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Optimisation of Group Consistency for Incomplete Uncertain Preference Relation

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ABSTRACT

An incomplete uncertain preference relation (UPR) is typical in group decision making (GDM) for decision makers (DMs) to express preference over alternatives because of the information interaction barrier between people and decision making environment. Completing missing values can guarantee individual consistency and consensus level effectively. The operation of traditional interval preference relations (IPRs) is based only on the end point transformation, which may cause interval discretisation and information distortion easily. To overcome these limitations, pairwise comparison of alternatives in an IPR is treated as an uncertain distribution function of the subjective preference of the DM which avoids discretisation operation and handles interval numbers collectively. A belief degree is used to maintain the original information as much as possible. It guarantees the extent how people believe the estimated value is close to the incomplete original value. An uncertain chance constrained programming model is proposed herein to estimate incomplete values based on a belief degree when the preference relation obeys a linear uncertain distribution. A distance measure is defined to compute the consistency index and consensus degree. Subsequently, an iterative algorithm is presented for GDM with linear UPRs, which adjusts inconsistent preference relations and uses an operator to aggregate all individual preference relations. Furthermore, it is proven that the operation of UPRs is an extension of that of traditional IPRs under a certain belief degree.

1. INTRODUCTION

Interval preference relations (IPRs) have been widely used in uncertain group decision making (GDM) to represent decision makers' (DMs') preference over alternatives. When encountering complex or emergent situations, DMs cannot express their complete preference information on alternatives and often present an incomplete (sparse) form of judgment because of knowledge reserve, information mastery, environment impact, and so on. Incomplete IPRs can be managed by two approaches. One is filling in incomplete values based on consistency [1–6], and the other is ranking alternatives directly with known elements [7–9]. Although the latter preserves the true preferences of DMs, it cannot guarantee the consistency between individuals, thus resulting in the distortion of decision making. Based on consistency, the iterative algorithm [10–13] and the optimisation model [4,14–17] are two typical methods to complete missing values. Based on the iterative algorithm, although it is easy to change the original preference relations and the convergence speed is slow, the process is robust and the consistency is good. Furthermore, the optimisation model with missing parameter constraints can obtain the optimal solutions of incomplete values.

The consistency index is typically used for measuring the consistency level of an individual IPR; subsequently, an iterative algorithm is established to achieve an acceptable level of consistency, such as in an interval fuzzy preference relation (IFPR) [18,19], interval intuitionistic preference relation [20–22], linguistic preference relation [23,24], and hesitant fuzzy preference relation (FPR) [25,26]. The compatibility [27–32] is used similar to the consistency index. To ensure the efficiency and consensus [11,33–35], GDM typically aggregates individual preference relations into a collective preference relation, which is often obtained by the weighted averaging operator [36,37], ordered weighted averaging operator [6,10,38], and weighted geometric averaging operator [39,40], followed by a consensus to measure the difference among all individuals.

In traditional GDM, an incomplete value is often determined by minimising the deviation between the incomplete value and a supplementary value obtained by the consistency property. However, the new supplementary value may not necessarily match the original preference information, and the DM cannot measure the authenticity between the supplementary value and original missing information. Belief degrees in the uncertainty theory proposed by Liu [41] can solve this problem. Moreover, when handling the interval information, only the two end points of the interval are used in the operation, and the internal information of the interval is

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completely ignored, which easily results in decision distortion caused by the discretisation operation of the intervals.

In fact, the pairwise comparison of alternatives is located in an uncertain interval range in the IPRs, which is an uncertain estimation based on subjective experience. It can be understood as the uncertainty distribution (UD) of the subjective preference of the DM [41–47]: for the linear uncertainty distribution (LUD), every value of the DM's preference in the interval is equally possible; for the normal uncertainty distribution (NUD), the preference value obeys the NUD under a certain belief degree, and so on. As a mathematical system dedicated to researching the belief degrees of experts, uncertainty theory [41] provides a new solution to GDM with IPRs. Using a belief degree to represent the extent of which DMs believe the small enough deviation between the supplementary values and original preference information will happen. This guarantees the authenticity of corresponding estimated values. Further, let the interval preference of DMs obeys a certain UD can handle the interval preference collectively in the process of GDM. That can effectively solve the distortion problem of traditional interval operations.

