Analysis of Geodetic Latitude Calculation Algorithms Using Chord Method

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Abstract— This article is an extension of investigation performed by the authors earlier, on the transformation of spatial rectangular coordinates X, Y, Z into geodetic curvilinear ones: latitude B, longitude L, height H. To calculate geodetic latitude B and reduced latitude U by solving a transcendental equation with variable coefficients f(t)=0, where t=tgU, a chord method was applied for the segment $[T_1, T_2]$; $T_1=tgu_1$; T-$=tgu2$. It is mathematically proved that the algorithm constructed on this segment using chord method will be valid only for points on the earth's surface with heights above sea level and with maximum error ΔB=0. 0017". To expand the limits of algorithm and to increase its accuracy, the segment of root location $[T_1, T_2]$ is reduced on the right to the value of $[T_1, T_6]$ where the latitude is determined not only for H>0 but also for H<0. The general formula of chords is transformed to a convenient for calculations form by introducing auxiliary quantities. Its accuracy was estimated and a comparative analysis with the previously obtained algorithm on segment $[T_3, T_2]$ was performed. As a result of a two-sided reduction of interval $[T_1, T_2]$, segment $[T3, T6]$ is obtained where the root of equation will be located only for H>0. It was proved that in order to calculate latitude for points of the earth's surface for |H| ≤ 10 km in non-iterative way using chord method, it was reasonable to use an algorithm constructed on segment $[T3, T2]$, and for near-Earth space points at H>a, calculations should be carried out according to the algorithm constructed on segment $[T1, T6]$.

Keywords—geodetic and rectangular spatial coordinates, ellipsoid, algorithms, chord method, latitude, formula errors.

I. INTRODUCTION

Scientific and technological progress in geodesy along with the development of near-Earth space makes the issue of determining the relative position of points in near-Earth space and on the surface of the Earth quite challenging. At the same time, aspects of solving this problem change significantly. The position of points in near-Earth space and on the Earth's surface is described by systems of geodetic spatial curvilinear coordinates of latitude B, longitude L, height H and spatial rectangular coordinates X, Y, Z. Transformation of spatial rectangular coordinates into their geodetic curvilinear coordinates is a fundamental challenge for geodesy.

Basic methods for solving the fundamental task of transforming spatial rectangular coordinates X, Y, Z of a point into its geodetic curvilinear coordinates – latitude B, longitude L, height H, and associated problems are considered in the following works [1-15].

One of the most promising non-iterative methods for calculating geodetic latitude is chord method. In article [1], for transforming coordinates in order to determine the tangent of reduced latitude tgU the equation

$$tgU = A + C \cdot SinU \qquad (1)$$

was solved with chord method, here the values

$$A = \frac{Z\sqrt{1-e^2}}{R}; \quad C = \frac{ae^2}{R}; \quad R = \sqrt{X^2 + Y^2} \neq 0, \qquad (2)$$

where

a – semi-major axis;

$(e = (\sqrt{a^2 - b^2}/a$ – first ellipsoid eccentricity;

b – semi-minor axis,

X, Y, Z – spatial rectangular coordinates.

As a part of study, the task was to construct a non-iterative algorithm for calculating latitude for points on the Earth's surface and near-Earth space using chord method with the error of latitude determining which is required by the Federal Agency on Technical Regulation and Metrology of the Russian Federation - ΔB of no more than 0.0001". In [1], the expediency of continuing such studies aimed to obtain more effective latitude calculation algorithms was mentioned.

II. METHODS

The solution of transcendental equation (1) with variable coefficients (2) using chord method became possible due to separation of root $\bar{t} = tg\bar{U}$ of equation (1), or what is equivalent to function

$$f(t) = t - A - C \cdot SinU; \quad t = tgU;$$

$$SinU = t/\sqrt{1 + t^2}. \qquad (3)$$

Using analytical approach, the interval $[T_1, T_2]$ of a very small length which contains only one root $\bar{t}$: $T_1 < \bar{t} < T_2$ was established where

$$T_1 = A = \frac{z\sqrt{1-e^2}}{R}; T_2 = \frac{z}{R\sqrt{1-e^2}} = \frac{T_1}{1-e^2}. \qquad (4)$$

For investigation and error estimation, approximate dependencies are used [1]

$$B = U; \quad R = (a + H) \cdot CosU; \quad Z = (a + H) \cdot SinU, \quad (5)$$

which determine the values of function at the endpoints of segment

$$f(T_1) = -\frac{ae^2}{a+H} \cdot t_0; \quad f(T_2) = \frac{He^2}{a+H} \cdot t_0, \tag{6}$$

where t_0 – any number in segment $[T_1, T_2]$.

