [23,24] have made great achievements. Some comprehensive overviews on 2-tuple linguistic representation model have been provided [25,26].

Uncertain 2-tuple linguistic representation model [27] (Interval 2-tuple linguistic representation model or Interval-valued 2-tuple linguistic representation model in the sense of some literatures) as an extension of 2-tuple linguistic representation model has attracted many scholars' attention since it can well express DMs' uncertain qualitative preferences when the uncertainty of decision problems and/or the lack of relevant experience with the evaluated alternatives. For example, a DM may give his/her assessment on an alternative as "between good and very good," which is a typical form of uncertain 2-tuple linguistic variable. Much work of uncertain 2-tuple linguistic variable is summarized from theories and applications for uncertain 2-tuple linguistic variable.

* Corresponding author. Email: shuiqiaozlg@126.com

(1) Theories of uncertain 2-tuple linguistic variables

(1.1) Uncertain 2-tuple linguistic fusion theory: It plays a vital role in decision-making problems since the group opinion is obtained by it. Some information aggregation operators are developed, including uncertain 2-tuple weighted averaging operator [27], uncertain 2-tuple order weighted averaging operator [27], uncertain 2-tuple ordered weighted harmonic operator and uncertain 2-tuple ordered weighted quadratic operator [28], uncertain 2-tuple correlated averaging operator and uncertain 2-tuple correlated geometric operator [29], uncertain 2-tuple linguistic Bonferroni mean operator [30], interval 2-tuple linguistic induced continuous ordered weighted averaging [31], uncertain 2-tuple linguistic Choquet integral operator [32].

(1.2) Uncertain 2-tuple linguistic measurement theory: Since measurement theory is foundation of decision-making methods, it plays an important role in decision-making theory. Some distance measurements are introduced such as uncertain 2-tuple linguistic induced continuous ordered weighted distance measure [31], generalized uncertain 2-tuple linguistic interval distance measures [33], generalized uncertain 2-tuple linguistic Shapley weighted uncertain distance measures [33].

(1.3) Uncertain 2-tuple linguistic decision-making theory: For decision-making problems with uncertain 2-tuple linguistic environment, some different decision-making methods are proposed. For example, uncertain 2-tuple linguistic VIKOR method [34,35], uncertain 2-tuple linguistic MULTIMOORA method [36], uncertain 2-tuple ELECTRE II [37].

(1.4) Uncertain 2-tuple linguistic preference relations theory: In order to introduce uncertain 2-tuple linguistic variable into decision-making process, Zhang and Guo [38] defined uncertain 2-tuple linguistic preference relation (U2TLPR). For incomplete U2TLPRs, two algorithms, which contain an iterative algorithm and an optimization-based algorithm, were developed to estimate the incomplete elements. Next, Zhang and Guo [39] investigated the consistency and consensus of U2TLPR by designing consistency improving algorithm and consensus reaching algorithm. Yao and Hu [40] studied the consistency and weighting vector of U2TLPR. In addition, a consistency improving algorithm was developed to derive U2TLPR with acceptable consistency from U2TLPR with unacceptable consistency.

(2) Applications of uncertain 2-tuple linguistic variables

The applications of uncertain 2-tuple linguistic variable have made great achievements in different fields, which include material selection [34], supplier selection [35,37], healthcare waste treatment technology evaluation and selection [36], emergency response capacity evaluation [41], failure mode and effect analysis [42], tacit knowledge [43], robot evaluation and selection [44], human-machine function allocation method for aircraft cockpit [45], energy planning [46], evaluating the risk of healthcare failure modes [47].

Theoretical research of uncertain 2-tuple linguistic variables not only has a strong theoretical research value, but also has wide application prospects in practice. Therefore, it is significant for us to investigate the decision-making problems with uncertain 2-tuple linguistic variables environment. Based on the above reviews, we find that the theory of uncertain 2-tuple linguistic variables is gradually completely and the applications of uncertain 2-tuple linguistic variables have wide scope. However, despite significant progress over the past years on decision-making methods under uncertain 2-tuple linguistic variables environment, only a few attempts have been made to deal with the consistency measure and priority weights of U2TLPRs. Moreover, the existing consistency measures are unstable since they were defined based on different consistent U2TLPRs derived from an original U2TLPR. Meanwhile, the real priority weights of U2TLPR may cause loss of uncertain 2-tuple linguistic information. Motivated by these, in this paper, we mainly discuss the consistency measure and priority weights of U2TLPRs. To accomplish these goals, the following points are considered:

- An additive consistency index is developed to measure the additive consistency level of U2TLPRs. The additive consistency index is different from the other indices [39,40]. It is reliable and stable since it only depends on the uncertain 2-tuple linguistic information of original U2TLPRs.
- For an U2TLPR with unacceptable additive consistency, an optimization-based model, which is a goal programming model, is presented to derive acceptable additive consistent U2TLPR from the U2TLPR with unacceptable additive consistency.
- To obtain the uncertain 2-tuple linguistic priority weights of U2TLPR, an optimization model is developed. The uncertain 2-tuple linguistic priority weighting vector is composed of uncertain 2-tuple linguistic variables. Being different with the crisp weights [40], the uncertain 2-tuple linguistic priority weights can keep the integrity of final decision-making information derived from the original decision-making information.
- In group decision-making (GDM) problems, the DMs' weights are determined based on the confidence degree, which is defined by the similarities among the DMs. Finally, a new method is proposed to solve GDM with U2TLPRs.

The remainder of the paper is processed as follows. In detail, Section 2 mainly reviews some preliminaries which are utilized in the following discussion. Section 3 first defines an additive consistency index of U2TLPR. Then, for unacceptable additive consistent U2TLPRs, a goal programming model is proposed to obtain U2TLPRs with acceptable additive consistency. In Section 4, an optimization model is developed to derive the uncertain 2-tuple linguistic priority weights. Section 5 provides a method to determine the DM's weights. Meanwhile, a method for addressing GDM problems is proposed. In Section 6, the application of our proposed methods is illustrated by using some examples and some comparisons are discussed simultaneously. Finally, Section 7 outlines the main work of this paper and the future research is given.

2. PRELIMINARIES

In this section, to introduced our work, some preliminaries are reviewed.

2.1. The 2-Tuple Linguistic Variable and 2-Tuple Linguistic Preference Relation (2TLPR)

Let $S = \{s_0, s_1, \cdots, s_\tau\}$ be a linguistic term set with granularity $\tau+1$. To CWWs, Herrera and Martínez [17] utilized a pair of values (s_i, α) called linguistic 2-tuple to propose the concept of 2-tuple linguistic representation model, where $s_i \in S$ represents the central value of the ith linguistic term and $\alpha \in [-0.5, 0.5)$ indicates the deviation to the central value of the ith linguistic term.

In traditional 2-tuple linguistic representation model, there is $\beta \in [0, \tau]$ represents the results of an aggregation of the indices of a set of labels in S. However, the range of β has a restriction in multi-granularity linguistic term sets. To overcome the restriction, Tai and Chen [48] proposed a generalized 2-tuple linguistic model and translation functions, which are defined as Definition 1.

Definition 1. [48] Let S be as before and $\beta \in [0, 1]$ be a value representing the result of a symbolic aggregation operation. To obtain the 2-tuple linguistic variable equivalent to β, the generalized translation function $\Delta: [0,1] \to S \times \left[-\frac{1}{2\tau}, \frac{1}{2\tau}\right)$ is defined as follows:

$$\Delta(\beta) = (s_i, \alpha) \quad with \quad \begin{cases} s_i, & i = round(\beta \cdot \tau) \\ \alpha = \beta - \frac{i}{\tau}, & \alpha \in \left[-\frac{1}{2\tau}, \frac{1}{2\tau}\right). \end{cases} \tag{1}$$

Conversely, there is a function $\Delta^{-1}: S \times \left[-\frac{1}{2\tau}, \frac{1}{2\tau}\right) \to [0, 1]$ used to transform 2-tuple linguistic variable into its equivalent crisp $\beta \in [0, 1]$ as follows:

$$\Delta^{-1}(s_i, \alpha) = \frac{i}{\tau} + \alpha = \beta. \tag{2}$$

In particular, a linguistic term s_i can be converted into a 2-tuple linguistic variable $(s_i, 0)$.

Definition 2. [22,49] Let S be as before and $L = \left(l_{ij}\right)_{n\times n}$ be a linguistic matrix. For $i, j = 1, 2, \cdots, n$, if $\Delta\left(\Delta^{-1}\left(l_{ij}\right) + \Delta^{-1}\left(l_{ji}\right)\right) = (s_\tau, 0), l_{ii} = \left(s_{\tau/2}, 0\right)$, then L is called 2TLPR, where $l_{ij} = \left(s_{ij}, \alpha_{ij}\right)$, $s_{ij} \in S$ and $\alpha_{ij} \in \left[-\frac{1}{2\tau}, \frac{1}{2\tau}\right)$.

2.2. The Uncertain 2-Tuple Linguistic Variable and U2TLPR

Definition 3. [27] Let S be as before. An uncertain 2-tuple linguistic variable $\left[(s_i, \alpha_1), (s_j, \alpha_2)\right]$ with $(s_i, \alpha_1) \le \left(s_j, \alpha_2\right)$ [39] is composed of two 2-tuple linguistic variables (s_i, α_1) and $\left(s_j, \alpha_2\right)$. An uncertain 2-tuple linguistic variable expressing the equivalent information to an interval value $[\beta_1, \beta_2]$ $(\beta_1 \le \beta_2 \in [0, 1], \beta_1 \le \beta_2)$ is obtained by the following function:

$$\Delta[\beta_1, \beta_2] = \left[(s_i, \alpha_1), \left(s_j, \alpha_2\right)\right], \tag{3}$$

in which $i = round(\beta_1 \cdot \tau)$, $j = round(\beta_2 \cdot \tau)$, $\alpha_1 = \beta_1 - \frac{i}{\tau}, \alpha_1 \in \left[-\frac{1}{2\tau}, \frac{1}{2\tau}\right)$ and $\alpha_2 = \beta_2 - \frac{j}{\tau}, \alpha_2 \in \left[-\frac{1}{2\tau}, \frac{1}{2\tau}\right)$.

Meanwhile, there exists a function Δ^{-1} used to transform uncertain 2-tuple linguistic variable $\left[(s_i, \alpha_1), \left(s_j, \alpha_2\right)\right]$ into an interval value $[\beta_1, \beta_2]$ as follows:

$$\Delta^{-1}\left[(s_i, \alpha_1)\left(s_j, \alpha_2\right)\right] = \left[\frac{i}{\tau} + \alpha_1, \frac{j}{\tau} + \alpha_2\right] = [\beta_1, \beta_2]. \tag{4}$$

Specially, an interval $\left[s_i, s_j\right]$ can be transformed into an uncertain 2-tuple linguistic variable $\left[(s_i, 0), \left(s_j, 0\right)\right]$.