Therefore, aiming at IPRs with incomplete values, an uncertain chance constrained programming model (UCCPM) is proposed herein to obtain the optimal solutions of incomplete values based on a certain belief degree and a LUD with its consistency condition. We also discover that the operation of uncertain preference relations (UPRs) is an extension of that of traditional IPRs. Based on the uncertainty theory, we define the consistency index which measures the deviation between IPRs and the formula for adjusting the inconsistent preference relations including all values in the interval. We use an induced hybrid weighted aggregation (IHWA) operator [16] to obtain the collective preference relations, which considers the weight of DMs and ordered positions simultaneously. Furthermore, we extend the GDM algorithm suggested by Meng and Chen [16] on FPRs to UPRs.

The paper is organised as follows. Section 2 discusses the basic definitions and properties of uncertainty theory and FPRs. Section 3 introduces the concept of UPRs. This section further defines the UCCPM, consistency index, and an algorithm to revise inconsistent preference relations. Additionally, an algorithm of GDM with UPRs which can address incomplete and inconsistent cases based on additive consistency is presented with an example and a comparative analysis. Section 4 summarises conclusions and discusses ideas for future research.

2. PRELIMINARIES

2.1. Uncertainty Theory

In uncertainty theory, a variable is called an uncertain variable, and a measure (distribution) function represents the degree with which we believe the uncertain variable falls into the left side of the current point [41].

Definition 1. Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Then $(\Gamma, \mathcal{L})$ is called a measurable space, and each element Λ in $\mathcal{L}$ is called an event. M represents an uncertain measure on the σ-algebra $\mathcal{L}$. $M\{\Lambda\}$ indicates the belief degree with which we believe Λ will happen.

Definition 2. Let Γ be a nonempty set, let $\mathcal{L}$ be a σ-algebra over Γ, and let M be an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathrm{M})$ is called an uncertainty space.

Axiom 1. *$M\{\Gamma\} = 1$ for the universal set Γ.*

Axiom 2. *$M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ.*

Axiom 3. *For each countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have* $M\left\{\bigcup_{i=1}^{n} \Lambda_i\right\} \leqslant \sum_{i=1}^{n} M\{\Lambda_i\}$.

Axiom 4. *Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure M is an uncertain measure satisfying $M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$, where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.*

Definition 3. An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers. Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables, and let f be a real-valued measurable function. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.

Definition 4. The UD Φ of an uncertain variable ξ is defined by $\Phi(x) = M\{\xi \leqslant x\}$ for any real number x. $\Phi(x)$ is a monotone increasing function and $0 \leqslant \Phi(x) \leqslant 1$ (Figure 1).

From Axiom 2, we have $M\{\xi \leqslant x\} + M\{\xi > x\} = 1$, then $M\{\xi > x\} = 1 - \Phi(x)$. When the UD Φ is continuous, we also have $M\{\xi < x\} = M\{\xi \leqslant x\} = \Phi(x)$, $M\{\xi \geqslant x\} = 1 - \Phi(x)$.

Definition 5. An UD $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and $\lim_{x \to -\infty} \Phi(x) = 0$, $\lim_{x \to +\infty} \Phi(x) = 1$.

Definition 6. Let ξ be an uncertain variable with regular UD $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution (IUD) of ξ. A function $\Phi^{-1} : (0, 1) \to R$ is the IUD of an uncertain variable ξ if and only if $M\{\xi \leqslant \Phi^{-1}(\alpha)\} = \Phi(\Phi^{-1}(\alpha)) = \alpha$, for all $\alpha \in (0, 1)$.

