

A Modified Negative Binomial Distribution: Properties, Overdispersion and Underdispersion

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ABSTRACT

In this paper, we introduce a new and useful discrete distribution (modified negative binomial distribution) and its statistical and probabilistic properties are discussed. This distribution is a three-parameter extension of the negative binomial distribution that generalizes some well-known discrete distributions (negative binomial and geometric). Various statistical and probabilistic properties were derived such as moments, probability and moment generating functions and maximum likelihood estimation of parameters. Modified negative binomial distribution is appealing from a theoretical point of view since it belongs to the exponential family as well as to the weighted negative binomial distributions family. It is a flexible distribution that can account for overdispersion or underdispersion that is commonly encountered in count data. Finally, a real numerical example is also considered for illustrative purpose.

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1. INTRODUCTION

Usually the binomial and Poisson and negative binomial distributions are used to analyze discrete data. However, it seems wise to consider flexible alternative models to take into account the overdispersion or underdispersion. It is well known that the negative binomial distribution has become increasingly popular as a more flexible alternative to the Poisson distribution especially when it is doubtful whether the strict requirements particularly independence for a Poisson distribution will be satisfied. For various applications of negative binomial distribution, see Johnson *et al.* [1].

Greenwood and Yule [2] presented the negative binomial distribution as a mixture of Poisson distribution where the mean λ of the Poisson distribution follows a gamma distribution. There are various extensions/modifications of negative binomial distribution in the literature including Engens extended negative binomial distribution [3,4], generalized negative binomial distribution of Jain and Consul [5], a new generalization of the negative binomial distribution of Gupta and Ong [6] and weighted negative binomial distribution; see Johnson *et al.* [1] for more details and explanations.

For many set of observed data, it is common to have the sample variance to be greater or smaller than the sample mean which are referred to as overdispersion and underdispersion, respectively. Several authors have worked on the case of overdispersion; see, for example, Gelfand and Dalal [7], Hougaard *et al.* [8] and Kokonendji *et al.* [9].

In this paper, we show that under some conditions the weighted geometric distributions are modified negative binomial (Mod-NB) distributions that provide a unified approach to handle both overdispersion and underdispersion. Interested readers may refer to Gupta and Kirmani [10] and Pakes *et al.* [11] for comprehensive discussions on weighted distributions. They were originally introduced by Fisher [12] through the method of ascertainment, which is just a method of adjustment applicable to many situations; see also Rao [13] and Patil [14].

The rest of this paper is organized as follows: In Section 2, we introduce the Mod-NB distribution along with its mathematical properties. Section 3 describes the maximum likelihood method for estimating the parameters. In Section 4, we investigate connections of the geometric weight function to overdispersion and underdispersion. A real numerical example is also considered for illustrative purposes in Section 5. Finally, some concluding remarks are made in Section 6.

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2. Mod-NB AND ITS PROPERTIES

Adding parameters to a known distribution is a useful way of constructing flexible families of distributions. In this paper, based on the negative binomial distribution, we introduce one new models, referred to as Mod-NB which include as special cases the well-known such as negative binomial and geometric distribution. We say that the random variable X follows Mod-NB distribution if its probability mass function be as follows:

$$P(X = x | p, \nu) = \frac{1}{\sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x (1-p)^r} \binom{x+r-1}{r-1}^{\nu} p^x (1-p)^r, \quad x = 0, 1, \dots, \quad (1)$$

for $r \in \mathbb{R}^+$ (set of known non-negative), $p \in (0, 1)$, and $\nu \in \mathbb{R}$. Throughout, we have considered r as a constant. For $\nu = 1$, we have the usual negative binomial distribution and for $\nu = 0$ and $r = 1$, we have the geometric distribution (see Figures 1 and 2).

