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Discrete Dynamical Analysis of Euler Scheme for Harvested *Predator-Prey Model* with Ratio-Dependent Response Function and Prey Refuge

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***Abstract*—In this research, Euler method is applied to discretize a harvested predator prey model with ratio-dependent response function and prey refuge. The existence and stability of fixed points has been analyzed. Stability of each fixed point shown that the fixed points are stable for small size of time step. Such dynamical properties are also confirmed by numerical simulations.**

***Keywords*—Dynamical Analysis, Discrete, Euler, Harvested Predator-Prey Model, Ratio-Dependent Response Function, Prey Refuge**

I. INTRODUCTION

The predator-prey model is firstly introduced by Lotka-Voltera which has been heavily modified by several researchers [1,2,3]. This is adapted based on the problems that occur in the predator-prey ecosystem. Lenzini and Rebaza [4] did a research on the predator-prey dynamic behavior by non-constant harvesting of predators by using a ratio-dependent response function, in which a predator function depends on both prey and predatory density. Furthermore Ilmiah [5] modified Kar's [6] model about predator-prey harvesting model by including prey protection with Holling Type II response function and model from Lenzini and Rebaza. Ilmiah modified it by adding prey refuge.

In this research, we use Euler method to discretize Ilmiah's model. Though, discrete models have a richer behavioural dynamic compared to continuous models [7]. However, it has disadvantaged that divergent approximation may be obtained. Hence, step-size plays an important role to avoid inconsistency [8]. Therefore in this study, Ilmiah's model will be constructed to be a discrete model. The numerical approach to convert the continuous model to the discrete model in this study is using the Euler scheme.

Furthermore, the fixed points, terms of existence, and stability of fixed points will be investigated.

In this research, harvested predator-prey model with ratio-dependent response function and prey refuge was discretized using the Euler method and resulting in the following discrete form of equation:

$$\begin{aligned} x_{n+1} &= x_n + hx_n\left((1-x_n) - \frac{a(1-m)x_n y_n}{y_n+(1-m)x_n} - h_1\right) = f(x_n,y_n) \\ y_{n+1} &= y_n + hy_n\left(\left(1-\frac{y_n}{c}\right) - \frac{b(1-m)x_n y_n}{y_n+(1-m)x_n} - h_2\right) = g(x_n,y_n) \end{aligned} \quad (1)$$

With x_n is a prey population density and y_n is a predator population density. All parameters are assumed to be positive. h_1 is the rate of harvesting of prey, h_2 is the rate of harvesting predators, mx is expressing the rate of the first species to shelter, then the rate of the first species that can be preyed by the second species is $(1-m)x$ by $m \in [0,1)$. While a, b and c is modification result's parameter in [5] and h is the step size

II. THE EXISTENCE AND STABILITY OF FIXED POINTS

The discrete dynamic system fixed point is the point (x^*, y^*) that meet $f(x^*,y^*) = x^*$ and $g(x^*,y^*) = y^*$, that is

$$x^* = x^* + hx^*\left((1-x^*) - \frac{a(1-m)x^*y^*}{y^*+(1-m)x^*} - h_1\right) = f(x^*,y^*)$$

and

$$y^* = y^* + hy^*\left(\left(1-\frac{y^*}{c}\right) - \frac{b(1-m)x^*y^*}{y^*+(1-m)x^*} - h_2\right) = g(x^*,y^*)$$

from that combination we get 2 fixed points which is:

(i) $E_1(1-h_1, 0)$ with existence condition is $h_1 < 1$ namely predator extinction.

(ii) $E_2(0, c(1-h_2))$ with existence condition is $h_2 < 1$ namely prey extinction.