Definition 7. An uncertain variable ξ is called linear if it has a LUD $\Phi(x)$:

$$\Phi(x) = \begin{cases} 0 & x \leqslant a \\ \dfrac{x-a}{b-a} & a \leqslant x \leqslant b \\ 1 & x \geqslant b \end{cases} \tag{1}$$

denoted by $\xi \sim \mathcal{L}(a, b)$ where a and b are real numbers with $a < b$ (Figure 2).

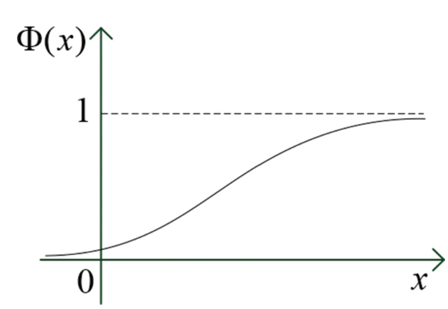

Figure 1 | Uncertainty distribution.

The IUD of linear uncertain variable (LUV) ξ is $\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$ (Figure 3). LUD is regular.

Theorem 1. *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variables with regular UDs $\Phi_1, \Phi_2, \dots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \dots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \dots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \dots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ has an IUD $\Psi^{-1}(\alpha)$. $\Psi^{-1}(\alpha)$ is defined as follows:*

$$f\left(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)\right) \tag{2}$$

2.2. Several Preference Relations

Let $X = \{x_i\}(i \in N = \{1, 2, \dots, n\}, n \geqslant 2)$ be a nonempty set of alternatives, $D = \{d_t\}(t \in M = \{1, 2, \dots, m\})$ be the set of DMs.

2.2.1. Fuzzy preference relations

Definition 8. [48] A FPR $R = (r_{ij})_{n\times n}$ is characterised by a function $\mu_R : X \times X \to [0,1]$, where $\mu_R(x_i, x_j) = r_{ij}$ indicates the preference intensity with which alternative x_i is preferred over x_j. R is additive reciprocal, if

$$r_{ij} + r_{ji} = 1, r_{ii} = 0.5, i, j \in N \tag{3}$$

$r_{ij} = 0.5$ indicates that there is no difference between x_i and x_j; $r_{ij} > 0.5$ represents that x_i is preferred to x_j; $r_{ij} < 0.5$ depicts that x_j is better than x_i.

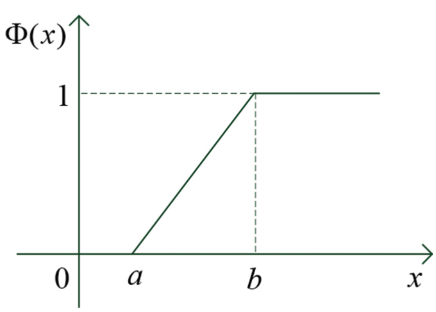

Figure 2 | Linear uncertainty distribution.

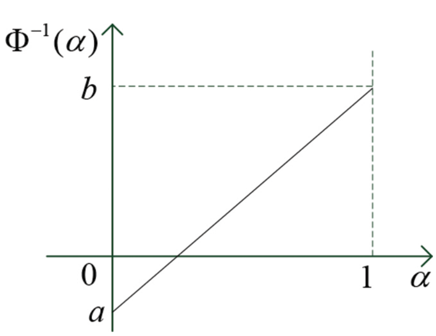

Figure 3 | Inverse linear uncertainty distribution.

Definition 9. [49] The FPR $R = (r_{ij})_{n\times n}$ is additively consistent, if it satisfies

$$r_{ij} = r_{ik} + r_{kj} - 0.5, i, j, k \in N \tag{4}$$

2.2.2. Interval fuzzy preference relations

Definition 10. [50] Let $\bar{a}_1 = [a_1^-, a_1^+]$, $\bar{a}_2 = [a_2^-, a_2^+]$ be any two positive intervals.