The Mod-NB distribution can be interpreted as a sum of non-independent geometric variables Z_i ($i = 1, \dots, r$) with joint distribution

$$P(Z_1 = z_1, \dots, Z_r = z_r) = \frac{1}{\sum_{z_1=0}^{\infty} \dots \sum_{z_r=0}^{\infty} \binom{x+r-1}{r-1}^{\nu-1} p^x (1-p)^r} \binom{x+r-1}{r-1}^{\nu-1} p^x (1-p)^r, \quad \left(x = \sum_{i=1}^r z_i\right).$$

If we set $Z(p, \nu) = \sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x (1-p)^r$, then Eq. (1) can be rewritten as follows:

$$\begin{aligned} P(X = x) &= \frac{1}{Z(p, \nu)} \binom{x+r-1}{r-1}^{\nu} p^x (1-p)^r \\ &= \frac{1}{Z(p, \nu)} \exp \left\{ x \log p + r \log (1-p) + \nu \log \left(\frac{(x+r-1)!}{x!} \right) + \nu \log (r-1)! \right\}. \end{aligned} \quad (2)$$

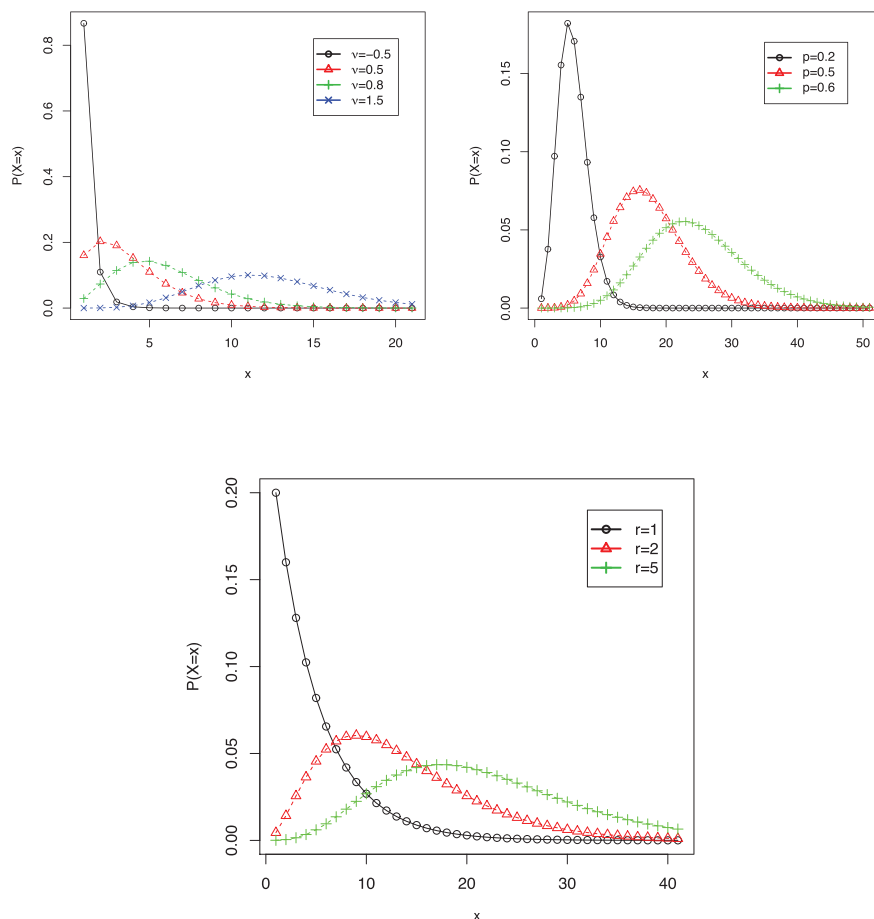


Figure 1 | Probability mass function. Left figure: $r = 10, p = 0.4$. Right figure: $r = 10, \nu = 1.5$. Down figure: $p = 0.8, \nu = 2$

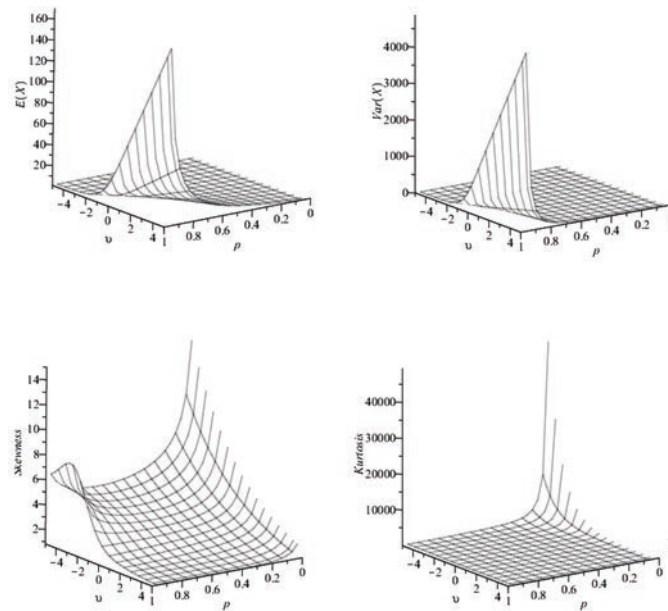


Figure 2 Surface of expectation, variance, skewness and kurtosis of modified negative binomial (Mod-NB) probability mass function with respect to parameter (v and p) while $r = 2$.