Definition 4. [37] Let $\tilde{a} = \left[(s_i, \alpha_1), \left(s_j, \alpha_2\right)\right]$ and $\tilde{b} = [(s_k, \beta_1), (s_l, \beta_2)]$ are two uncertain 2-tuple variables. The Manhattan distance between $\tilde{a}$ and $\tilde{b}$ is defined as

$$\begin{aligned} d_M\left(\tilde{a}, \tilde{b}\right) = \frac{1}{2}\Big(&\left|\Delta^{-1}(s_i, \alpha_1) - \Delta^{-1}(s_k, \beta_1)\right| \\ &+ \left|\Delta^{-1}\left(s_j, \alpha_2\right) - \Delta^{-1}(s_l, \beta_2)\right|\Big). \end{aligned} \tag{5}$$

In order to rank the uncertain 2-tuple linguistic variables, Wan *et al.* [37] proposed the possibility degree of uncertain 2-tuple linguistic variables. Let $\tilde{u}_i = \left[u_i^-, u_i^+\right] (i = 1, 2, \cdots, n)$ be uncertain 2-tuple linguistic variable, the possibility degree between $\tilde{u}_i$ and $\tilde{u}_j$ is defined as

$$p\left(\tilde{u}_i \ge \tilde{u}_j\right) = \frac{1}{2}\left(1 + \frac{DE}{NU}\right), \tag{6}$$

where $DE = \left(\Delta^{-1}u_i^+ - \Delta^{-1}u_j^+\right) + \left(\Delta^{-1}u_i^- - \Delta^{-1}u_j^-\right)$, $NU = \left|\Delta^{-1}u_i^+ - \Delta^{-1}u_j^+\right| + \left|\Delta^{-1}u_i^- - \Delta^{-1}u_j^-\right| + l_{\tilde{u}_i\tilde{u}_j}$, $l_{\tilde{u}_i\tilde{u}_j}$ represents the length of intersection of interval values $\Delta^{-1}(\tilde{u}_i)$ and $\Delta^{-1}\left(\tilde{u}_j\right)$. The possibility degree has the features: (1) $0 \le p\left(\tilde{u}_i \ge \tilde{u}_j\right) \le 1$; (2) $p\left(\tilde{u}_i \ge \tilde{u}_j\right) + p\left(\tilde{u}_j \ge \tilde{u}_i\right) = 1$.

Let $p_{ij} = p\left(\tilde{u}_i \ge \tilde{u}_j\right)$, matrix $P = \left(p_{ij}\right)_{n\times n}$ is called possible degree matrix. It is obvious that P is a fuzzy preference relation. By summing each row of P, we can obtain $q_i = \sum_{j=1}^n p_{ij}$. Then, by $\overline{p}_{ij} = \frac{q_i - q_j}{2(n-1)} + \frac{1}{2}$, we can obtain a consistent fuzzy preference relation $\overline{P} = \left(\overline{p}_{ij}\right)_{n\times n}$.

Then, by normalizing of the sum of each row for $\overline{P}$, the dominant index of uncertain 2-tuple linguistic variable $\tilde{u}_i$ is defined as follows:

$$DI_i = \frac{1}{n(n-1)}\left(\sum_{j=1}^n p\left(\tilde{u}_i \ge \tilde{u}_j\right) + \frac{n}{2} - 1\right). \tag{7}$$

It represents that the average possible degree of $\tilde{u}_i$ preferred to $\tilde{u}_j \left(j = 1, 2, \cdots, n\right)$. The bigger the value of dominant index DI_i, the larger the uncertain 2-tuple linguistic variable $\tilde{u}_i$. Therefore, based on the dominant indices DI_i, uncertain 2-tuple linguistic variables can be ranked.

Definition 5. [39] An U2TLPR is defined as $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n} = \left(\left[\tilde{u}_{ij}^-, \tilde{u}_{ij}^+\right]\right)_{n\times n}$. For $\forall i, j = 1, 2, \cdots, n$, $\tilde{U}$ satisfies $\tilde{u}_{ii}^+ = \tilde{u}_{ii}^- = (s_{\tau/2}, 0)$, $\Delta\left(\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right)\right) = \Delta\left(\Delta^{-1}\left(\tilde{u}_{ji}^-\right) + \Delta^{-1}\left(\tilde{u}_{ij}^+\right)\right) = (s_\tau, 0)$ and $\tilde{u}_{ij}^- \le \tilde{u}_{ij}^+$. Where $\tilde{u}_{ij}$ is an uncertain 2-tuple linguistic variable, which indicates the interval linguistic preference degree of the alternative x_i over x_j.

Particularly, if $\tilde{u}_{ij}^- = \tilde{u}_{ij}^+$, then an U2TLPR degenerates into a 2TLPR.

3. ADDITIVE CONSISTENCY OF U2TLPR

In this section, the additive consistency index of an U2TLPR is presented based on the additive consistency measure. Furthermore, a goal programming model is constructed to improve the additive consistency of an U2TLPR with unacceptable additive consistency until it satisfies acceptable additive consistency.

3.1. Additive Consistency Index of U2TLPR

Motivated by the definition of interval fuzzy preference relations [50], the additive consistency of an U2TLPR is defined as Definition 6.

Definition 6. Let $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ with $\tilde{u}_{ij} = \left[\tilde{u}_{ij}^-, \tilde{u}_{ij}^+\right]$ be an U2TLPR. For $i, j, k = 1, \cdots, n$, if $\tilde{U}$ satisfies the following condition:

$$\Delta^{-1}\left(\tilde{u}_{ij}^\dagger\right) + \Delta^{-1}\left(\tilde{u}_{jk}^\dagger\right) + \Delta^{-1}\left(\tilde{u}_{ki}^\dagger\right) = \Delta^{-1}\left(\tilde{u}_{kj}^\dagger\right) + \Delta^{-1}\left(\tilde{u}_{ji}^\dagger\right) + \Delta^{-1}\left(\tilde{u}_{ik}^\dagger\right), \quad (8)$$

then $\tilde{U}$ is an additive consistent U2TLPR for $\dagger = +, -$.

According to Definition 6, we have the following theorem.

Theorem 1. *An U2TLPR $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ is defined as before. $\tilde{U}$ has additive consistency if and only if*

$$\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right). \quad (9)$$

Proof. Sufficiency (⇒). Since $\tilde{U}$ is an additive consistent U2TLPR, as Definition 6, we have

$$\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) = \Delta^{-1}\left(\tilde{u}_{kj}^+\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right) + \Delta^{-1}\left(\tilde{u}_{ik}^+\right), \quad (10)$$

$$\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = \Delta^{-1}\left(\tilde{u}_{kj}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^-\right) + \Delta^{-1}\left(\tilde{u}_{ik}^-\right). \quad (11)$$

Based on Definition 5, we obtain $\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right) = \Delta^{-1}\left(s_\tau, 0\right)$. Then, according to Eqs. (10) and (11), we have

$$\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right).$$

Necessity (⇐). As $\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right)$ and $\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right) = \Delta^{-1}\left(s_\tau, 0\right)$, we have

$$\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) = \Delta^{-1}\left(\tilde{u}_{kj}^+\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right) + \Delta^{-1}\left(\tilde{u}_{ik}^+\right),$$

$$\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = \Delta^{-1}\left(\tilde{u}_{kj}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^-\right) + \Delta^{-1}\left(\tilde{u}_{ik}^-\right).$$

It means that $\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right)$, i.e., $\tilde{U}$ has additive consistency.

Theorem 2. *Let $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ be an additive consistent U2TLPR. Then, the following conditions are equivalent:*

(I) $\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right), i, j, k = 1, 2, \cdots, n$;

(II) $\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right), i < j < k$.

Proof. It is obvious that (I) ⇒ (II).

(II) ⇒ (I). Based on Definition 6, we know that (I) holds true when any two or three of indices i, j, k are equal. Therefore, we only consider that $i \neq j \neq k$, which contains six possible index orders.

1. $i < j < k$. Obviously, (I) holds true.
2. $i < k < j$. We have $\Delta^{-1}\left(\tilde{u}_{ik}^+\right) + \Delta^{-1}\left(\tilde{u}_{ik}^-\right) + \Delta^{-1}\left(\tilde{u}_{kj}^+\right) + \Delta^{-1}\left(\tilde{u}_{kj}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right) + \Delta^{-1}\left(\tilde{u}_{ji}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right)$. Since $\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right) = \Delta^{-1}\left(s_\tau, 0\right)$, we obtain

$$\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right).$$

Similarity, (I) holds true in the rest of four cases: $j < k < i$, $j < i < k$, $k < i < j$ and $k < j < i$. Then, we have Thus, we have (II) ⇒ (I). Based on the above analyses, we obtain (I) ⇔ (II).

Theorem 2 provides two ways to check the additive consistency level of an U2TLPR. That is to say, any one of (I) and (II) can be used to judge whether U2TLPR has additive consistency. Considering the computational complexity of (I) and (II), we select (II) to define the additive consistency index of U2TLPR.

In fact, the U2TLPRs provided by DMs do not satisfy additive consistency since the lack of cognition of DMs. Namely, based on (II), there is the deviation between $\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right)$ and $3\Delta^{-1}\left(s_\tau, 0\right)$.

The total deviation can be defined by means of Manhattan distance as

$$MD\left(\tilde{U}\right) = \sum_{i<j<k}^{n} \mid \Delta^{-1}\left(\tilde{u}_{ij}^{+}\right) + \Delta^{-1}\left(\tilde{u}_{ij}^{-}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{+}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{-}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{+}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{-}\right) - 3\Delta^{-1}\left(s_{\tau}, 0\right) \mid .$$

In the following, the additive consistency index of an U2TLPR $\tilde{U}$ is defined as Definition 7.

Definition 7. Based on the total deviation $MD\left(\tilde{U}\right)$, the additive consistency index of an U2TLPR $\tilde{U}$ is defined as follows:

$$ACI\left(\tilde{U}\right) = \frac{2}{n(n-1)(n-2)} \sum_{i<j<k}^{n} \mid \Delta^{-1}\left(\tilde{u}_{ij}^{+}\right) + \Delta^{-1}\left(\tilde{u}_{ij}^{-}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{+}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{-}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{+}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{-}\right) - 3\Delta^{-1}\left(s_{\tau}, 0\right) \mid . \quad (12)$$

The additive consistency index indicates the reliability of the uncertain 2-tuple linguistic preference information. The smaller the value of $ACI\left(\tilde{U}\right)$, the more consistent and reliable the uncertain preference information in U2TLPR $\tilde{U}$.

Remark 1. Zhang and Guo [39] introduced the consistency index of an U2TLPR by calculating the deviation between an U2TLPR and its corresponding consistent U2TLPR. Yao and Hu [40] defined a consistency index of an U2TLPR based on the distance between an U2TLPR and its corresponding consistent U2TLPR. In general, different methods would derive various different consistent U2TLPRs, which would lead to diverse consistency indices of an U2TLPR. In this paper, the additive consistency index of an U2TLPR defined in Definition 7 is reliable and stable since it only depends on the uncertain preference information of original U2TLPR.

Example 1. [39]. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of alternatives, a DM provides his preference information using U2TLPR, which is shown as follows:

$$\tilde{A} = \begin{pmatrix}
[(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] & [(s_2,0),\ (s_4,0)] & [(s_5,0),\ (s_6,0)] & [(s_4,0),\ (s_6,0)] & [(s_2,0),\ (s_3,0)] \\
[(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)] & [(s_2,0),\ (s_3,0)] & [(s_6,0),\ (s_8,0)] & [(s_2,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] \\
[(s_4,0),\ (s_6,0)] & [(s_5,0),\ (s_6,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] & [(s_3,0),\ (s_5,0)] & [(s_6,0),\ (s_7,0)] \\
[(s_2,0),\ (s_3,0)] & [(s_0,0),\ (s_2,0)] & [(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)] & [(s_5,0),\ (s_6,0)] & [(s_5,0),\ (s_6,0)] \\
[(s_2,0),\ (s_4,0)] & [(s_4,0),\ (s_6,0)] & [(s_3,0),\ (s_5,0)] & [(s_2,0),\ (s_3,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] \\
[(s_5,0),\ (s_6,0)] & [(s_3,0),\ (s_4,0)] & [(s_1,0),\ (s_2,0)] & [(s_2,0),\ (s_3,0)] & [(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)]
\end{pmatrix}.$$

By Eq. (12), the additive consistency of $\tilde{A}$ is calculated as $ACI\left(\tilde{A}\right) = 0.1792$.