(a) $\bar{a}_1 \oplus \bar{a}_2 = [a_1^-, a_1^+] \oplus [a_2^-, a_2^+] = [a_1^- + a_2^-, a_1^+ + a_2^+]$

(b) $\bar{a}_1 \ominus \bar{a}_2 = [a_1^-, a_1^+] \ominus [a_2^-, a_2^+] = [a_1^- - a_2^+, a_1^+ - a_2^-]$

(c) $\lambda\bar{a}_1 = \lambda[a_1^-, a_1^+] = \begin{cases} [\lambda a_1^-, \lambda a_1^+] & \lambda \geqslant 0 \\ [\lambda a_1^+, \lambda a_1^-] & \lambda < 0 \end{cases}$

Definition 11. [51] Let $\overline{R} = (\bar{r}_{ij})_{n\times n}$ be a FPR. $\overline{R}$ is called an IFPR if $\bar{r}_{ij} = [r_{ij}^l, r_{ij}^r]$ and $\bar{r}_{ji} = [r_{ji}^l, r_{ji}^r]$ satisfy

$$r_{ij}^l + r_{ji}^r = r_{ij}^r + r_{ji}^l = 1 \tag{5}$$

$$r_{ii}^l = r_{ii}^r = 0.5 \tag{6}$$

$r_{ij}^r \geqslant r_{ij}^l \geqslant 0$, Eq. (5) is the additive reciprocity of $\overline{R}$.

Definition 12. [1] Let $\overline{R} = (\bar{r}_{ij})_{n\times n}$ be an IFPR. $\overline{R}$ is additively consistent if

$$\begin{cases} r_{ik}^l + r_{kj}^l = r_{ij}^l + 0.5 \\ r_{ik}^r + r_{kj}^r = r_{ij}^r + 0.5 \end{cases}, i < k < j, i, j, k \in N \tag{7}$$

is true. Additionally, it can be represented as $\bar{r}_{ik} \oplus \bar{r}_{kj} = \bar{r}_{ij} \oplus [0.5, 0.5]$.

2.2.3. Incomplete fuzzy preference relations

Definition 13. [10] Let $R = (r_{ij})_{n\times n}$ be a FPR. If at least an unknown preference value r_{ij} exists in R, then R is called an incomplete FPR. The incomplete FPR $R = (r_{ij})_{n\times n}$ can be completed based on the additive consistency if $n-1$ nonleading diagonal preference values are known.

3. FPRs BASED ON UNCERTAINTY THEORY

3.1. Uncertain Preference Relations

Definition 14. Let $\overline{R} = (\mathcal{L}(r_{ij}^l, r_{ij}^r))_{n\times n}$ be a non-negative matrix, where $\bar{r}_{ij}$ is an uncertain variable. The UD of $\bar{r}_{ij}$ is Φ_{ij}, and the IUD is Φ_{ij}^{-1}. $\overline{R}$ is an UPR if it satisfies

$$\Phi_{ij}^{-1}(\beta) + \Phi_{ji}^{-1}(1-\beta) = 1 \tag{8}$$

$$\Phi_{ii}^{-1}(\beta) = 0.5 \tag{9}$$

for any β in $[0,1]$, $i, j \in N$.

Eq. (8) is the additive reciprocity of $\overline{R}$. The judgment element $\overline{r}_{ij}$ in an UPR indicates the degree to which the alternative x_i is superior to x_j. $\Phi_{ij}^{-1}(\beta) = 0.5$ indicates that there is no difference between x_i and x_j; $\Phi_{ij}^{-1}(\beta) > 0.5$ indicates that x_i is superior to x_j; $\Phi_{ij}^{-1}(\beta) < 0.5$ depicts that x_j is better than x_i.

For example, if $\overline{r}_{ij}$ obeys the LUD, $\overline{r}_{ij} \sim \mathcal{L}(r_{ij}^l, r_{ij}^r)$, $\overline{r}_{ji} \sim \mathcal{L}(1 - r_{ij}^r, 1 - r_{ij}^l)$, then $\overline{R} = (\overline{r}_{ij})_{n\times n}$ is called a linear uncertainty preference relation.