It follows from (2) that the Mod-NB distribution belongs to the exponential family on $\mathbb{N}$, where $T(X) = \left(X, \log \left(\frac{(X+r-1)!}{X!} \right) \right)$ is the sufficient statistic and $(\log p, v) \in \mathbb{R}^2$ is the corresponding natural parameters (see Barndorff-Nielsen [15]). For a set of n independent and identically distributed variables $\mathbf{X} = (X_1, \dots, X_n)$, the sufficient statistic based on $\mathbf{X}$ is $T(\mathbf{X}) = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n \log \left(\frac{(X_i+r-1)!}{X_i!} \right) \right)$. Now, we also establish the Mod-NB distribution belongs to the family of weighted geometric distributions defined as follows:

$$P(X = x|p, v) = \frac{w(x; v) P^*(X = x|1-p)}{E_{1-p}(w(X; v))}, \quad (3)$$

where $w(\cdot; v)$ is a non-negative weight function with parameter, $P^*(\cdot|1-p)$ is the probability mass function of a geometric distribution with parameter $(1-p)$, and $E_{1-p}(\cdot)$ indicates that the expectation is taken with respect to the geometric distribution with parameter $1-p$.

Therefore, if we take the weight function as $w(x; v) = \left(\frac{(x+r-1)!}{(r-1)! x!} \right)^v$, then the Mod-NB distribution in (1) can be expressed as the weighted geometric distribution in (3).

The probability generating function of this weighted geometric variable is given by

$$\begin{aligned} \psi_X(s) &= E(s^X) = \sum_{x=0}^{\infty} s^x \frac{w(x; v) P^*(X = x|1-p)}{E_{1-p}(w(X; v))} \\ &= \frac{1}{E_{1-p}(w(X; v))} \sum_{x=0}^{\infty} w(x; v) (1-p) (sp)^x \\ &= \frac{1-p}{1-sp} \frac{E_{1-sp}(w(X; v))}{E_{1-p}(w(X; v))} \\ &= \frac{Z(sp, v)}{Z(p, v)}, \quad 0 \leq s \leq 1. \end{aligned}$$

The moment generating function of X , say $M_X(s)$, is immediately derived from $\psi_X(s)$ as

$$M_X(s) = E(e^{sX}) = \psi_X(e^s) = \frac{Z(p e^s, v)}{Z(p, v)}, \quad 0 \leq s \leq 1.$$

3. MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random sample size of n with observed values $\mathbf{x} = (x_1, \dots, x_n)$ from a Mod-NB distribution with parameters $1-p$ and ν . Let us define

$$t_1(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{and} \quad t_2(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \log \left(\frac{(x_i + r - 1)!}{x_i!} \right).$$

It is of interest to note that $t_1(\mathbf{x})$ and $t_2(\mathbf{x})$ are, respectively, the sample mean and the log-geometric mean.

The log-likelihood function for the Mod-NB distribution model based on the observed sample $\mathbf{x}$ is

$$l(p, \nu) = n(-\log Z(p, \nu) + t_1 \log p + r \log(1-p) + \nu t_2 + \nu \log(r-1)!). \quad (4)$$