(iii) $E_3(x^*, y^*)$ where $y^* = \frac{((1-x^*)-h_1)(1-m)x^*}{a(1-m)-((1-x^*)-h_1)}$ and x^* is positive root from equation:

$$A{x^*}^2 + Bx^* + C = 0$$

with

$A = (1-m) + \frac{bc}{a(1-m)}$,

$B = c(1-h_2) + \frac{2bc}{a(1-m)}\big(a(1-m) - (1-h_1)\big) - (1-h_1)(1-m)$,

and

$C = c\big(a(1-m) - (1-h_1)\big)\left(\frac{b}{a(1-m)}\big(a(1-m) - (1-h_1)\big) + 1 - h_2\right)$,

$D = B^2 - 4AC$,
$D = \left(c(1-h_2) - (1-h_1)(1-m)\right)^2 - 4c(1-m)\left(a(1-m) - (1-h_1)\right)\left((1-h_2) + b\right)$.

The solution of this equation is:

$$x_{1,2}^* = \frac{-B \pm \sqrt{D}}{2A}.$$

By analyzing sign of B, C and discriminant $D$the researchers get the cases:

Case 1: If $C < 0$, then system (1) has unique interior fixed point

$E_{3,1} = (x_{3,1}^*, y_{3,1}^*) = \left(\frac{-B+\sqrt{B^2-4AC}}{2A}, \frac{((1-x^*)-h_1)(1-m)x^*}{a(1-m)-((1-x^*)-h_1)}\right)$.

Case 2: If $B < 0$ and $D = B^2 - 4AC = 0$, then system (1) has unique interior fixed point

$E_{3,2} = (x_{3,2}^*, y_{3,2}^*) = \left(\frac{-B}{2A}, \frac{((1-x^*)-h_1)(1-m)x^*}{a(1-m)-((1-x^*)-h_1)}\right)$.

Case 3: If $C = 0$ and $B < 0$, then system (1) has unique interior fixed point

$$E_{3,3} = (x_{3,3}^*, y_{3,3}^*) = \left(\frac{-B}{A}, \frac{\left((1-x^*)-h_1\right)(1-m)x^*}{a(1-m)-\left((1-x^*)-h_1\right)}\right)$$

Case 4: If $B < 0, C > 0$ and $D = B^2 - 4AC > 0$, then system (1) has two interior fixed points

a. $E_{3,4+} = (x_{3,4+}^*, y_{3,4+}^*) = \left(\frac{-B+\sqrt{B^2-4AC}}{2A}, \frac{((1-x^*)-h_1)(1-m)x^*}{a(1-m)-((1-x^*)-h_1)}\right)$

b. $E_{3,4-} = (x_{3,4-}^*, y_{3,4-}^*) = \left(\frac{-B-\sqrt{B^2-4AC}}{2A}, \frac{((1-x^*)-h_1)(1-m)x^*}{a(1-m)-((1-x^*)-h_1)}\right)$.

Case 5:

If no condition in case 1-4 holds, then system (1) has no positive fixed point.

A. Stability of Fixed Point

To learn the stability of fixed points of the model, we can see that the system (1) is in 2-dimesion, so that we can use this Lemma [9], which can be easily proved by the relations between roots and coefficients of a quadratic equation

Lemma 1. Let $F(\lambda) = \lambda^2 - p\lambda + q$. Suppose that $F(1) > 0$, $\lambda_1$andλ_2 are the two roots of $F(\lambda) = 0$. Then

1. $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only $F(-1) > 0$, and $q < 1$,
2. $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$,
3. $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$, and $q > 1$,
4. $\lambda_1$andλ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if$p^2 - 4q < 0$ and $q = 1$.

Let λ_1 and λ_2 are two eigen values of the fixed points. We recall some definitions of topological types for a fixed point(x^*, y^*). The fixed point (x^*, y^*) is called a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The fixed point (x^*, y^*) is called *saddle* if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). The fixed point (x^*, y^*) is called *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. And (x^*, y^*) is called *non-hyperbolic* if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