The root of equation (3) by chord method on segment $[T_1, T_2]$ is determined using general formula

$$tgU = T_1 - \frac{(T_1 - T_2) \cdot f(T_1)}{f(T_1) - f(T_2)}, \tag{7}$$

and for accuracy assessment, the error of argument of function tgU is found using rule (18) from [1]

$$\Delta U = -\frac{3}{2} \cdot \frac{ae^2}{a+H} \cdot SinU \cdot Cos^5U \cdot f(T_1) \cdot f(T_2) \tag{8}$$

Using the values of function $f(t)$ in accordance with formula (6), it is established at the endpoints of root location segment that finding latitude using formula (7) is mathematically valid only for points on the earth's surface with heights above sea level H>0. For accuracy assessment, it was found that the error calculated by formula (8), which is equal to $\Delta U = 0{,}0017"$, reaches its maximal value at the latitude $U = 45°$, at the height $H = a/2$.

To expand the limits of formula (7) and to increase its accuracy which is calculated by formula (8), root location segment $[T_1, T_2]$ is reduced. For this purpose, equation (3) is brought to the form (1) or is represented in the form (12) from [1]:

$$tgU = \frac{A}{1 - C\,CosU} \tag{9}$$

Substitution $t=T_2$ according to formula (9) in [1] results in:

$$T_3 = tgU_3 = \frac{Z\sqrt{1-e^2}}{R(1 - \frac{ae^2}{a_0})}; \quad a_0 = R\sqrt{1 + T_2^2} \tag{10}$$

If Z>0, the inequalities $T_1 < T_3 < T_2$ are satisfied and root $\bar{t}$ is on the segment $[T_3, T_2]$ at $(-a + ae^2) < H < +\infty$.

As a result of left-sided reducing the segment $[T_1, T_2]$ the area of root location decreased to

$T_3 < \bar{t} < T_2$, and the accuracy of latitude calculating increased.

This is graphically shown in Fig. 1. where we can see that value t^* is much closer to the true value $\bar{t}$, than t_1.
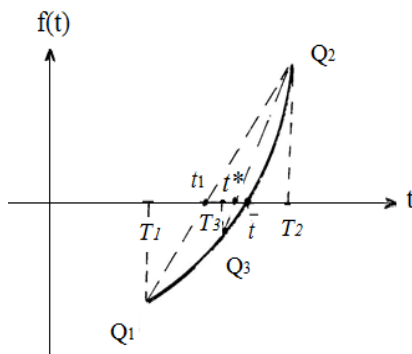

Fig. 1. Reducing of left root location area

In fact, to calculate the latitude for all $-a + ae^2 < H < +\infty$, formula (8) is used where T_1 is replaced with T_2 and T_2 is replaced with T_3:

$$tgU = T_2 - \frac{(T_2 - T_3) \cdot f(T_2)}{f(T_2) - f(T_3)} \tag{11}$$

Accordingly, confidence evaluation of results obtained by formula (8) is performed using expression

$$\Delta_3 U = -\frac{3}{2} \cdot \frac{ae^2}{a+H} \cdot SinU \cdot Cos^5U \cdot f(T_2) \cdot f(T_3) \quad \text{and}$$

values $f(T_2)$ (4) and $f(T_3)$ from [1]:

$$f(T_3) = -\frac{aHe^4}{(a+H)^2} \cdot t \cdot Sin^2U \tag{12}$$

Result error is

$$\Delta_3 U = \frac{3}{2} \cdot \frac{a^2 H^2 e^8}{(a+H)^4} \cdot Sin^5U \cdot Cos^3U; \quad a + H \neq 0 \tag{13}$$

Accuracy evaluation revealed that the error calculated by formula (13) reaches its maximal value at $H=a$, $U \approx 52°$ and amounts to $\Delta_3 U = 2{,}8" \cdot 10^{-6}$.