According to the additive reciprocity, $(1-\beta)r_{ij}^l + \beta r_{ij}^r + \beta(1-r_{ij}^r) + (1-\beta)(1-r_{ij}^l) = 1$ (Figure 4). When $\beta = 0$, $r_{ij}^l + (1-r_{ij}^l) = 1$. When $\beta = 1$, $r_{ij}^r + (1-r_{ij}^r) = 1$. These are the definitions of additive reciprocity of IFPRs. That is, when β moves in $[0, 1]$, $\Phi_{ij}^{-1}(\beta)$ corresponds to each value in the interval.

Definition 15. Let $\overline{R} = \left(\overline{r}_{ij}\right)_{n\times n}$ be an UPR. $\overline{R}$ is additively consistent if it satisfies

$$\Phi_{ik}^{-1}(\beta) + \Phi_{kj}^{-1}(\beta) = \Phi_{ij}^{-1}(\beta) + 0.5 \tag{10}$$

for any β in $[0, 1]$, $i < k < j$, $i, j, k \in N$.

3.2. Uncertain Chance Constrained Programming Model

Let $\overline{R}^t = (\overline{r}_{ijt})_{n\times n}$ be the incomplete preference relation given by the DM d_t $(t \in M)$. The DMs are independent. $\overline{r}_{ijt}$ is an uncertain variable and its UD is Φ_{ijt}. We have $\Phi_{ijt}^{-1}(\beta) + \Phi_{jit}^{-1}(1-\beta) = 1$, $\Phi_{iit}^{-1}(\beta) = 0.5$ for any β $(\beta \in [0, 1])$.

In $\overline{R}^t$, let $\overline{r}_{ijt}$ be the unknown value and $\overline{r}'_{ijt}$ be the ideal value of $\overline{r}_{ijt}$ obtained from Eq. (10). There are $(n-2)$ equations calculating $\overline{r}'_{ijt}$ in the upper triangular matrix without considering equations composed of diagonal elements and incomplete values. $\overline{r}'_{ijt}$ is determined using the mean value, and the incomplete values in the lower triangular matrix can be obtained from the additive reciprocity.

For higher accuracy, the deviation between the ideal value $\overline{r}'_{ijt}$ and the estimated value $\overline{r}_{ijt}$ should be as small as possible. An UCCPM is established as follows:

$$\min \sum_{i<j} \varepsilon_{ijt} \qquad (M-1)$$

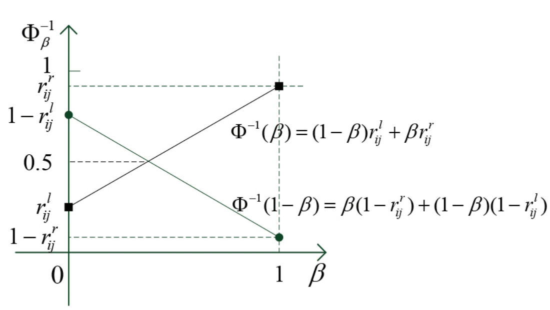

Figure 4 | Additive reciprocity of inverse LUD.

$$\text{s.t.}\begin{cases} M\left\{\left|\overline{r}_{ijt} - \overline{r}'_{ijt}\right| \leqslant \varepsilon_{ijt}\right\} \geqslant \alpha & (M-1-1) \\ \overline{r}'_{ijt} = \dfrac{1}{n-2} \displaystyle\sum_{k=1, k\neq i,j}^{n} \left(\overline{r}_{ikt} + \overline{r}_{kjt} - 0.5\right) & (M-1-2) \\ \overline{r}_{ijt} \sim \mathcal{L}(0, 1) & (M-1-3) \\ \varepsilon_{ijt} \geqslant 0 & (M-1-4) \\ i < j, i, j \in N, t \in M & (M-1-5) \end{cases}$$

In the model, ε_{ijt} represents the deviation between the ideal value and the estimated value; α represents the belief degree and $\alpha \in [0, 1]$; $(M-1-1)$ signifies that when the absolute value of deviation between the estimated value and ideal value is less than or equal to ε_{ijt} and its uncertain measure is no less than the belief degree α, the estimated value is useful and credible; $(M-1-2)$ indicates that the ideal value is the average of all possible values except for the equations comprising diagonal elements, and incomplete values; $(M-1-3)$, $(M-1-4)$ and $(M-1-5)$ are range constraints.