We can show that

$$\begin{aligned} E(X) &= \frac{1}{Z(p, \nu)} \sum_{x=0}^{\infty} x \binom{x+r-1}{r-1}^{\nu} p^x (1-p)^r \\ &= p \frac{1}{\sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x} \sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} \left(\frac{d}{dp} p^x \right) \\ &= p \frac{1}{\sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x} \frac{d}{dp} \left(\sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x \right) \\ &= p \frac{1}{((1-p)^{-r} Z(p, \nu))} \frac{d}{dp} ((1-p)^{-r} Z(p, \nu)) \\ &= p \frac{d \log((1-p)^{-r} Z(p, \nu))}{dp}, \end{aligned}$$

and

$$\begin{aligned} E \left(\log \left(\frac{(X+r-1)!}{X! (r-1)!} \right) \right) &= \frac{1}{Z(p, \nu)} \sum_{x=0}^{\infty} \log \left(\frac{(x+r-1)!}{x! (r-1)!} \right) \binom{x+r-1}{r-1}^{\nu} p^x (1-p)^r \\ &= \frac{1}{\sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x} \sum_{x=0}^{\infty} \log \left(\frac{(x+r-1)!}{x! (r-1)!} \right) \binom{x+r-1}{r-1}^{\nu} p^x \\ &= \frac{1}{\sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x} \sum_{x=0}^{\infty} \left(\frac{d}{d\nu} \binom{x+r-1}{r-1}^{\nu} \right) p^x \\ &= \frac{1}{\sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x} \frac{d}{d\nu} \sum_{x=0}^{\infty} \binom{x+r-1}{r-1}^{\nu} p^x \\ &= \frac{1}{((1-p)^{-r} Z(p, \nu))} \frac{d}{d\nu} ((1-p)^{-r} Z(p, \nu)) \\ &= \frac{1}{Z(p, \nu)} \frac{d}{d\nu} Z(p, \nu) \\ &= \frac{d \log(Z(p, \nu))}{d\nu}, \end{aligned}$$

and then, we have

$$E \left(\log \left(\frac{(X+r-1)!}{X!} \right) \right) = \frac{d \log(Z(p, \nu))}{d\nu} + \log((r-1)!).$$

As mentioned earlier since Mod-NB belongs to the exponential distributions family therefore the likelihood equations, based on the observed sample $\mathbf{x}$, may be written as

$$\begin{cases} p \frac{d \log ((1-p)^{-r} Z(p, \nu))}{dp} = E(X) = t_1(\mathbf{x}), \\ \frac{d \log (Z(p, \nu))}{d\nu} + \log((r-1)!) = E\left(\log\left(\frac{(X+r-1)!}{X!}\right)\right) = t_2(\mathbf{x}). \end{cases} \quad (5)$$

Since these equations cannot be solved analytically, an iterative method such as the Newton-Raphson method can be used (see Gelman *et al.* [16], pages 272–273). In each iteration, the expectations, variance and covariance $\log\left(\frac{(X+r-1)!}{X!}\right)$ are computed by plugging the estimates of p and ν obtained from the previous iterations in the expression

$$E(g(X)) = \frac{1}{Z(p, \nu)} \sum_{x=0}^{\infty} g(x) \binom{x+r-1}{r-1}^{\nu} p^x (1-p)^r.$$

Notice that the maximum likelihood estimators of the parameters can also be obtained by direct maximization of the log-likelihood function in (4) by using the SAS (PROC NLMIXED) or MaxBFGS routine of the Ox program (Doornik [17]) or optim routine of the R package (R Development Core Team [18]).

4. OVERDISPERSION AND UNDERDISPERSION IN WEIGHTED GEOMETRIC DISTRIBUTION

In this section, we discuss connection between weighted geometric distribution and its overdispersion and underdispersion. We generalize the Mod-NB distribution to the general weight $w(x; \nu)$, when weights are not necessary $w(x; \nu) = \left(\frac{(x+r-1)!}{(r-1)! x!}\right)^{\nu}$. We say that the random variable X^w follows weighted geometric distribution if its probability mass function be as follows:

$$P(X^w = x | p, \nu) = \frac{w(x; \nu)}{E(w(X; \nu))} (1-p) p^x \quad x = 0, 1, \dots$$

Theorem 4.1. Let X be a geometric random variable with mean $p/(1-p) > 0$, and let $w(x, \nu)$, $x \in \mathbb{N}$, be a weight function not depending on the geometric distribution parameter. Then,

$$\text{Var}_{1-p}(X^w) = E_{1-p}(X^w) + \frac{p^2}{(1-p)^2} + p^2 \frac{d^2}{dp^2} E_{1-p}(w(X, \nu)).$$