The Jacobian matrix of system (1) is

$$J(x^*, y^*) = \begin{bmatrix} 1 + h - 2hx^* - \frac{ha(1-m)y^{*2}}{(y^* + (1-m)x^*)^2} - hh_1 & -\frac{ha(1-m)^2 x^{*2}}{(y^* + (1-m)x^*)^2} \\ \frac{hb(1-m)y^{*2}}{(y^* + (1-m)x^*)^2} & 1 + h - \frac{2hy^*}{c} + \frac{hb(1-m)^2 x^{*2}}{(y^* + (1-m)y^*)^2} - hh_2 \end{bmatrix}$$

Jacobian matrix used to analyze stability of each fixed point. The Jacobian matrix in $E_1(1-h_1, 0)$ is

$$J(1-h_1, 0) = \begin{pmatrix} 1 - h(1+h_1) & -ha \\ 0 & 1 + h(1+b-h_2) \end{pmatrix} \quad (2)$$

The stability type of the fixed point E_1 is expressed in terms of the following theorem.

Theorem 1. Let $h_a = \frac{2}{(1+h_1)}$, $h_b = \frac{2}{-1-b+h_2}$, and $h_2 > 1 + b$

1. If$h < min(h_a, h_b)$ then E_2 is a sink.
2. If $(h_a, h_b) < h < max(h_a, h_b)$ then E_1 is saddle.
3. If $h > max(h_a, h_b)$ then E_2 is source.
4. If $h = h_a$ or $h = h_b$ then E_2 is *non-hyperbolic.*

Proof: The Eigen value of Jacobian matrix (2) is $\lambda_1 = 1 - h(1+h_1)$ and $\lambda_2 = 1 + h(1+b-h_2)$.

1. E_1 is stable (*sink*) if $-1 < 1 - h(1+\mathrm{h}_1) < 1$ and $-1 < 1 + h(1+b-h_2) < 1$.
   The first condition causes
   $$-2 < -h(1+h_1) < 1.$$
   Which is equivalence with
   $$0 < h < \frac{2}{(1+h_1)} = h_a.$$
   Second condition causes
   $$-2 < h(-1-b+h_2) < 2,$$
   So that
   $$0 < h(-1-b+h_2) < 2$$
   Because $h_2 > 1 + b$ so it obtains,
   $$0 < h < \frac{2}{(-1-b+h_2)} = h_b.$$
   That clear E_1 is stable (*sink*) if $h < min(h_a, h_b)$.
2. It's clear that $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) equivalence with $h > h_a$and $h < h_b$ and $h < h_a$and $h > h_b$.Therefore E_1 is unstable saddle (*saddle*), if $(h_a, h_b) < h < max(h_a, h_b)$.
3. It's clear that E_0 is unstable (*source*) if $|\lambda_1| > 1$ and $|\lambda_2| > 1$ that equivalence with $h > max(h_a, h_b)$.
4. It's clear that E_1 is *non-hyperbolic if* $|\lambda_1| = |\lambda_2| = 1$ that equivalence with $h = h_a$or $h = h_b$.

The Jacobian matrix in $E_1\left(0, c(1-h_2)\right)$ is

$$J(0, c(1-h_2)) = \begin{pmatrix} 1 + h(1-h_1) - ha(1-m) & 0 \\ hb(1-m) & 1 - h(1-h_2) \end{pmatrix} \quad (3)$$

The stability type of the fixed point E_2 is expressed in terms of the following theorem.

Theorem 2 Let $h_c = \frac{2}{(a(1-m)-(1-h_1))}$, $h_d = \frac{2}{1-h_2}$, and $a(1-m) > (1-h_1)$

1. If $h < min(h_c, h_d)$ then E_2 is a sink.
2. If $(h_c, h_d) < h < max(h_c, h_d)$ then E_2 is a saddle.
3. If $h > max(h_c, h_d)$ then E_2 is a source
4. If $h = h_a$ or $h = h_d$ then E_2 is a *non-hyperbolic.*

The Jacobian matrix in fixed point $E_3(x^*, y^*)$ is

$$J(x^*,y^*)=\begin{pmatrix} 1-hx^* & -\dfrac{ha(1-m)^2x^{*2}}{(y^*+(1-m)x^*)^2} \\ -\dfrac{hb(1-m)^2y^{*2}}{(y^*+(1-m)x^*)^2} & 1-\dfrac{hy^*}{c} \end{pmatrix} \quad (4)$$