Additionally, $M\left\{\left|\overline{r}_{ijt} - \overline{r}'_{ijt}\right| \leqslant \varepsilon_{ijt}\right\} \geqslant \alpha \Leftrightarrow \begin{cases} M\left\{\overline{r}_{ijt} - \overline{r}'_{ijt} \leqslant \varepsilon_{ijt}\right\} \geqslant \alpha, \overline{r}_{ijt} \geqslant \overline{r}'_{ijt} \\ M\left\{\overline{r}_{ijt} - \overline{r}'_{ijt} \geqslant -\varepsilon_{ijt}\right\} \geqslant \alpha, \overline{r}_{ijt} < \overline{r}'_{ijt} \end{cases}$. Let Υ be the joint distribution of $\overline{r}_{ijt} - \overline{r}'_{ijt}$, Φ_{ijt} be the distribution of $\overline{r}_{ijt}$, and Ψ_{ijt} be the distribution of $\overline{r}'_{ijt}$. Based on Eq. (10), we have $\Psi_{ijt}^{-1}(\beta) = \Phi_{ikt}^{-1}(\beta) + \Phi_{kjt}^{-1}(\beta) - 0.5$, $\beta \in [0, 1]$, $i < k < j$.

- $\overline{r}_{ijt} \leqslant \overline{r}'_{ijt}$,

$$\begin{aligned} & M\left\{\overline{r}_{ijt} - \overline{r}'_{ijt} \leqslant \varepsilon_{ijt}\right\} \geqslant \alpha \\ & \Rightarrow \Upsilon\left(\varepsilon_{ijt}\right) \geqslant \alpha \\ & \Rightarrow \varepsilon_{ijt} \geqslant \Upsilon^{-1}(\alpha) \\ & \Rightarrow \varepsilon_{ijt} \geqslant \Phi_{ijt}^{-1}(\alpha) - \Psi_{ijt}^{-1}(1-\alpha) \\ & \Rightarrow \varepsilon_{ijt} \geqslant \Phi_{ijt}^{-1}(\alpha) - \frac{\sum_{k=1, k\neq i,j}^{n} \left[\Phi_{ikt}^{-1}(1-\alpha) + \Phi_{kjt}^{-1}(1-\alpha) - 0.5\right]}{\text{n} - 2} \end{aligned}$$

- $\overline{r}_{ijt} \leqslant \overline{r}'_{ijt}$,

$$\begin{aligned} & M\left\{\overline{r}_{ijt} - \overline{r}'_{ijt} \geqslant -\varepsilon_{ijt}\right\} \geqslant \alpha \\ & \Rightarrow 1 - M\left\{\overline{r}_{ijt} - \overline{r}'_{ijt} \leqslant -\varepsilon_{ijt}\right\} \geqslant \alpha \\ & \Rightarrow 1 - \Upsilon\left(-\varepsilon_{ijt}\right) \geqslant \alpha \\ & \Rightarrow \Upsilon\left(-\varepsilon_{ijt}\right) \leqslant 1 - \alpha \\ & \Rightarrow -\varepsilon_{ijt} \leqslant \Upsilon^{-1}(1-\alpha) \\ & \Rightarrow -\varepsilon_{ijt} \leqslant \Phi_{ijt}^{-1}(1-\alpha) - \Psi_{ijt}^{-1}(\alpha) \\ & \Rightarrow -\varepsilon_{ijt} \leqslant \Phi_{ijt}^{-1}(1-\alpha) - \frac{\sum_{k=1, k\neq i,j}^{n} \left[\Phi_{ikt}^{-1}(\alpha) + \Phi_{kjt}^{-1}(\alpha) - 0.5\right]}{\text{n} - 2} \end{aligned}$$