Proof. Let $\theta = \log p$. Then, the canonical probability mass function of X^w is given by

$$P(X = x | \theta, \nu) = w(x; \nu) \exp\{x\theta + \log(1 - e^{\theta}) - \log E(w(X; \nu))\}, \quad x \in \mathbb{N}.$$

For fixed ν , the probability mass function X^w is therefore an element of the natural exponential family on $\mathbb{N}$ with cumulant function $K(\theta, \nu) = \log E_{\theta}(w(X; \nu)) - \log(1 - e^{\theta})$; see Jørgensen [19]. So, the mean $E_{1-p}(X^w) = dK(\theta, \nu)/d\theta$ of X^w is

$$\begin{aligned} E_{1-p}(X^w) &= \frac{e^{\theta}}{1 - e^{\theta}} + \frac{d}{d\theta} \log E_{\theta}(w(X; \nu)) \\ &= \frac{p}{1-p} + p \frac{d}{dp} \log E_{1-p}(w(X; \nu)), \quad \left(\frac{d}{d\theta} = \frac{d/dp}{d\theta/dp} = p \frac{d}{dp}\right). \end{aligned}$$

Next, the characteristic variance $Var_{1-p}(X^w) = d^2 K(\theta, \nu) / d\theta^2$ of X^w is

$$\begin{aligned} Var_{1-p}(X^w) &= \frac{d}{d\theta} \left(\frac{d}{d\theta} K(\theta, \nu) \right) \\ &= \frac{d}{d\theta} \left(\frac{e^\theta}{1 - e^\theta} + \frac{d}{d\theta} \log E_\theta(w(X; \nu)) \right) \\ &= \frac{e^\theta}{(1 - e^\theta)^2} + \frac{d}{d\theta} \log E_\theta(w(X; \nu)) \\ &= \frac{p}{(1 - p)^2} + p \frac{d}{dp} \left(p \frac{d}{dp} \log E_{1-p}(w(X; \nu)) \right) \\ &= \frac{p}{(1 - p)^2} + p \frac{d}{dp} \log E_{1-p}(w(X; \nu)) + p^2 \frac{d^2}{dp^2} \log E_{1-p}(w(X; \nu)) \\ &= E_{1-p}(X^w) + \frac{p^2}{(1 - p)^2} + p^2 \frac{d^2}{dp^2} \log E_{1-p}(w(X; \nu)). \end{aligned}$$

Hence, the theorem is proved. □

The following corollary is a direct consequence of Theorem 4.1 which enable us to compare weighted geometric distributions in terms of overdispersion and underdispersion.

Corollary 4.1. Let X be a geometric random variable with mean $p / (1 - p) > 0$, and let $w(x, \nu)$, $x \in \mathbb{N}$ be a weight function not depending on the geometric mean parameter. Then,

- If $\frac{d^2}{dp^2} \log E_{1-p}(w(X; \nu)) > -\frac{1}{(1 - p)^2}$, then the weighted version X^w of X is overdispersed.
- If $\frac{d^2}{dp^2} \log E_{1-p}(w(X; \nu)) < -\frac{1}{(1 - p)^2}$, then the weighted version X^w of X is underdispersed.

Corollary 4.1 will be clearly useful in cases when $E_{1-p}(w(X; \nu))$ is available in an explicit form as in the case of the size-biased geometric distribution with $w(x; \nu) = x$, $x \in \mathbb{N}$.

It is of interest to use a statistical test for detecting the overdispersion or underdispersion in observed count data, (Mizère *et al.* [20]), and it is therefore useful to have a family of count distributions possessing both overdispersion and underdispersion properties with respect to the parameters. In this case, the parameter estimation would lead to an appropriate model within the family for overdispersed or underdispersed count data.

5. REAL DATA EXAMPLE

To illustrate the usefulness and flexibility of the Mod-NB distribution, we consider a real data set. Jaggia and Thosar (1993) model the number of bids received by 126 U.S. firms that were targets of tender offers during the period from 1978 through 1985 and were actually taken over within 52 weeks of the initial offer. The count variable is the number of bids after the initial bid (NUMBIDS) received by the target firm (See Table 1). Here, we have $k = 126$. We fit the Mod-NB, Negative Binomial (NB), Binomial (B), Beta Negative Binomial (BNB), Wang [21], and COM Poisson Binomial (COMP) Borges *et al.* [22] distributions by using the maximum likelihood method for estimate parameter of models.