In Zhang and Guo [39], the method of constructing the consistent U2TLPR was developed. Using the method, the consistent U2TLPR is obtained

$$\tilde{A}^{*}_{Zhang} = \begin{pmatrix}
[(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] & [(s_2,0),\ (s_4,0)] & [(s_2,0),\ (s_5,0)] & [(s_3,0),\ (s_7,0)] & [(s_3,0),\ (s_8,0)] \\
[(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)] & [(s_2,0),\ (s_3,0)] & [(s_2,0),\ (s_4,0)] & [(s_3,0),\ (s_6,0)] & [(s_3,0),\ (s_7,0)] \\
[(s_4,0),\ (s_6,0)] & [(s_5,0),\ (s_6,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] & [(s_5,0),\ (s_7,0)] & [(s_5,0),\ (s_8,0)] \\
[(s_3,0),\ (s_6,0)] & [(s_4,0),\ (s_6,0)] & [(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)] & [(s_5,0),\ (s_6,0)] & [(s_5,0),\ (s_7,0)] \\
[(s_1,0),\ (s_5,0)] & [(s_2,0),\ (s_5,0)] & [(s_1,0),\ (s_3,0)] & [(s_2,0),\ (s_3,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] \\
[(s_0,0),\ (s_5,0)] & [(s_1,0),\ (s_5,0)] & [(s_0,0),\ (s_3,0)] & [(s_1,0),\ (s_3,0)] & [(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)]
\end{pmatrix}.$$

Then, based on the concept of consistency index [39], we obtain $CI\left(\tilde{A}\right)_{Zhang} = d\left(\tilde{A}, \tilde{A}^{*}_{Zhang}\right) = 0.1375$.

In Yao and Hu [40], another method for deriving the consistent U2TLPR was presented. According to their method, the corresponding consistent U2TLPR $\tilde{A}^{*}_{Yao}$ is obtained as follows:

$$\tilde{A}^{*}_{Yao} = \begin{pmatrix}
[(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] & [(s_2,0),\ (s_4,0)] & [(s_4,0),\ (s_6,0)] & [(s_4,0),\ (s_6,0)] & [(s_2,0),\ (s_5,0)] \\
[(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)] & [(s_2,0),\ (s_3,0)] & [(s_4,0),\ (s_8,0)] & [(s_2,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] \\
[(s_4,0),\ (s_6,0)] & [(s_5,0),\ (s_6,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] & [(s_3,0),\ (s_5,0)] & [(s_6,0),\ (s_7,0)] \\
[(s_2,0),\ (s_4,0)] & [(s_0,0),\ (s_4,0)] & [(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_6,0)] & [(s_5,0),\ (s_6,0)] \\
[(s_2,0),\ (s_4,0)] & [(s_4,0),\ (s_6,0)] & [(s_3,0),\ (s_5,0)] & [(s_2,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,0),\ (s_5,0)] \\
[(s_3,0),\ (s_6,0)] & [(s_3,0),\ (s_4,0)] & [(s_1,0),\ (s_2,0)] & [(s_2,0),\ (s_3,0)] & [(s_3,0),\ (s_4,0)] & [(s_4,0),\ (s_4,0)]
\end{pmatrix}.$$

Then, we have $CI\left(\tilde{A}\right)_{Yao} = d\left(\tilde{A}, \tilde{A}^{*}_{Yao}\right) = 0.0250$.

It is worthwhile mentioning that $CI\left(\tilde{A}\right)_{Zhang} \neq CI\left(\tilde{A}\right)_{Yao}$ and U2TLPRs $\tilde{A}^{*}_{Zhang}$ and $\tilde{A}^{*}_{Yao}$ derived from the same original U2TLPR $\tilde{A}$ are consistent. It means that different methods for obtaining consistent U2TLPRs would derive diverse additive consistency based estimated U2TLPR. However, the consistency level of U2TLPR should be not change with its corresponding consistent U2TLPR. Because the additive consistency index defined in Definition 7 only depends on the uncertain 2-tuple linguistic information of original U2TLPR, it is reliable and stable.

Based on Definition 7, the following theorem is apparent.

Theorem 3. *Let U2TLPR $\tilde{U}$ be as before. Then,*

1. $ACI\left(\tilde{U}\right) \in [0, 1]$;
2. $ACI\left(\tilde{U}\right) = 0$ if and only if is an additive consistent U2TLPR.

Providing a predefined threshold $\overline{CI} \in [0, 1]$, the concept of acceptable additive consistent U2TLPR is defined as follows:

Definition 8. Let $\tilde{U}$ be an U2TLPR and $\overline{CI} \in [0,1]$ be a predefined consistency threshold. If

$$ACI\left(\tilde{U}\right) \leq \overline{CI}, \tag{13}$$

then $\tilde{U}$ has acceptable additive consistency. Otherwise, $\tilde{U}$ has unacceptable additive consistency.

In general, the consistency threshold $\overline{CI}$ is determined on the basis of the practical decision-making problems. As Wan *et al.* [51] said, if the decision-making problem is significant, the consistency threshold should be allocated a small value; if the decision-making problem is urgent, a lager value should be assigned.

3.2. Obtaining U2TLPR with Acceptable Additive Consistency

Due to the complexity of decision-making environment and the diversity of DMs' cognition, it is hard for DMs to provide U2TLPRs that satisfy additive consistency. Hence, for an unacceptable additive consistent U2TLPR, a goal programming model will be developed to derive an U2TLPR with acceptable additive consistency from the unacceptable additive consistent U2TLPR. Based on the minimum deviation between an acceptable additive consistent U2TLPR $\tilde{U}^* = \left(\tilde{u}^*_{ij}\right)_{n\times n} = \left(\left[\tilde{u}^{-*}_{ij}, \tilde{u}^{+*}_{ij}\right]\right)_{n\times n}$ and an U2TLPR $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n} = \left(\left[\tilde{u}^-_{ij}, \tilde{u}^+_{ij}\right]\right)_{n\times n}$ with unacceptable additive consistency, a mathematical optimization model (M-1) is constructed as follows:

$$(M-1)\ \min \sum_{i<j}^{n} dev_{ij}$$
$$s.t. \begin{cases} \dfrac{2}{n(n-1)(n-2)} \displaystyle\sum_{i<j<k}^{n} \left|Dev^*_{ijk}\right| \leq \overline{CI}, \\ 0 \leq \Delta^{-1}\left(\tilde{u}^{-*}_{ij}\right) \leq \Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right) \leq 1, \\ i,j = 1,\cdots,n, \\ i<j, \end{cases}$$

where $dev_{ij} = \left(\left|\Delta^{-1}\left(\tilde{u}^{-*}_{ij}\right) - \Delta^{-1}\left(\tilde{u}^{-}_{ij}\right)\right| + \left|\Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right) - \Delta^{-1}\left(\tilde{u}^{+}_{ij}\right)\right|\right)$ and $Dev^*_{ijk} = \Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right) + \Delta^{-1}\left(u^{-*}_{ij}\right) + \Delta^{-1}\left(\tilde{u}^{+*}_{jk}\right) + \Delta^{-1}\left(\tilde{u}^{-*}_{jk}\right) + \Delta^{-1}\left(\tilde{u}^{+*}_{ki}\right) + \Delta^{-1}\left(\tilde{u}^{-*}_{ki}\right) - 3\Delta^{-1}(s_\tau, 0)$. The first constraint condition ensures the obtained U2TLPR $\tilde{U}^*$ is acceptable additive consistency. Meanwhile, the second condition guarantees the elements located in the upper triangular of $\tilde{U}^*$ are uncertain 2-tuple linguistic variables.

To better solve model (M-1), it can be transformed into a goal programming model by introducing some parameters. Suppose that

$$\varepsilon_{ij} = \Delta^{-1}\left(\tilde{u}^{-*}_{ij}\right) - \Delta^{-1}\left(\tilde{u}^{-}_{ij}\right), \delta_{ij} = \Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right) - \Delta^{-1}\left(\tilde{u}^{+}_{ij}\right),$$

$$\varepsilon^+_{ij} = \begin{cases} \varepsilon_{ij}, \varepsilon_{ij} \geq 0 \\ 0, \varepsilon_{ij} < 0 \end{cases}, \ \varepsilon^-_{ij} = \begin{cases} 0, \varepsilon_{ij} \geq 0 \\ -\varepsilon_{ij}, \varepsilon_{ij} < 0 \end{cases},$$

$$\delta^+_{ij} = \begin{cases} \delta_{ij}, \delta_{ij} \geq 0 \\ 0, \delta_{ij} < 0 \end{cases}, \ \delta^-_{ij} = \begin{cases} 0, \delta_{ij} \geq 0 \\ -\delta_{ij}, \delta_{ij} < 0 \end{cases},$$

$$\lambda_{ijk} = \Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right) + \Delta^{-1}\left(\tilde{u}^{-*}_{ij}\right) + \Delta^{-1}\left(\tilde{u}^{+*}_{jk}\right) + \Delta^{-1}\left(\tilde{u}^{-*}_{jk}\right) + \Delta^{-1}\left(\tilde{u}^{+*}_{ki}\right) + \Delta^{-1}\left(\tilde{u}^{-*}_{ki}\right) - 3\Delta^{-1}(s_\tau, 0),$$

$$\lambda^+_{ijk} = \begin{cases} \lambda_{ijk}, \lambda_{ijk} \geq 0 \\ 0, \lambda_{ijk} < 0 \end{cases}, \ \lambda^-_{ijk} = \begin{cases} 0, \lambda_{ijk} \geq 0 \\ -\lambda_{ijk}, \lambda_{ijk} < 0 \end{cases}$$

Then, $|\varepsilon_{ij}| = \varepsilon^+_{ij} + \varepsilon^-_{ij}$, $|\delta_{ij}| = \delta^+_{ij} + \delta^-_{ij}$, $|\lambda_{ijk}| = \lambda^+_{ijk} + \lambda^-_{ijk}$, $\varepsilon_{ij} = \varepsilon^+_{ij} - \varepsilon^-_{ij}$, $\delta_{ij} = \delta^+_{ij} - \delta^-_{ij}$ and $\lambda_{ijk} = \lambda^+_{ijk} - \lambda^-_{ijk}$.

Afterward, a goal programming model (M-2) can be constructed as follows:

$$(M-2)\ \min \sum_{i<j}^{n} \left(\varepsilon^+_{ij} + \varepsilon^-_{ij} + \delta^+_{ij} + \delta^-_{ij}\right)$$
$$s.t. \begin{cases} \dfrac{2}{n(n-1)(n-2)} \displaystyle\sum_{i<j<k}^{n} \left(\lambda^+_{ijk} + \lambda^-_{ijk}\right) \leq \overline{CI} \\ \Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right) + \Delta^{-1}\left(\tilde{u}^{-*}_{ij}\right) + \Delta^{-1}\left(\tilde{u}^{+*}_{jk}\right) + \\ \Delta^{-1}\left(\tilde{u}^{-*}_{jk}\right) + \Delta^{-1}\left(\tilde{u}^{+*}_{ki}\right) + \Delta^{-1}\left(\tilde{u}^{-*}_{ki}\right) - \\ 3\Delta^{-1}(s_\tau, 0) = \lambda^+_{ijk} - \lambda^-_{ijk}, \\ 0 \leq \Delta^{-1}\left(\tilde{u}^{-*}_{ij}\right) \leq \Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right) \leq 1, \\ i,j = 1,\cdots,n; i<j. \end{cases}$$

Solving the model (M-2) yields the optimal solutions $\tilde{u}^{+*}_{ij}$ and $\tilde{u}^{-*}_{ij}$ for all $i,j = 1,2,\cdots,n$ and $i<j$. According to Definition 5, the acceptable additive consistent U2TLPR $\tilde{U}^*$ derived from the original U2TLPR $\tilde{U}$ can be generated by

$$\tilde{u}^*_{ij} = \begin{cases} \left[\tilde{u}^{-*}_{ij}, \tilde{u}^{+*}_{ij}\right] & i<j, \\ \left[\left(s_{\tau/2}, 0\right), \left(s_{\tau/2}, 0\right)\right] & i=j, \\ \left[\Delta\left(1-\Delta^{-1}\left(\tilde{u}^{+*}_{ij}\right)\right), \Delta\left(1-\Delta^{-1}\left(\tilde{u}^{-*}_{ij}\right)\right)\right] & i>j. \end{cases} \tag{14}$$

4. DERIVING THE UNCERTAIN 2-TUPLE LINGUISTIC PRIORITY VECTOR FROM U2TLPR

This section develops an optimization model to derive the uncertain 2-tuple linguistic priority weights from U2TLPR.