III. Results

As it was in studies performed in [1], by substituting $t=T_1$ according to formula (9) we obtain

$$T_6 = tgU_6 = \frac{A}{1 - C/\sqrt{1+T_1^2}} = \frac{Z\sqrt{1-e^2}}{R(1 - \frac{ae^2}{b_0})}; \quad b_0 = R\sqrt{1 + T_1^2} \tag{14}$$

From a geometric point of view, in chord method as can be seen in Fig. 2, the arc $Q_1 Q_2$ of f(t) curve on segment $[T_1, T_2]$ is replaced with chord that contracts it., Abscissa t_1 of the intersection of chord $Q_1 Q_2$ with axis 0t is taken as approximate value of the root $\bar{t}$ of intersection $Q_1 Q_2$ of f(t) curve with axis 0t.
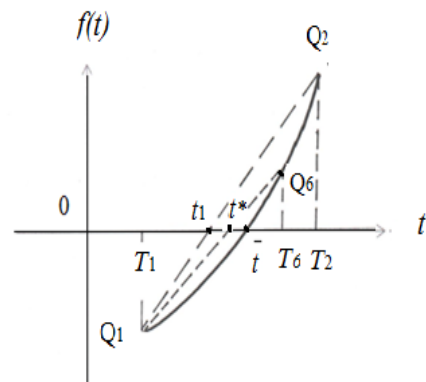

Fig. 2. Approximation of root on the right

Since approximation to the root by iteration method according to formula (9) is carried out from two sides, number T_6 on coordinate axis will be located on the other side from T_1, that is, $T_1 < \bar{t} < T_6$ and at the same time $T_1 < T_6 < T_2$. Thus, the root separation part $[T_1, T_2]$ decreased from the right to segment $[T_1, T_6]$. On segment $[T_1, T_6]$, abscissa $t^* = tgU$ of the intersection of chord $Q_1 Q_6$ with axis 0t is taken as an approximate root value being more accurate than approximate value t_1

Taking into account approximate expressions calculated by formulas (5), with an error of order magnitude e^6, by formula (3) we find

$$f(T_6) = \left(\frac{ae^2}{a+H}\right)^2 \cdot t^* \cdot Sin^2 U \qquad (15)$$

Using values $f(T_1)$ и $f(T_6)$ calculated by formulas (6) and (15), respectively, we establish that the root $\bar{t}$ of equation (3) will be on segment $[T_1, T_6]$ for all $(-a + ae^2) < H < +\infty$. The tangent of reduced latitude on this segment of the root branch is found by formula (7) replacing T_2 with T_6, that is:

$$tgU = T_1 - \frac{(T_1 - T_6) \cdot f(T_1)}{f(T_1) - f(T_6)}. \qquad (16)$$

Under this condition, on the basis of dependence (8), we find the error in latitude calculating

$$\Delta U = -\frac{3}{2} \frac{ae^2}{a+H} \cdot SinU \cdot Cos^5 U \cdot f(T_1) \cdot f(T_6) \qquad \text{by}$$

formula (16). Taking into account expressions $f(T_1)$ и $f(T_6)$ calculated by formulas (6) and (15), respectively, we obtain the formula for error in latitude calculating

$$\Delta_6 U = \frac{3}{2} \cdot \left(\frac{ae^2}{a+H}\right)^4 \cdot Sin^5 U \cdot Cos^3 U. \qquad (17)$$

Estimates presented by expressions (13) and (17) allow establishing a relationship between the errors of results obtained by formulas (11) and (16)

$$\Delta_3 U = \left(\frac{H}{a}\right)^{2.} \cdot \Delta_6 U \qquad (18)$$

Analysis of accuracy evaluation using relation (18) shows that, according to formulas (11) and (16), for $H > a$ equal results are obtained which in compliance with studies according to formula (13) are found with the maximal error of $\Delta U = 2,8'' \cdot 10^{-6}$ at latitude U=52°.

For $H < a$ the errors of these formulas satisfy the inequality $\Delta_1 U < \Delta_6 U$. Therefore, in this case, in order to find the latitude, it is reasonable to use the algorithm proposed in [1] which is based on formula (11). And conversely, for $H > a$, more accurate results are obtained when calculating the latitude according to formula (16).

Based on formula (16), we construct an algorithm which is convenient for calculations. For that purpose, value T_6, calculated by formulas (14) can be represented as $T_6 = C_0 \cdot T_1$, where C_0 is obtained as

$C_0 = \frac{1}{1 - \frac{ae^2}{b_0}}$; $b_0 = R \cdot \sqrt{1 + T_1^2}$ and a new variable $P = \frac{b_0}{R\sqrt{T_1^2 + 1/C_0^2}}$.is introduced. According to the dependences for C_0 and b_0 we obtain equality $b_0 = \frac{ae^2 C_0}{C_0 - 1}$.