Let $B_0=\frac{x^*y^*}{c}+\frac{ab(1-m)^3x^{*2}y^{*2}}{(y^*+(1-m)x^*)^4}$, $B^*=\frac{1}{4}\left(x^*+\frac{y^*}{c}\right)^2$ and
$h^*=\frac{x^*c^2(y^*+(1-m)x^*)^4+cy^*(y^*+(1-m)x^*)^4}{cx^*y^*(y^*+(1-m)x^*)^4+cab(1-m)^3x^{*2}y^{*2}}$,
$R=\frac{x^*y^*}{c}+\frac{ab(1-m)x^{*2}y^{*2}}{(y^*+(1-m)x^*)^4}$, $S=-2\left(x^*+\frac{y^*}{c}\right)$,
$T=4$, $h_e=-\frac{S}{2R}-\frac{\sqrt{S^2-4RT}}{2R}=h^*-\frac{\sqrt{S^2-4RT}}{2R}$,
and
$h_f=\frac{-S}{2R}+\frac{\sqrt{S^2-4RT}}{2R}=h^*+\frac{\sqrt{S^2-4RT}}{2R}$ then

1. Interior fixed point (x^*,y^*) in case 1, 3 and 4(a) is a sink if $1-m>0$and one of the following conditions holds.
   (a). $0<h<h^*$ and $B_0\geq B^*$, or
   (b). $0<h<h_e$ and $B_0<B^*$.
2. Interior fixed point (x^*,y^*) case 1, 3 and 4(a) is a saddle if $1-m>0$, $h_e<h<h_f$ and $B_0<B^*$.
3. Interior fixed point (x^*,y^*) in case 1, 3 and 4(a) is a source if $1-m>0$ and one of the following conditions holds.
   (a). $h>h^*$ and $B_0\geq B^*$, or
   (b). $h>h_f$ and $B_0<B^*$.
4. Interior fixed point (x^*,y^*) in case 1, 3 and 4(a) is a non-hyperbolic if $1-m>0$ and one of the following conditions holds.
   (a). $h=h^*$ and $B_0\geq B^*$, or
   (b). $h=h_e$ or $h=h_f$ and $B_0<B^*$.

III. Numerical Simulation

This simulation is done to illustrate the existence of E_1 and E_3 by using parameter $a=1.8, b=0.1, c=0.2, m=0.6, h_1=0.4$ and $h_2=1.2$. Based on these parameters the fixed point is generated into $E_1=(0.60)$, and fixed point $E_3=(0.3854,0.0655)$. This simulation generates $h_a=1.4286, h_b=20$. So, according to the previous analysis, it can be concluded that the fixed point E_1 will be (1) Sink if $h<1.4286$, (2) Saddle if $1.4286<h<20$.(3) Source if $h>20$.(4) *Non-hyperbolic* if $h=1.4286$ and $h=20$. To demonstrate the suitability between numerical simulation and the results of the analysis, a step size $h=0.1$ was obtained and a simulation result is shown in Fig. 1 below.

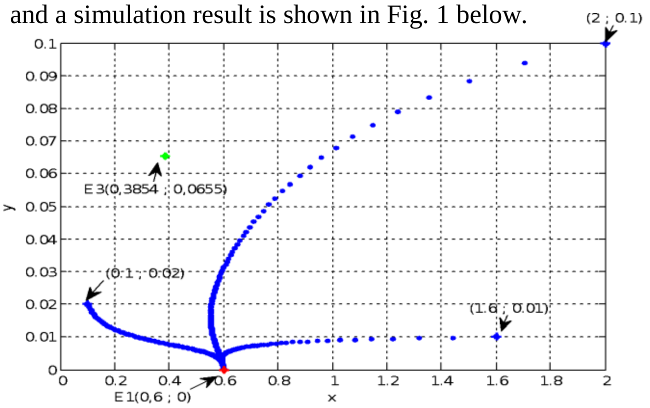

Fig. 1. The model simulation for stability of fixed point E_1

In the Fig. 1, it appears that with some initial values almost the solution converges for E_1. This is in accordance with the results of the analysis which states that the fixed point E_1 sink if $h<1.4286$. Next, in Fig. 2 it will be simulated stability of fixed point E_2 with parameter values $a=1.8, b=1.2, c=0.2, m=0.6, dan\ h_1=h_2=0.4$.