Let $\tilde{V} = (\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n)^T$ be a weighting vector in which $\tilde{v}_i = \left[v^-_i, v^+_i\right]$ is an uncertain 2-tuple linguistic variable, then $\tilde{V}$ is called uncertain 2-tuple linguistic priority vector if the following conditions hold:

$$\Delta^{-1}\left(v^+_i\right) + \sum_{i\neq j, j=1}^{n} \Delta^{-1}\left(v^-_j\right) \leq 1, \tag{15}$$

$$\Delta^{-1}\left(v^-_i\right) + \sum_{i\neq j, j=1}^{n} \Delta^{-1}\left(v^+_j\right) \geq 1, \tag{16}$$

$$0 \leq \Delta^{-1}\left(v^-_i\right) \leq \Delta^{-1}\left(v^+_i\right) \leq 1. \tag{17}$$

Note that the above inequations are inspired by Sugihara *et al.* [52].

For an uncertain 2-tuple linguistic priority vector $\tilde{V}$ defined as before, let

$$\tilde{u}_{ij} = \left[\tilde{u}_{ij}^-, \tilde{u}_{ij}^+\right] = \begin{cases} \left[\left(s_{\frac{\tau}{2}}, 0\right), \left(s_{\frac{\tau}{2}}, 0\right)\right] & i = j, \\ [vl, vu] & i \neq j, \end{cases} \quad (18)$$

where

$$vl = \Delta\left(\Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right) + \Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right)\left(\Delta^{-1}\left(v_i^-\right) - \Delta^{-1}\left(v_j^+\right)\right)\right)$$

and

$$vu = \Delta\left(\Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right) + \Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right)\left(\Delta^{-1}\left(v_i^+\right) - \Delta^{-1}\left(v_j^-\right)\right)\right).$$

Based on Eq. (18), the following theorem is obtained and it is shown as follows:

Theorem 4. *Let $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ be a matrix, where $\tilde{u}_{ij}$ is defined by Eq. (18), then*

1. $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ is an U2TLPR;
2. $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ has additive consistency.

Proof. (1). Based on Eq. (18), we have $\Delta^{-1}\left(\tilde{u}_{ij}^-\right) = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right)\left(\Delta^{-1}\left(v_i^-\right) - \Delta^{-1}\left(v_j^+\right)\right)$ and $\Delta^{-1}\left(\tilde{u}_{ji}^+\right) = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right)\left(\Delta^{-1}\left(v_j^+\right) - \Delta^{-1}\left(v_i^-\right)\right)$. Then, we have $\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right) = \Delta^{-1}\left(s_\tau, 0\right)$. Namely, $\Delta\left(\Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{ji}^+\right)\right) = \left(s_\tau, 0\right)$. Since $\tilde{u}_{ii}^+ = \tilde{u}_{ii}^- = \left(s_{\tau/2}, 0\right)$ and $\tilde{u}_{ij}^- \leq \tilde{u}_{ij}^+$. Therefore, we obtain $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ is an U2TLPR.

(2). According to Eq. (18), we have $\Delta^{-1}\left(\tilde{u}_{ij}^+\right) + \Delta^{-1}\left(\tilde{u}_{ij}^-\right) + \Delta^{-1}\left(\tilde{u}_{jk}^+\right) + \Delta^{-1}\left(\tilde{u}_{jk}^-\right) + \Delta^{-1}\left(\tilde{u}_{ki}^+\right) + \Delta^{-1}\left(\tilde{u}_{ki}^-\right) = 3\Delta^{-1}\left(s_\tau, 0\right)$. It means that $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ is of additive consistency.

In fact, the Theorem 4 indicates that U2TLPR $\tilde{U} = \left(\tilde{u}_{ij}\right)_{n\times n}$ with additive consistency can be constructed based on uncertain 2-tuple linguistic priority vector $\tilde{V}$ and Eq. (18).

For $i, j = 1, 2, \cdots, n$, $i \neq j$, Eq. (18) is equivalent to the following formulas:

$$\Delta^{-1}\left(\tilde{u}_{ij}^-\right) = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right)\left(\Delta^{-1}\left(v_i^-\right) - \Delta^{-1}\left(v_j^+\right)\right), \quad (19)$$

$$\Delta^{-1}\left(\tilde{u}_{ij}^+\right) = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right)\left(\Delta^{-1}\left(v_i^+\right) - \Delta^{-1}\left(v_j^-\right)\right). \quad (20)$$

In practical decision-making problems, the U2TLPRs provided by DMs are not additive consistency. Namely, there are deviation between the left and right of Eqs. (19) and (20), respectively. Then, the deviation d_{ij}^+ and d_{ij}^- are introduced as follows:

$$d_{ij}^- = \Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right) + \Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right) \times \left(\Delta^{-1}\left(v_i^-\right) - \Delta^{-1}\left(v_j^+\right)\right) - \Delta^{-1}\left(\tilde{u}_{ij}^-\right), \quad (21)$$

$$d_{ij}^+ = \Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right) + \Delta^{-1}\left(s_{\frac{\tau}{2}}, 0\right) \times \left(\Delta^{-1}\left(v_i^+\right) - \Delta^{-1}\left(v_j^-\right)\right) - \Delta^{-1}\left(\tilde{u}_{ij}^+\right), \quad (22)$$

The smaller the deviation d_{ij}^+ and d_{ij}^-, the better additive consistency of U2TLPR $\tilde{U}$. Motivated by the idea, the uncertain 2-tuple linguistic priority weights of U2TLPR can be obtained by solving the following the optimization model (M-3):

$$(M-3)\ \min \sum_{i<j}^{n} \left(\left|d_{ij}^+\right| + \left|d_{ij}^-\right|\right)$$

$$s.t. \begin{cases} d_{ij}^- = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right) \\ \left(\Delta^{-1}\left(v_i^-\right) - \Delta^{-1}\left(v_j^+\right)\right) - \Delta^{-1}\left(u_{ij}^-\right), \\ d_{ij}^+ = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right) \\ \left(\Delta^{-1}\left(v_i^+\right) - \Delta^{-1}\left(v_j^-\right)\right) - \Delta^{-1}\left(u_{ij}^+\right), \\ \Delta^{-1}\left(v_i^+\right) + \sum_{j=1, i\neq j}^{n} \Delta^{-1}\left(v_j^-\right) \leq 1, \\ \Delta^{-1}\left(v_i^-\right) + \sum_{j=1, i\neq j}^{n} \Delta^{-1}\left(v_j^+\right) \geq 1, \\ 0 \leq \Delta^{-1}\left(v_i^-\right) \leq \Delta^{-1}\left(v_i^+\right) \leq 1. \end{cases}$$

For model (M-3), suppose that

$$\phi_{ij}^+ = \begin{cases} d_{ij}^+, d_{ij}^+ \geq 0 \\ 0, d_{ij}^+ < 0 \end{cases}; \phi_{ij}^- = \begin{cases} 0, d_{ij}^+ \geq 0 \\ -d_{ij}^+, d_{ij}^+ < 0 \end{cases};$$

$$\varphi_{ij}^+ = \begin{cases} d_{ij}^-, d_{ij}^- \geq 0 \\ 0, d_{ij}^- < 0 \end{cases}; \varphi_{ij}^- = \begin{cases} 0, d_{ij}^- \geq 0 \\ -d_{ij}^-, d_{ij}^- < 0 \end{cases}.$$

Then

$$|d_{ij}^+| = \phi_{ij}^+ + \phi_{ij}^-, d_{ij}^+ = \phi_{ij}^+ - \phi_{ij}^-,$$
$$|d_{ij}^-| = \varphi_{ij}^+ + \varphi_{ij}^-, d_{ij}^- = \varphi_{ij}^+ - \varphi_{ij}^-.$$

Based on the above transformations, the model (M-3) is equivalent to a linear programming model (M-4)

$$(M-4)\ \min \sum_{i<j}^{n} \left(\phi_{ij}^+ + \phi_{ij}^- + \varphi_{ij}^+ + \varphi_{ij}^-\right)$$

$$s.t. \begin{cases} \varphi_{ij}^+ - \varphi_{ij}^- = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right) \\ \left(\Delta^{-1}\left(v_i^-\right) - \Delta^{-1}\left(v_j^+\right)\right) - \Delta^{-1}\left(u_{ij}^-\right), \\ \phi_{ij}^+ - \phi_{ij}^- = \Delta^{-1}\left(s_{\tau/2}, 0\right) + \Delta^{-1}\left(s_{\tau/2}, 0\right) \\ \left(\Delta^{-1}\left(v_i^+\right) - \Delta^{-1}\left(v_j^-\right)\right) - \Delta^{-1}\left(u_{ij}^+\right), \\ \Delta^{-1}\left(v_i^+\right) + \sum_{j=1, i\neq j}^{n} \Delta^{-1}\left(v_j^-\right) \leq 1, \\ \Delta^{-1}\left(v_i^-\right) + \sum_{j=1, i\neq j}^{n} \Delta^{-1}\left(v_j^+\right) \geq 1, \\ 0 \leq \Delta^{-1}\left(v_i^-\right) \leq \Delta^{-1}\left(v_i^+\right) \leq 1. \end{cases}$$

By solving model (M-4), the optimal solutions $\tilde{V} = \left(\tilde{v}_1^*, \cdots, \tilde{v}_n^*\right)^T = \left(\left[v_1^{-*}, v_1^{+*}\right], \cdots, \left[v_n^{-*}, v_n^{+*}\right]\right)^T$ can be obtained.

5. METHOD FOR GDM WITH U2TLPRs

In this section, the DM's weights are determined by the similarity measure of DMs. Then, a new method is proposed to solve GDM problems with U2TLPRs.

5.1. Deriving the Weights of DMs in GDM with U2TLPRs

Consider a GDM where the $X = \{x_1, x_2, \cdots, x_n\}$ and $D = \{d_1, d_2, \cdots, d_m\}$ are set of the alternatives and set of DMs, respectively. Let $W = (w_1, w_2, \cdots, w_m)^T$ be DM's weighting vector in which $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$. Assume that $\tilde{U}^{(l)} = \left(\tilde{u}_{ij}^{(l)}\right)_{n\times n}$ with $\tilde{u}_{ij}^{(l)} = \left[\tilde{u}_{ij}^{(l)-}, \tilde{u}_{ij}^{(l)+}\right] = \left[\left(u_{ij}^{(l)-}, \alpha_{ij}^{(l)-}\right), \left(u_{ij}^{(l)+}, \alpha_{ij}^{(l)+}\right)\right]$ is the individual U2TLPR given by DM d_l.

In GDM, the key of solving GDM problems with preference relations is how to allocate the weights to DMs. Once the weights are assigned, all individual preference relations can be aggregated into a collective one.

For each pair of DM (d_l, d_p), similarity degree between two DMs d_l and d_p can be measured by $\tilde{U}^{(l)} = \left(\tilde{u}_{ij}^{(l)}\right)_{n\times n}$ and $\tilde{U}^{(p)} = \left(\tilde{u}_{ij}^{(p)}\right)_{n\times n}$, which is defined as follows:

$$S_{lp} = 1 - \frac{2}{n(n-1)} \sum_{i<j}^{n} d_M\left(\tilde{u}_{ij}^{(l)}, \tilde{u}_{ij}^{(p)}\right), \tag{23}$$

where $d_M\left(\tilde{u}_{ij}^{(l)}, \tilde{u}_{ij}^{(p)}\right) = \frac{1}{2}\left(|\Delta^{-1}\left(u_{ij}^{(l)-}, \alpha_{ij}^{(l)-}\right) - \Delta^{-1}\left(u_{ij}^{(p)-}, \alpha_{ij}^{(p)-}\right)| + |\Delta^{-1}\left(u_{ij}^{(l)+}, \alpha_{ij}^{(l)+}\right) - \Delta^{-1}\left(u_{ij}^{(p)+}, \alpha_{ij}^{(p)+}\right)|\right)$ is defined by using Eq. (5).