Then, taking into account variable P and the expression for b_0, we perform the following transformations

$$\frac{\sqrt{1+T_1^2}}{C} = \frac{b_0}{ae^2} = \frac{C_0}{C_0 - 1}; \quad f(T_1) = \frac{1 - C_0}{C_0} \cdot T_1;$$

$$f(T_6) = T_6 - T_1 - \frac{C \cdot T_6}{\sqrt{T_6^2 + 1}} = (C_0 - 1)T_1 - \frac{C_0 - 1}{C_0} \cdot T_1 \cdot P;$$

$$f(T_1)(T_1 - T_6) = \frac{(1 - C_0)^2}{C_0} \cdot T_1^2;$$

$$f(T_1) - f(T_6) = \frac{1 - C_0}{C_0} \cdot T_1 \cdot (1 + C_0 - P).$$

With the expressions obtained, the formula (16) is as follows

$tgU = T_1 - \frac{(1 - C_0)T_1}{1 + C_0 - P} = T_1 \frac{2C_0 - P}{1 + C_0 - P}$. Since $tg\,U = tg\,B \cdot \sqrt{1 - e^2}$, we get the following formula for calculating geodetic latitude

$$B = arctg\left(\frac{Z}{R}\frac{2C_0 - P}{1 + C_0 - P}\right) \qquad (19)$$

For calculations by formula (19), we use constant values obtained from the parameters of ellipsoid with the semi-major axis a и compression denominator f, while $\alpha = \frac{a-b}{a} = 1/f$:

$$K_0 = \sqrt{1 - e^2} = \frac{f - 1}{f}; \quad K_1 = ae^2 = \frac{a(2f - 1)}{f^2}.$$

The solution by formulas (19) is performed in the following sequence:

1) $R = \sqrt{X^2 + Y^2} \neq 0$;

2) $T_1 = K_0 \cdot Z/R$;

3) $b_0 = R \cdot \sqrt{1 + T_1^2}$;

4) $C_0 = \frac{1}{1 - K_1/b_0}$

5) $P = \frac{b_0}{R\sqrt{T_1^2 + \frac{1}{C_0^2}}} = \frac{\sqrt{T_1^2 + 1}}{\sqrt{T_1^2 + \frac{1}{C_0^2}}}$;

6) $B = arctg\left(\frac{Z}{R} \cdot \frac{2C_0 - P}{1 + C_0 - P}\right),$ \qquad (20)

latitude calculation error $\Delta B = \Delta_6 U$ is estimated by formula (17).

The reliability of proposed algorithm was established by mutually inverse coordinate transformations followed by matching of assessments (17). As a result of a two-sided reduction of segment $[T_1, T_2]$ root location received segment $[T_3, T_6]$. However, according to the values of $f(t)$ function in these points (12) и (15), it follows that the root of equation will be located on this segment only for $H > 0$. Therefore, the possibility of using trapezoidal formula on segment $[T_3, T_6]$ to calculate the latitude of points of the earth's surface with heights below sea level is not mathematically proven.

Therefore, for points on the earth's surface at $|H| \leq 10$ km to calculate latitude in an iterative way, the chord method can be used on root location segments $[T_3, T_2]$ and $[T_1, T_6]$. However, according to the algorithm of formula (11), the results of studies presented in [1] showed that latitude is calculated with limiting error $\Delta B = 1''10^{-10}$, whereas according to the algorithm described by formulas (20), $\Delta B = 4,4''10^{-5}$.

IV. CONCLUSION

The analysis of non-iterative algorithms for calculating geodetic latitude using chord method on segments $[T_3, T_2]$ and $[T_1, T_6]$ revealed the following:

1) For points on the Earth's surface and in near-Earth space, their geodetic latitude is found with the accuracy required by the Federal Agency on Technical Regulation and Metrology of the Russian Federation, that is, the error in calculating latitude ΔB does not exceed 0.0001";

2) However, according to the algorithm of formula (11), the latitude of points on the earth's surface is calculated more accurately than according to formula (16) using algorithm (20);

3) For satellite orbits in near-Earth space, the latitude value is calculated more accurately using algorithm described by formulas (20);

In further studies, in order to obtain a more accurate algorithm for calculating geodetic latitudes, it is reasonable to consider the solution of equation (1) using chord method and according to the technique proposed for segment *[T₁, T₆]* (Fig. 2).

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