The form of probability mass functions of BNB and COMP that be used in Table 2, respectively, as follows:

$$P(X = x) = \binom{k}{x} \frac{B(a + k, n - x - b)}{B(a, b)}, \quad x = 0, 1, \dots, k,$$

$$P(X = x) = \frac{1}{Z(\theta, \nu)} \frac{\theta^x}{[x! (k - x)!]^\nu}, \quad x = 0, 1, \dots, k,$$

Table 1 | Takeover bids: Actual frequency distribution.

Count	0	1	2	3	4	5	6	7	8	9	10
Frequency	9	63	31	12	6	1	2	1	0	0	1
Relative frequency	.071	.500	.246	.095	.048	.008	.016	.008	.000	.000	.008

Table 2 | The goodness of fit of the Mod-NB, NB, B, BNB and COMP distributions for real data.

Number of Takeover Bids	Observed Frequency	Expected Frequency				
		Mod-NB	NB	B	BNB	COMP
0	9	9.91	23.92	18.67	23.30	23.82
1	63	61.55	38.02	39.28	37.48	36.82
2	31	30.78	31.84	37.19	32.45	32.45
3	12	13.76	18.69	20.86	19.53	19.78
4	6	5.88	8.62	7.68	8.92	8.97
5	1	2.45	3.33	1.94	3.19	3.12
6	2	1.00	1.12	0.34	0.90	0.84
7	1	0.41	0.34	0.04	0.19	0.17
8	0	0.16	0.09	0.00	0.03	0.03
9	0	0.07	0.02	0.00	0.00	0.00
10	1	0.03	0.01	0.00	0.00	0.00
Parameter	—	$\hat{p} = 0.3818$	$\hat{p} = 0.0855$	$\hat{p} = 0.1738$	$\hat{a} = 6.0821$	$\hat{\nu} = 0.7037$
Estimates	—	$\hat{\nu} = -0.3898$	$\hat{\tau} = 18.6020$	—	$\hat{b} = 28.8135$	$\hat{\theta} = 0.3058$
	—	$\hat{r} = 0.0008$	—	—	—	—
Kernel of the log-likelihood	-180.1697	-184.5131	-201.1203	-208.3501	-205.4349	-206.4554
Chi-square	—	3.0104	29.7659	29.3274	31.0962	33.0807
P-value	—	0.9812	0.0009	0.0011	0.0005	0.0002

Mod-NB, modified negative binomial; NB, negative binomial; B, binomial; BNB, beta negative binomial; COMP, COM Poisson binomial.

where $B(\cdot, \cdot)$ is the Beta function and $Z(\theta, \nu) = \sum_{j=0}^k \frac{\theta^j}{[j! (m-j)!]^\nu}$.

Suppose $p_i = P(X = i)$ and $O_i =$ observed frequency of $(X = i)$. Then, in Table 1 the kernel of the log-likelihood of any model is given by

$$\sum_{i=0}^k O_i \log(p_i) = \sum_{i=0}^k O_i \log\left(\frac{n\hat{p}_i}{n}\right) = \sum_{i=0}^k O_i \log(E_i) - n \log(n),$$

where $n = \sum_{i=0}^k$ is total frequency, E_i is the expected frequency of $X = i$ and K is the number level of data. Also it follows that the kernel of the log-likelihood for data is $\sum_{i=0}^k O_i \log(O_i) - n \log(n)$.

In Table 2, it used the Chi-square statistics for testing the goodness of fit of model on real data. By p -value of this test is shown that the Mod-NB probability mass function, significantly is the best choice for distribution of real data.

6. CONCLUDING REMARKS

We study and discuss here the mathematical properties of the Mod-NB distribution as a extension of the negative binomial distribution. The main advantage of this model is its flexibility to handle overdispersion or underdispersion commonly encountered in count datasets. The Mod-NB distribution is appealing from a theoretical point of view since it belongs to the exponential family as well as to the weighted negative binomial distributions family. Various statistical and probabilistic properties were derived such as moments, probability and moment generating functions and maximum likelihood estimation of parameters. Since the Mod-NB distribution belongs to exponential family, it will also be possible to develop a subjective or objective Bayesian analysis for this model. Work in this direction is currently under progress and we hope to report these findings in a future paper.

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