The larger the value of S_{lp}, the higher the degree of similarity between d_l and d_p.

Based on Eq. (23), the following theorem is apparent.

Theorem 5. *Let $\tilde{U}^{(l)} = \left(\tilde{u}_{ij}^{(l)}\right)_{n\times n}$ and $\tilde{U}^{(p)} = \left(\tilde{u}_{ij}^{(p)}\right)_{n\times n}$ be as before. Then,*

1. $0 \leq S_{lp} \leq 1$;
2. $S_{lp} = S_{pl}$;
3. $S_{ll} = 1$.

In line with the similarity, the confidence degree CD_l of DM d_l can be defined as follows:

$$CD_l = \sum_{q=1, q\neq l}^{m} S_{lq}. \tag{24}$$

The confidence degree reflects the support degree of a DM from the others. The higher the confidence degree of a DM, the larger the support degree of a DM from the other DMs. Therefore, DM with higher confidence degree should be assigned a larger weight; otherwise, DM should be assigned a smaller weight. Thus, the DMs' weights can be determined by

$$w_l = \frac{CD_l}{\sum_{l=1}^{m} CD_l}, l = 1, 2, \cdots, m. \tag{25}$$

Based on the uncertain 2-tuple linguistic weighted averaging ($ULWA_{2-tuple}$) [53], the collective matrix $\tilde{U}^c$ of $\tilde{U}^{(l)}$ $(l = 1, 2, \cdots, m)$ is defined as Definition 9.

Definition 9. Let $\tilde{U}^{(l)} = \left(\left[\tilde{u}_{ij}^{(l)-}, \tilde{u}_{ij}^{(l)+}\right]\right)_{n\times n}$ be as before. Then, the the collective matrix $\tilde{U}^c$ is defined by

$$\tilde{U}^c = \left(\left[\tilde{u}_{ij}^{c-}, \tilde{u}_{ij}^{c+}\right]\right)_{n\times n}, \tag{26}$$

where $\tilde{u}_{ij}^{c-} = \Delta\left(\sum_{l=1}^{m} w_l \Delta^{-1}\left(\tilde{u}_{ij}^{(l)-}\right)\right)$, $\tilde{u}_{ij}^{c+} = \Delta\left(\sum_{l=1}^{m} w_l \Delta^{-1}\left(\tilde{u}_{ij}^{(l)+}\right)\right)$ and $\sum_{k=1}^{m} w_l = 1$ and $w_l \geq 0$ defined in Eq. (25).

From Eq. (26), the following theorem is obtained.

Theorem 6. *Let $\tilde{U}^c$ and $\tilde{U}^{(l)}$ be as before. Then, the $\tilde{U}^c$ have the following properties:*

1. $\tilde{U}^c$ is an U2TLPR.
2. If $\tilde{U}^{(l)}$ $(l = 1, 2, \cdots, m)$ are additive consistency, then the $\tilde{U}^c$ is additive consistency.
3. If $\tilde{U}^{(l)}$ $(l = 1, 2, \cdots, m)$ are acceptable additive consistency, then the $\tilde{U}^c$ is acceptable additive consistency.

Proof. (1) Based on the Definition 9, we have $\Delta^{-1}\tilde{u}_{ij}^{c-} = \sum_{l=1}^{m} w_k \Delta^{-1}\tilde{u}_{ij}^{(l)-}$ and $\Delta^{-1}\tilde{u}_{ji}^{c+} = \sum_{l=1}^{m} w_l \Delta^{-1}\tilde{u}_{ji}^{(l)+}$. For $i \neq j$, it follows that $\Delta\left(\Delta^{-1}\tilde{u}_{ij}^{c-} + \Delta^{-1}\tilde{u}_{ji}^{c+}\right) = \Delta\left(\sum_{l=1}^{m} w_l\left(\Delta^{-1}\tilde{u}_{ij}^{(l)-} + \Delta^{-1}\tilde{u}_{ji}^{(l)+}\right)\right) = (s_\tau, 0)$. On the other hand, for $i = j$, we have $\tilde{u}_{ii}^{c-} = \tilde{u}_{ii}^{c+} = (s_{\tau/2}, 0)$. Thus, the collective matrix $\tilde{U}^c$ is an U2TLPR.

(2) Assume $\tilde{U}^{(l)}$ $(l = 1, 2, \cdots, m)$ are additive consistency. Then, we have the following:

$$\Delta^{-1}\tilde{u}_{ij}^{(l)\dagger} + \Delta^{-1}\tilde{u}_{jk}^{(l)\dagger} + \Delta^{-1}\tilde{u}_{ki}^{(l)\dagger} = \Delta^{-1}\tilde{u}_{kj}^{(l)\dagger} + \Delta^{-1}\tilde{u}_{ji}^{(l)\dagger} + \Delta^{-1}\tilde{u}_{ik}^{(l)\dagger}.$$

It follows that

$$\begin{aligned}
&\Delta^{-1}\left(\tilde{u}_{ij}^{c\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{c\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{c\dagger}\right) \\
&\quad - \Delta^{-1}\left(\tilde{u}_{kj}^{c\dagger}\right) - \Delta^{-1}\left(\tilde{u}_{ji}^{c\dagger}\right) - \Delta^{-1}\left(\tilde{u}_{ik}^{c\dagger}\right) \\
&= \sum_{l=1}^{m} w_l\left(\Delta^{-1}\left(\tilde{u}_{ij}^{(l)\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{(l)\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{(l)\dagger}\right)\right. \\
&\quad \left. - \Delta^{-1}\left(\tilde{u}_{kj}^{(l)\dagger}\right) - \Delta^{-1}\left(\tilde{u}_{ji}^{(l)\dagger}\right) - \Delta^{-1}\left(\tilde{u}_{ik}^{(l)\dagger}\right)\right) \\
&= 0.
\end{aligned}$$

Therefore, $\Delta^{-1}\left(\tilde{u}_{ij}^{c\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{c\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{c\dagger}\right) = \Delta^{-1}\left(\tilde{u}_{kj}^{c\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{ji}^{c\dagger}\right) + \Delta^{-1}\left(\tilde{u}_{ik}^{c\dagger}\right)$. It means that $\tilde{U}^c$ has additive consistency.

(3) Assume $\tilde{U}^{(l)}$ $(l = 1, 2, \cdots, m)$ are acceptable additive consistency. According to Definition 9, we have the following:

$$
\begin{aligned}
ACI\left(\tilde{U}^{c}\right) &= \frac{2}{n(n-1)(n-2)} \sum_{i<j<k}^{n} \left|\Delta^{-1}\left(\tilde{u}_{ij}^{c+}\right)\right. \\
&\quad + \Delta^{-1}\left(\tilde{u}_{ij}^{c-}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{c+}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{c-}\right) \\
&\quad \left. + \Delta^{-1}\left(\tilde{u}_{ki}^{c+}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{c-}\right) - 3\Delta^{-1}\left(s_{\tau}, 0\right)\right| \\
&= \frac{2}{n(n-1)(n-2)} \sum_{i<j<k}^{n} \left|\sum_{l=1}^{m} w_l \left(\Delta^{-1}\left(\tilde{u}_{ij}^{(l)+}\right)\right.\right. \\
&\quad + \Delta^{-1}\left(\tilde{u}_{ij}^{(l)-}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{(l)+}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{(l)-}\right) \\
&\quad \left.\left. + \Delta^{-1}\left(\tilde{u}_{ki}^{(l)+}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{(l)-}\right) - 3\Delta^{-1}\left(s_{\tau}, 0\right)\right)\right| \\
&\leq \frac{2}{n(n-1)(n-2)} \sum_{i<j<k}^{n} \sum_{l=1}^{m} w_l \left|\Delta^{-1}\left(\tilde{u}_{ij}^{(l)+}\right)\right. \\
&\quad + \Delta^{-1}\left(\tilde{u}_{ij}^{(l)-}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{(l)+}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{(l)-}\right) \\
&\quad \left. + \Delta^{-1}\left(\tilde{u}_{ki}^{(l)+}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{(l)-}\right) - 3\Delta^{-1}\left(s_{\tau}, 0\right)\right| \\
&= \sum_{l=1}^{m} w_l \left(\frac{2}{n(n-1)(n-2)} \sum_{i<j<k}^{n} \left|\Delta^{-1}\left(\tilde{u}_{ij}^{(l)+}\right)\right.\right. \\
&\quad + \Delta^{-1}\left(\tilde{u}_{ij}^{(l)-}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{(l)+}\right) + \Delta^{-1}\left(\tilde{u}_{jk}^{(l)-}\right) \\
&\quad \left.\left. + \Delta^{-1}\left(\tilde{u}_{ki}^{(l)+}\right) + \Delta^{-1}\left(\tilde{u}_{ki}^{(l)-}\right) - 3\Delta^{-1}\left(s_{\tau}, 0\right)\right|\right) \\
&\leq \sum_{l=1}^{m} w_l \overline{CI} \\
&= \overline{CI}
\end{aligned}
$$

5.2. New Approach for GDM with U2TLPRs

Based on the above analyses, a new method for GDM with U2TLPRs is graphically depicted in Figure 1.

The GDM approach with U2TLPRs based on our proposed approaches can be formally described in Algorithm 1.

Remark 2. The aforesaid Algorithm 1 used to deal with the GDM problems with U2TLPRs. If there is one DM in decision-making problems, then the process for deriving DMs' weights and collective U2TLPR is omitted. Therefore, Algorithm 1 is reduced to the method for individual decision-making with an U2TLPR. In other words, Algorithm 1 can not only be implemented to deal with GDM with U2TLPRs, but also be used to solve the individual decision-making with an U2TLPR.

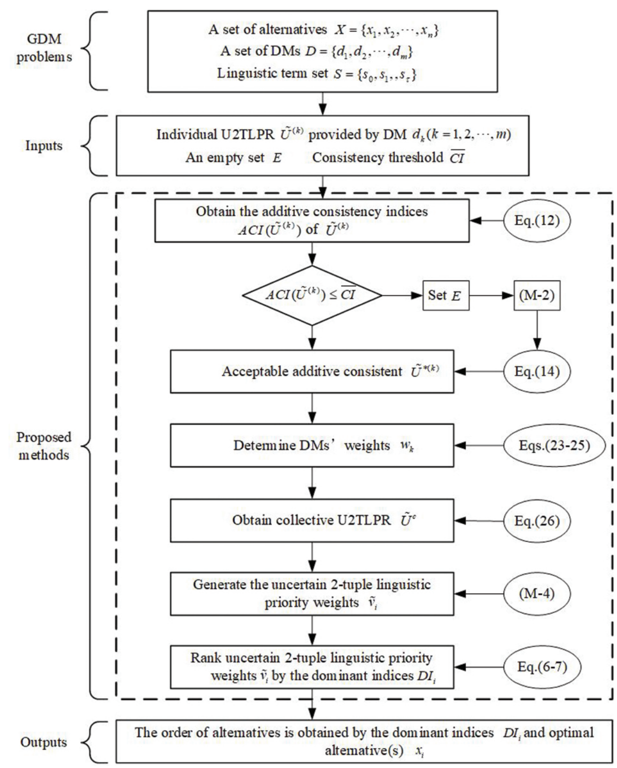

Figure 1 Decision-making process of group decision-making (GDM) problems with uncertain 2-tuple linguistic preference relations (U2TLPRs).