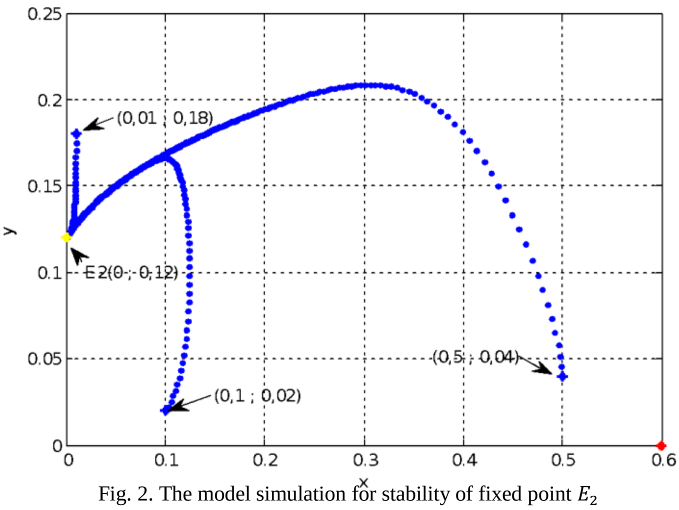

Fig. 2. The model simulation for stability of fixed point E_2

The third simulation uses parameters $a=1, b=1, c=0.2, m=0.6, h_1=0.6$, and $h_2=0.4$.
Based on these parameters the fixed point is generated $E_3(0.04,0.142)$ is sink. Fixed point $E_1(0,0.12)$ and $E_2(0.4,0)$ exist but not stable. The third simulation illustrate in Fig. 3.

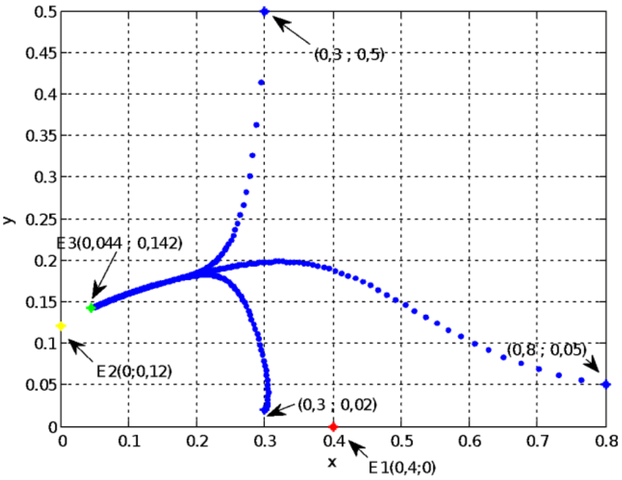

Fig. 3. The model simulation for stability of fixed point E_3

The Fig. 3 shows that by using some convergent initial values toward an interior fixed point (x^*,y^*) that satisfies cases 1, 3 and 4 (a). This means that the interior fixed points (x^*,y^*) that meet cases 1, 3 and 4 (a) are stable. This shows that in the long run, with the above parameters, the two ecosystems will be balanced

IV. Discussion and Conclusion

The predator-prey model was obtained by discretizing the model using the Euler method approach. The results of the analysis show that the predator-prey model has three fixed points, i.e. the fixed point of the predator extinction pointE_1, the prey extinction point E_2 and the fixed point of the interior E_3. The nature of the stability of the fixed point E_1, E_2 and E_3 determined by a certain terms and conditions.

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