Algorithm 1 The algorithm for the GDM approach with U2TLPRs.

Input: The U2TLPR $\tilde{U}^{(l)}$ $(l = 1, 2, \cdots, m)$ provided by DMs in GDM problem and the predefined consistency
threshold $\overline{CI}$ and an empty set E.
Output: The best alternative(s).
Begin:
for $k = 1, 2, \cdots, m$ **do**
$ACI\left(\tilde{U}^{(l)}\right) \leftarrow$ Eq. (12).
if $ACI\left(\tilde{U}^{(l)}\right) > \overline{CI}$ **then**
$E = \left\{\tilde{U}^{(l)}\right\}$.
$\tilde{U}^{*(l)} \leftarrow$ model (M-2) and Eq. (14).
else
$\tilde{U}^{*(l)} \leftarrow \tilde{U}^{(l)}$.
end
end
DMs' weights w_l $(l = 1, 2, \cdots, m) \leftarrow$ Eqs. (23–25).
The collective U2TLPR $\tilde{U}^c \leftarrow$ Eq. (26).
The uncertain 2-tuple linguistic priority vector $\tilde{V} \leftarrow$ model (M-4).
for $i = 1, 2, \cdots, n$ **do**
$DI_i \leftarrow$ Eqs. (6–7).
end
By ranking the dominant indices DI_i $(i = 1, 2, \cdots, n)$, the best alternative(s) is (are) obtained.
End

6. ILLUSTRATIVE EXAMPLES

In this section, the new approach proposed in Section 5.2 is applied to an individual decision-making problem and a GDM problem with U2TLPRs, respectively. Meanwhile, some comparative analyses between our proposed methods and other existing methods are provided.

6.1. Application to Individual Decision-Making with U2TLPR

Example 1. This example is taken from Yao and Hu [40] about risk management in construction projects. Based on the linguistic term set $S_{\text{Example}} = \{s_0 = \textit{extremely low}, s_1 = \textit{very low}, s_2 = \textit{low}, s_3 = \textit{slightly low}, s_4 = \textit{fair}, s_5 = \textit{slightly high}, s_6 = \textit{high}, s_7 = \textit{very high}, s_8 = \textit{extremely highg}\}$, a DM provide his/her uncertain preference opinion by utilizing U2TLPR, which is listed as follows:

$$\tilde{U} = \begin{pmatrix} [(s_4,0),\ (s_4,0)] & [(s_1,0),\ (s_2,0)] & [(s_6,0),\ (s_7,0)] & [(s_4,0),\ (s_5,0)] & [(s_5,0),\ (s_6,0)] \\ [(s_6,0),\ (s_7,0)] & [(s_4,0),\ (s_4,0)] & [(s_3,0),\ (s_4,0)] & [(s_2,0),\ (s_3,0)] & [(s_6,0),\ (s_7,0)] \\ [(s_1,0),\ (s_2,0)] & [(s_4,0),\ (s_5,0)] & [(s_4,0),\ (s_4,0)] & [(s_6,0),\ (s_7,0)] & [(s_4,0),\ (s_5,0)] \\ [(s_3,0),\ (s_4,0)] & [(s_5,0),\ (s_6,0)] & [(s_1,0),\ (s_2,0)] & [(s_4,0),\ (s_4,0)] & [(s_2,0),\ (s_3,0)] \\ [(s_2,0),\ (s_3,0)] & [(s_1,0),\ (s_2,0)] & [(s_3,0),\ (s_4,0)] & [(s_5,0),\ (s_6,0)] & [(s_4,0),\ (s_4,0)] \end{pmatrix}.$$

Obviously, this example is an individual decision-making problem. Therefore, Algorithm 1 proposed in Section 5.2 omitted the process for deriving DMs' weights and collective U2TLPR is applied to solve this example.

Set consistency threshold $\overline{CI} = 0.08$ [39] and the U2TLPR $\tilde{U}$ has been provided.

Based on Eq. (12), the additive consistency index $ACI\left(\tilde{U}\right) = 0.2667$. We have $ACI\left(\tilde{U}\right) > \overline{CI}$, it means that U2TLPR $\tilde{U}$ has unacceptable additive consistency.

Based on (M-2), a goal programming model is constructed as follows:

$$\min\left(\varepsilon_{12}^{+}+\varepsilon_{12}^{-}+\delta_{12}^{+}+\delta_{12}^{-}+\varepsilon_{13}^{+}+\varepsilon_{13}^{-}+\delta_{13}^{+}+\delta_{13}^{-}\right.$$
$$+\varepsilon_{14}^{+}+\varepsilon_{14}^{-}+\delta_{14}^{+}+\delta_{14}^{-}+\varepsilon_{15}^{+}+\varepsilon_{15}^{-}+\delta_{15}^{+}+\delta_{15}^{-}$$
$$+\varepsilon_{23}^{+}+\varepsilon_{23}^{-}+\delta_{23}^{+}+\delta_{23}^{-}+\varepsilon_{24}^{+}+\varepsilon_{24}^{-}+\delta_{24}^{+}+\delta_{24}^{-}$$
$$+\varepsilon_{25}^{+}+\varepsilon_{25}^{-}+\delta_{25}^{+}+\delta_{25}^{-}+\varepsilon_{34}^{+}+\varepsilon_{34}^{-}+\delta_{34}^{+}+\delta_{34}^{-}$$
$$\left.+\varepsilon_{35}^{+}+\varepsilon_{35}^{-}+\delta_{35}^{+}+\delta_{35}^{-}+\varepsilon_{45}^{+}+\varepsilon_{45}^{-}+\delta_{45}^{+}+\delta_{45}^{-}\right)$$

$$s.t.\begin{cases}
\frac{1}{30}\left(\lambda_{123}^{+}+\lambda_{123}^{-}+\lambda_{124}^{+}+\lambda_{124}^{-}+\lambda_{125}^{+}+\right.\\
\lambda_{125}^{-}+\lambda_{134}^{+}+\lambda_{134}^{-}+\lambda_{135}^{+}+\lambda_{135}^{-}+\lambda_{145}^{+}\\
+\lambda_{145}^{-}+\lambda_{234}^{+}+\lambda_{234}^{-}+\lambda_{235}^{+}+\lambda_{235}^{-}\\
\left.+\lambda_{245}^{+}+\lambda_{245}^{-}+\lambda_{345}^{+}+\lambda_{345}^{-}\right)\le\overline{CI},\\
\delta_{14}^{+}-\delta_{14}^{-}=\Delta^{-1}\left(\tilde{u}_{14}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{14}^{+}\right),\\
\delta_{15}^{+}-\delta_{15}^{-}=\Delta^{-1}\left(\tilde{u}_{15}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{15}^{+}\right),\\
\delta_{23}^{+}-\delta_{23}^{-}=\Delta^{-1}\left(\tilde{u}_{23}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{23}^{+}\right),\\
\delta_{24}^{+}-\delta_{24}^{-}=\Delta^{-1}\left(\tilde{u}_{24}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{24}^{+}\right),\\
\delta_{25}^{+}-\delta_{25}^{-}=\Delta^{-1}\left(\tilde{u}_{25}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{25}^{+}\right),\\
\delta_{34}^{+}-\delta_{34}^{-}=\Delta^{-1}\left(\tilde{u}_{34}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{34}^{+}\right),\\
\delta_{35}^{+}-\delta_{35}^{-}=\Delta^{-1}\left(\tilde{u}_{35}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{35}^{+}\right),\\
\delta_{45}^{+}-\delta_{45}^{-}=\Delta^{-1}\left(\tilde{u}_{45}^{+*}\right)-\Delta^{-1}\left(\tilde{u}_{45}^{+}\right),
\end{cases}$$

$$s.t.\begin{cases}
\Delta^{-1}\left(\tilde{u}_{12}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{12}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{23}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{23}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{31}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{31}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{123}^{+}-\lambda_{123}^{-},\\
\Delta^{-1}\left(\tilde{u}_{12}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{12}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{24}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{24}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{41}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{41}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{124}^{+}-\lambda_{124}^{-},\\
\Delta^{-1}\left(\tilde{u}_{12}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{12}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{25}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{25}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{51}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{51}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{125}^{+}-\lambda_{125}^{-},\\
\Delta^{-1}\left(\tilde{u}_{13}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{13}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{34}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{34}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{41}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{41}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{134}^{+}-\lambda_{134}^{-},\\
\Delta^{-1}\left(\tilde{u}_{13}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{13}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{35}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{35}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{51}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{51}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{135}^{+}-\lambda_{135}^{-},\\
\Delta^{-1}\left(\tilde{u}_{14}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{14}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{45}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{45}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{51}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{51}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{145}^{+}-\lambda_{145}^{-},\\
\Delta^{-1}\left(\tilde{u}_{23}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{23}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{34}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{34}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{42}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{42}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{234}^{+}-\lambda_{234}^{-},\\
\Delta^{-1}\left(\tilde{u}_{23}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{23}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{35}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{35}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{52}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{52}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{235}^{+}-\lambda_{235}^{-},\\
\Delta^{-1}\left(\tilde{u}_{24}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{24}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{45}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{45}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{52}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{52}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{245}^{+}-\lambda_{245}^{-},\\
\Delta^{-1}\left(\tilde{u}_{34}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{34}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{45}^{+*}\right)\\
+\Delta^{-1}\left(\tilde{u}_{45}^{-*}\right)+\Delta^{-1}\left(\tilde{u}_{53}^{+*}\right)+\Delta^{-1}\left(\tilde{u}_{53}^{-*}\right)\\
-3\Delta^{-1}\left(s_{\tau},0\right)=\lambda_{345}^{+}-\lambda_{345}^{-},\\
0\le\Delta^{-1}\left(\tilde{u}_{12}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{12}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{13}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{13}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{14}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{14}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{15}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{15}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{23}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{23}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{24}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{24}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{25}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{25}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{34}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{34}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{35}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{35}^{+*}\right)\le 1,\\
0\le\Delta^{-1}\left(\tilde{u}_{45}^{-*}\right)\le\Delta^{-1}\left(\tilde{u}_{45}^{+*}\right)\le 1.
\end{cases}$$

Solving the model by LINGO 11.0, the optimal solutions $\tilde{u}_{ij}^{+*}\ (i<j)$ and $\tilde{u}_{ij}^{-*}\ (i<j)$ are derived. Then, according to Eq. (14), we obtain

the acceptable additive consistent

$$\tilde{U}^* = \begin{pmatrix} [(s_4,0),\ (s_4,0)] & [(s_1,0),\ (s_2,0)] & [(s_0,0),\ (s_7,0)] & [(s_6,-0.0625),\ (s_6,-0.0625)] & [(s_5,0),\ (s_6,0)] \\ [(s_6,0),\ (s_7,0)] & [(s_4,0),\ (s_4,0)] & [(s_3,0),\ (s_4,0)] & [(s_6,-0.0625),\ (s_6,-0.0625)] & [(s_6,0),\ (s_7,0)] \\ [(s_1,0),\ (s_8,0)] & [(s_4,0),\ (s_5,0)] & [(s_4,0),\ (s_4,0)] & [(s_4,-0.0625),\ (s_4,-0.0625)] & [(s_4,0),\ (s_5,0)] \\ [(s_2,0.0625),\ (s_2,0.0625)] & [(s_2,0.0625),\ (s_2,0.0625)] & [(s_4,-0.025),\ (s_4,-0.025)] & [(s_4,0),\ (s_4,0)] & [(s_2,0),\ (s_3,0)] \\ [(s_2,0),\ (s_3,0)] & [(s_1,0),\ (s_2,0)] & [(s_3,0),\ (s_4,0)] & [(s_5,0),\ (s_6,0)] & [(s_4,0),\ (s_4,0)] \end{pmatrix}.$$

Solving (M-3), the uncertain 2-tuple linguistic priority weights $\tilde{v}_i$ of $\tilde{U}$ is obtained as

$$\begin{aligned} \tilde{v}_1 &= [(s_2,0),(s_4,-0.0175)], \\ \tilde{v}_2 &= [(s_4,-0.0625),(s_4,-0.0625), \\ \tilde{v}_3 &= [(s_0,0.0175),(s_2,0)], \\ \tilde{v}_4 &= [(s_1,-0.0625),(s_1,-0.0625)], \\ \tilde{v}_5 &= [(s_0,0),(s_0,0)]. \end{aligned}$$

Based on Eqs. (6) and (7), the dominant indices DI_i of x_i are generated as

$$\begin{aligned} &DI_1 = 0.2597, DI_2 = 0.2903, \\ &DI_3 = 0.1903, DI_4 = 0.1597, \\ &DI_5 = 0.1000. \end{aligned}$$

In accordance with the DI_i, the ranking order of alternatives is produced as $x_2 \succ x_1 \succ x_3 \succ x_4 \succ x_5$.

With different values of consistency thresholds $\overline{CI}$, the ranking results is derived and shown in Table 1.

As seen from Table 1, the raking order of alternatives is different based on different consistency thresholds. When $\overline{CI} = 0.03$, the raking order of alternatives is $x_2 \succ x_1 \succ x_3 \succ x_5 \succ x_4$ provided by the proposed method for individual decision-making, which is different from $x_2 \succ x_3 \succ x_1 \succ x_4 \sim x_5$ that obtained by Yao and Hu [40].

Compared with the method proposed by Yao and Hu [40], some differences are analyzed in the following:

- The additive consistency index of U2TLPR is developed based on the uncertain preference information of original U2TLPR. It is different from the consistency index [40] of U2TLPR constructed by computing the deviation between original U2TLPR and its corresponding consistent U2TLPR.

Table 1 Ranking orders of alternatives with different consistency thresholds.

$\overline{CI}$	Ranking Orders of Alternatives	Optimal Alternative
0.03	$x_2 \succ x_1 \succ x_3 \succ x_5 \succ x_4$	x_2
0.04	$x_2 \succ x_3 \succ x_1 \succ x_5 \succ x_4$	x_2
0.05	$x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_5$	x_3
0.06	$x_3 \succ x_1 \succ x_2 \succ x_4 \succ x_5$	x_3
0.07	$x_1 \succ x_3 \succ x_2 \succ x_4 \succ x_5$	x_1
0.08	$x_2 \succ x_1 \succ x_3 \succ x_4 \succ x_5$	x_2

- In this paper, in order to keep balance between consistency improvement and uncertain preference preservation, a goal programming model is proposed. However, the consistency improving algorithm proposed in Yao and Hu [40] was based on the relationship between U2TLPR and its crisp priority weights.
- The optimization model is proposed to derive the uncertain 2-tuple linguistic priority vector from U2TLPR. The uncertain 2-tuple linguistic priority vector is composed of uncertain 2-tuple linguistic variables. However, the model was developed in Yao and Hu [40] to obtain the priority vector, which is composed of crisp priority weights. The uncertain 2-tuple linguistic priority vector can keep the integrity of final decision-making information.

6.2. Application to GDM with U2TLPRs

In this subsection, an investment problem, which is taken from Zhang and Guo [39], is addressed. An investment company wants to invest a sum of money in the best option(s). To reduce the risks involved in making decisions in this uncertain and highly competitive environment, the leader of the company invites a group of experts to participate in the decision and hopes to achieve a consensus solution. A panel with four alternatives is given as follows: A_1 is a car industry, A_2 is a food company, A_3 is a computer company and A_4 is an arms manufacturer. There are four DMs, d_1, d_2, d_3, d_4 from four consultancy departments: the risk analysis department, the growth analysis department, the social political analysis department and the environmental impact analysis department. These DMs provide their preferences over the alternatives using U2TLPRs as follows:

$$\tilde{L}^{(1)} = \begin{pmatrix} [(s_4,0),(s_4,0)] & [(s_3,0),(s_4,0)] & [(s_2,0),(s_3,0)] & [(s_5,0),(s_6,0)] \\ [(s_4,0),(s_5,0)] & [(s_4,0),(s_4,0)] & [(s_2,0),(s_4,0)] & [(s_4,0),(s_6,0)] \\ [(s_5,0),(s_6,0)] & [(s_4,0),(s_6,0)] & [(s_4,0),(s_4,0)] & [(s_6,0),(s_7,0)] \\ [(s_2,0),(s_3,0)] & [(s_2,0),(s_4,0)] & [(s_1,0),(s_2,0)] & [(s_4,0),(s_4,0)] \end{pmatrix},$$

$$\tilde{L}^{(2)} = \begin{pmatrix} [(s_4,0),(s_4,0)] & [(s_0,0),(s_1,0)] & [(s_2,0),(s_3,0)] & [(s_3,0),(s_4,0)] \\ [(s_7,0),(s_8,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_4,0),(s_6,0)] \\ [(s_5,0),(s_6,0)] & [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] \\ [(s_4,0),(s_5,0)] & [(s_2,0),(s_4,0)] & [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] \end{pmatrix},$$

$$\tilde{L}^{(3)} = \begin{pmatrix} [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_4,0),(s_6,0)] & [(s_1,0),(s_2,0)] \\ [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_2,0),(s_4,0)] \\ [(s_2,0),(s_4,0)] & [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_2,0),(s_3,0)] \\ [(s_6,0),(s_7,0)] & [(s_4,0),(s_6,0)] & [(s_5,0),(s_6,0)] & [(s_4,0),(s_4,0)] \end{pmatrix},$$

$$\tilde{L}^{(4)} = \begin{pmatrix} [(s_4,0),(s_4,0)] & [(s_5,0),(s_7,0)] & [(s_6,0),(s_8,0)] & [(s_4,0),(s_6,0)] \\ [(s_1,0),(s_3,0)] & [(s_4,0),(s_4,0)] & [(s_5,0),(s_6,0)] & [(s_3,0),(s_4,0)] \\ [(s_0,0),(s_2,0)] & [(s_2,0),(s_3,0)] & [(s_4,0),(s_4,0)] & [(s_1,0),(s_3,0)] \\ [(s_2,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_5,0),(s_7,0)] & [(s_4,0),(s_4,0)] \end{pmatrix}.$$

Set consistency threshold $\overline{CI} = 0.09$ [39] and the U2TLPR $\tilde{L}^{(k)}$ $(k = 1, 2, 3, 4)$ have been provided.

Based on Eq. (12), the additive consistency indices $ACI\left(\tilde{L}^{(k)}\right)$ $(k = 1, 2, 3, 4)$ of U2TLPR $\tilde{L}^{(k)}$ are obtained as follows:

$$ACI\left(\tilde{L}^{(1)}\right) = 0.0417,$$
$$ACI\left(\tilde{L}^{(2)}\right) = 0.0833,$$
$$ACI\left(\tilde{L}^{(3)}\right) = 0.0833,$$
$$ACI\left(\tilde{L}^{(4)}\right) = 0.0208.$$

Since $ACI\left(\tilde{L}^{(k)}\right) < \overline{CI}$, $k = 1, 2, 3, 4$, all individual U2TLPR $\tilde{L}^{(k)}$ have acceptable additive consistency.

By Eqs. (23–25), the weights w_k $(k = 1, 2, 3, 4)$ of four DMs are obtained:

$$w_1 = 0.2482,$$

$$w_2 = 0.2530,$$

$$w_3 = 0.2553,$$

$$w_4 = 0.2435.$$

According to Eq. (26), the collective U2TLPR $\tilde{L}^c$ is obtained as follows:

$$\begin{pmatrix} [(s_4,0),(s_4,0)] & [(s_3,-0.0021),(s_4,0.0284)] & [(s_3,0.0606),(s_5,-0.0021)] & [(s_3,0.0287),(s_4,0.0591)] \\ [(s_4,-0.0284),(s_5,0.0021)] & [(s_4,0),(s_4,0)] & [(s_4,-0.0316),(s_5,-0.0006)] & [(s_3,0.0307),(s_5,0.0003)] \\ [(s_3,0.0021),(s_5,-0.0606)] & [(s_3,0.0006),(s_4,0.0316)] & [(s_4,0),(s_4,0)] & [(s_3,0.0319),(s_4,0.0624)] \\ [(s_4,-0.0591),(s_5,-0.0287)] & [(s_3,-0.0003),(s_5,-0.0307)] & [(s_4,-0.0624),(s_5,-0.0319)] & [(s_4,0),(s_4,0)] \end{pmatrix}.$$

Solving model (M-4), the uncertain 2-tuple linguistic priority weights $\tilde{v}_i$ $(i = 1, 2, 3, 4)$ are obtained:

$$\tilde{v}_1 = [(s_1, -0.0023), (s_2, -0.0049)],$$
$$\tilde{v}_2 = [(s_2, -0.0617), (s_3, 0.0019)],$$
$$\tilde{v}_3 = [(s_1, 0.0031), (s_2, 0.0015)],$$
$$\tilde{v}_4 = [(s_1, 0.0017), (s_2, -0.0597)].$$

Based on Eqs. (6) and (7), the dominant indices DI_i of x_i $(i = 1, 2, 3, 4)$ as generated as follows:

$$DI_1 = 0.1978, DI_2 = 0.3339,$$
$$DI_3 = 0.2090, DI_4 = 0.2593.$$

In accordance with the DI_i $(i = 1, 2, 3, 4)$, the ranking order of alternatives is

$$x_2 \succ x_4 \succ x_3 \succ x_1.$$

Based on the method in Zhang and Guo [39], the ranking of the four alternatives is $x_2 \succ x_4 \succ x_3 \succ x_1$, which is the same with the ranking results provided by our proposed method. Compared with the method proposed by Zhang and Guo [39], we observe:

- In Zhang and Guo [39], a method for constructing a consistent U2TLPR from the original U2TLPR was introduced. Then, a consistency index of U2TLPR was defined by calculating the deviation between an U2TLPR and its consistent U2TLPR. However, the additive consistency index presented in this paper is developed based on the uncertain preference information of original U2TLPR, which is reliable and stable.
- The consistency improving algorithm [39] was developed, which takes the consistent U2TLPR as the object of the improvement. The algorithm is automatic and iterative. But in this paper, to adjust the U2TLPR with unacceptable additive consistency into U2TLPR with acceptable additive consistency, a goal programming model is constructed, which can guarantee the minimum deviation between the original U2TLPR and adjusted U2TLPR.
- In this paper, the DMs' weights are defined by the confidence degree. Nevertheless, the DMs' weights are given by subjective assignment with equal weights [39]. On the other hand, the final decision-making results depend on the uncertain 2-tuple linguistic priority vector given by a optimization model. However, the uncertain 2-tuple linguistic weighed averaging ($ULWA_{2-tuple}$) [39] operator was used to obtain the ranking order of alternatives.

7. CONCLUSIONS

For decision-making problems with U2TLPRs, the additive consistency and uncertain 2-tuple linguistic priority weights are investigated. The main contributions of this paper are summarized as follows:

1. Based on the uncertain 2-tuple linguistic preference information of the original U2TLPR, an additive consistency index is defined to check the additive consistency level of U2TLPR.
2. For improving the additive consistency of the U2TLPRs, a goal programming model is developed to derive the U2TLPRs with acceptable additive consistency from the unacceptable additive consistent U2TLPRs.
3. To keep the integrity of final decision-making information, an optimization model is constructed to obtain the uncertain 2-tuple linguistic priority weights of U2TLPR.
4. To determine the DMs' weights in GDM, the similarities and confidence degrees are defined, respectively. Combing the similarities and confidence degrees, the method for determining the DMs' weights is provided.

For an unacceptably additive consistent U2TLPR with large dimension, the solving process of deriving the revised U2TLPR with acceptably additive consistency may be time-consuming. Hence, it is interesting future study to utilize some intelligence algorithms with less time complexity to solve the additive consistency improvement process. In fact, these methods proposed in this paper can be extended to GDM with other decision-making information environment. In addition, future research will be concentrated on the consensus of U2TLPRs and dealing with the consistency level and priority vectors of incomplete U2TLPRs.

CONFLICT OF INTEREST

The authors have no competing financial, professional, or personal interests from other parties that are related to this paper.

AUTHORS' CONTRIBUTIONS

Peng Wu and Ligang Zhou conceived the study and were responsible for the design and development of the data analysis. Jiaming Zhu, Huayou Chen and Yu Chen responsible for data interpretation. Peng Wu and Ligang Zhou wrote the first draft of the article.

ACKNOWLEDGMENTS

The authors would like to thank the editors and the anonymous reviewers for their insightful and constructive comments and suggestions that have led to this improved version of the paper. The work was supported by National Natural Science Foundation of China (Nos.71771001, 71871001, 71701001, 71501002, 71901001, 71901088), Natural Science Foundation for Distinguished Young Scholars of Anhui Province (No. 1908085J03), The Academic and Technical Leaders Reserve Talents Research Activities Funding Project of Anhui Province (No. 2018H179), College Excellent Youth Talent Support Program (gxyq2019236) and Key Research Project of Humanities and Social Sciences in Colleges and Universities of Anhui Province (SK2019A0013).

REFERENCES

[1] T.L. Saaty, The Analytic Hierarchy Process, McGraw-Hill, New York, 1980.
[2] T. Tanino, Fuzzy preference orderings in group decision making, Fuzzy Sets Syst. 12 (1984), 117–131.
[3] T.L. Saaty, L.G. Vargas, Uncertainty and rank order in the analytic hierarchy process, Eur. J. Oper. Res. 32 (1987), 107–117.
[4] L.G. Zhou, J.M. Merigó, H.Y. Chen, J.P. Liu, The optimal group continuous logarithm compatibility measure for interval multiplicative preference relations based on the COWGA operator, Inf. Sci. 328 (2016), 250–269.
[5] Y.C. Dong, C.C. Li, F. Chiclana, E. Herrera-Viedma, Average-case consistency measurement and analysis of interval-valued reciprocal preference relations, Knowl. Based Syst. 114 (2016), 108–117.
[6] Z.S. Xu, J. Chen, Some models for deriving the priority weights from interval fuzzy preference relations, Eur. J. Oper. Res. 184 (2008), 266–280.
[7] P. van Laarhoven, W. Pedrycz, A fuzzy extension of saaty's priority theory, Fuzzy Sets Syst. 11 (1983), 229–241.
[8] Z.J. Wang, X.Y. Tong, Consistency analysis and group decision making based on triangular fuzzy additive reciprocal preference relations, Inf. Sci. 361–362 (2016), 29–47.
[9] F. Liu, Z.L. Liu, Y.H. Wu, A group decision making model based on triangular fuzzy additive reciprocal matrices with additive approximation-consistency, Appl. Soft Comput. 65 (2018), 349–359.
[10] J.J. Buckley, Fuzzy hierarchical analysis, Fuzzy Sets Syst. 17 (1985), 233–247.
[11] P. Wu, S.H. Liu, L.G. Zhou, H.Y. Chen, A fuzzy group decision making model with trapezoidal fuzzy preference relations based on compatibility measure and cowga operator, Appl. Intell. 48 (2018), 46–67.
[12] P. Wu, L.G. Zhou, T. Zheng, H.Y. Chen, A fuzzy group decision making and its application based on compatibility with multiplicative trapezoidal fuzzy preference relations, Int. J. Fuzzy Syst. 19 (2017), 683–701.
[13] Y.Y. Zhou, L.H. Cheng, L.G. Zhou, H.Y. Chen, J.Q. Ge, A group decision making approach for trapezoidal fuzzy preference relations with compatibility measure, Soft Comput. 21 (2017), 2709–2721.
[14] Z.S. Xu, Intuitionistic preference relations and their application in group decision making, Inf. Sci. 177 (2007), 2363–2379.
[15] H.C. Liao, Z.S. Xu, M.M. Xia, Multiplicative consistency of hesitant fuzzy preference relation and its application in group decision making, Int. J. Inf. Technol. Decis. Making. 13 (2014), 47–76.
[16] Z. Zhang, X.Y. Kou, Q.X. Dong, Additive consistency analysis and improvement for hesitant fuzzy preference relations, Expert Syst. Appl. 98 (2018), 118–128.
[17] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (2000), 746–752.
[18] Y.C. Dong, Y.F. Xu, H.Y. Li, B. Feng, The OWA-based consensus operator under linguistic representation models using position indexes, Eur. J. Oper. Res. 203 (2010), 455–463.
[19] Z.F. Tao, H.Y. Chen, L.G. Zhou, J.P. Liu, On new operational laws of 2-tuple linguistic information using archimedean t-norm and s-norm, Knowl. Based Syst. 66 (2014), 156–165.
[20] J.M. Merigó, A.M. Gil-Lafuente, Induced 2-tuple linguistic generalized aggregation operators and their application in decision-making, Inf. Sci. 236 (2013), 1–16.
[21] S.P. Wan, 2-tuple linguistic hybrid arithmetic aggregation operators and application to multi-attribute group decision making, Knowl. Based Syst. 45 (2013), 31–40.
[22] Z.W. Gong, J. Forrest, Y.J. Yang, The optimal group consensus models for 2-tuple linguistic preference relations, Knowl. Based Syst. 37 (2013), 427–437.
[23] L. Martínez, Sensory evaluation based on linguistic decision analysis, Int. J. Approx. Reason. 44 (2007), 148–164.
[24] L. Martínez, M. Espinilla, L.G. Pérez, A linguistic multigranular sensory evaluation model for olive oil, Int. J. Comp. Intell. Syst. 1 (2008), 148–158.
[25] L. Martínez, F. Herrera, An overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges, Inf. Sci. 207 (2012), 1–18.
[26] L. Martínez, R.M. Rodríguez, F. Herrera, The 2-tuple Linguistic Model Computing with Words in Decision Making, Springer, Berlin, 2015.
[27] H.M. Zhang, The multiattribute group decision making method based on aggregation operators with interval-valued 2-tuple linguistic information, Math. Comput. Model. 56 (2012), 27–35.
[28] H.M. Zhang, Some interval-valued 2-tuple linguistic aggregation operators and application in multiattribute group decision making, Appl. Math. Model. 37 (2013), 4269–4282.
[29] J.Q. Wang, D.D. Wang, H.Y. Zhang, X.H. Chen, Multi-criteria group decision making method based on interval 2-tuple linguistic information and choquet integral aggregation operators, Soft Comput. 19 (2015), 389–405.

[30] X. Liu, Z.F. Tao, H.Y. Chen, L.G. Zhou, A new interval-valued 2-tuple linguistic bonferroni mean operator and its application to multiattribute group decision making, Int. J. Fuzzy Syst. 19 (2017), 86–108.
[31] X. Liu, B. Han, H.Y. Chen, L.G. Zhou, Interval-valued 2-tuple linguistic induced continuous ordered weighted distance measure and its application to multiple attribute group decision making. Informatica. 29 (2018), 321–352.
[32] B. Liu, M. Fu, S. Zhang, B. Xue, Q. Zhou, S. Zhang, An interval-valued 2-tuple linguistic group decision-making model based on the choquet integral operator, Int. J. Syst. Sci. 49 (2018), 407–424.
[33] F.Y. Meng, Y. Yuan, X.H. Chen, Several generalized interval-valued 2-tuple linguistic interval distance measures and their application, Int. J. Uncertain. Fuzz. Knowl. Based Syst. 25 (2017), 759–786.
[34] H.C. Liu, L. Liu, J. Wu, Material selection using an interval 2-tuple linguistic VIKOR method considering subjective and objective weights, Mater. Design. 52 (2013), 158–167.
[35] X.Y. You, J.X. You, H.C. Liu, L. Zhen, Group multi-criteria supplier selection using an extended VIKOR method with interval 2-tuple linguistic information, Expert Syst. Appl. 42 (2015), 1906–1916.
[36] H.C. Liu, J.X. You, C. Lu, M.M. Shan, Application of interval 2-tuple linguistic MULTIMOORA method for health-care waste treatment technology evaluation and selection, Waste Manage. 34 (2014), 2355–2364.
[37] S.P. Wan, G.L. Xu, J.Y. Dong, Supplier selection using ANP and ELECTRE II in interval 2-tuple linguistic environment, Inf. Sci. 385–386 (2017), 19–38.
[38] Z. Zhang, C. Guo, Consistency-based algorithms to estimate missing elements for uncertain 2-tuple linguistic preference relations, Int. J. Comput. Int. Syst. 7 (2014), 924–936.
[39] Z. Zhang, C. Guo, Consistency and consensus models for group decision-making with uncertain 2-tuple linguistic preference relations, Int. J. Syst. Sci. 47 (2016), 2572–2587.
[40] S.B. Yao, J.Q. Hu, A minimum deviation approach for improving the consistency of uncertain 2-tuple linguistic preference relations, Comput. Ind. Eng. 117 (2018), 181–190.
[41] K.X. Qi, Q. Wang, Q. Duan, L. Gong, J. Sun, K. Liew, L. Jiang, A multi criteria comprehensive evaluation approach for emergency response capacity with interval 2-tuple linguistic information, Appl. Soft Comput. 72 (2018), 419–441.
[42] H.C. Liu, P. Li, J.X. You, Y.Z. Chen, A novel approach for FMEA: combination of interval 2-tuple linguistic variables and gray relational analysis, Qual. Reliab. Eng. Int. 31 (2015), 761–772.
[43] H. Li, J.X. You, H.C. Liu, G. Tian, Acquiring and sharing tacit knowledge based on interval 2-tuple linguistic assessments and extended fuzzy petri nets, Int. J. Uncertain. Fuzz. Knowl. Based Syst. 26 (2018), 43–65.
[44] H.C. Liu, M.L. Ren, J. Wu, Q.L. Lin, An interval 2-tuple linguistic mcdm method for robot evaluation and selection, Int. J. Prod. Res. 52 (2014), 2867–2880.
[45] A. Zhang, W. Bi, Z. Tang, C. Zhang, Human-machine function allocation method for aircraft cockpit based on interval 2-tuple linguistic information, J. Syst. Eng. Electron. 27 (2016), 1291–1302.
[46] A. Singh, A. Gupta, A. Mehra, Energy planning problems with interval-valued 2-tuple linguistic information, Oper. Res. 17 (2017), 821–848.
[47] H.C. Liu, J.X. You, X.Y. You, Evaluating the risk of healthcare failure modes using interval 2-tuple hybrid weighted distance measure, Comput. Ind. Eng. 78 (2014), 249–258.
[48] W.S. Tai, C.T. Chen, A new evaluation model for intellectual capital based on computing with linguistic variable, Expert Syst. Appl. 36 (2009), 3483–3488.
[49] S. Alonso, F.J. Cabrerizo, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group decision making with incomplete fuzzy linguistic preference relations, Int. J. Intell. Syst. 24 (2010), 201–222.
[50] Z.J. Wang, A note on incomplete interval fuzzy preference relations and their applications, Comput. Ind. Eng. 77 (2014), 65–69.
[51] S.P. Wan, F. Wang, J.Y. Dong, A group decision making method with interval valued fuzzy preference relations based on the geometric consistency, Inf. Fusion. 40 (2018), 87–100.
[52] K. Sugihara, H. Ishii, H. Tanaka, Interval priorities in ahp by interval regression analysis, Eur. J. Oper. Res. 158 (2004), 745–754.
[53] J.P. Xu, Z.B. Wu, A maximizing consensus approach for alternative selection based on uncertain linguistic preference relations, Comput. Ind. Eng. 64 (2013), 999–1008.