Therefore, the equivalent model of $(M-1)$ represented by an inverse distribution is as follows:

$$\min \sum_{i<j,i,j=N} \varepsilon_{ijt} \qquad (M-2)$$

$$\text{s.t.}\begin{cases} \varepsilon_{ijt} \geqslant \Phi_{ijt}^{-1}(\alpha) - \dfrac{\sum_{k=1,k\neq i,j}^{n}\left[\Phi_{ikt}^{-1}(1-\alpha)+\Phi_{kjt}^{-1}(1-\alpha)-0.5\right]}{n-2} & (M-2-1)\\ -\varepsilon_{ijt} \leqslant \Phi_{ijt}^{-1}(1-\alpha) - \dfrac{\sum_{k=1,k\neq i,j}^{n}\left[\Phi_{ikt}^{-1}(\alpha)+\Phi_{kjt}^{-1}(\alpha)-0.5\right]}{n-2} & (M-2-2)\\ \Phi_{ijt}^{-1}(\theta)+\Phi_{jit}^{-1}(1-\theta)=1, \theta\in[0,1] & (M-2-3)\\ \varepsilon_{ijt}\geqslant 0, i<j, i,j\in N, t\in M & (M-2-4) \end{cases}$$

Let $\bar{r}_{ijt}$ be a LUV, $\bar{r}_{ijt} \sim \mathcal{L}(\bar{r}_{ijt}^{l}, \bar{r}_{ijt}^{r})$. $(M-2-1)$ equals to $\varepsilon_{ijt} \geqslant (1-\alpha)\bar{r}_{ijt}^{l} + \alpha\bar{r}_{ijt}^{r} - \frac{1}{n-2}\sum_{k=1,k\neq i,j}^{n}[\alpha\bar{r}_{ikt}^{l} + (1-\alpha)\bar{r}_{ikt}^{r} + \alpha\bar{r}_{kjt}^{l} + (1-\alpha)\bar{r}_{kjt}^{r} - 0.5]$, $(M-2-2)$ equals to $-\varepsilon_{ijt} \leqslant \alpha\bar{r}_{ijt}^{l} + (1-\alpha)\bar{r}_{ijt}^{r} - \frac{1}{n-2}\sum_{k=1,k\neq i,j}^{n}\left[(1-\alpha)\bar{r}_{ikt}^{l} + \alpha\bar{r}_{ikt}^{r} + (1-\alpha)\bar{r}_{kjt}^{l} + \alpha\bar{r}_{kjt}^{r} - 0.5\right]$.

Therefore, the equivalent model of $(M-2)$ is

$$\min \sum_{i<j,i,j=N} \varepsilon_{ijt} \qquad (M-3)$$

$$\text{s.t.}\begin{cases} \varepsilon_{ijt} \geqslant (1-\alpha)\bar{r}_{ijt}^{l} + \alpha\bar{r}_{ijt}^{r} - \dfrac{\sum_{k=1,k\neq i,j}^{n}\left[\alpha\bar{r}_{ikt}^{l}+(1-\alpha)\bar{r}_{ikt}^{r}+\alpha\bar{r}_{kjt}^{l}+(1-\alpha)\bar{r}_{kjt}^{r}-0.5\right]}{n-2}\\ -\varepsilon_{ijt} \leqslant \alpha\bar{r}_{ijt}^{l} + (1-\alpha)\bar{r}_{ijt}^{r} - \dfrac{\sum_{k=1,k\neq i,j}^{n}\left[(1-\alpha)\bar{r}_{ikt}^{l}+\alpha\bar{r}_{ikt}^{r}+(1-\alpha)\bar{r}_{kjt}^{l}+\alpha\bar{r}_{kjt}^{r}-0.5\right]}{n-2}\\ (1-\beta)\bar{r}_{ijt}^{l}+\beta\bar{r}_{ijt}^{r}+\beta\bar{r}_{jit}^{l}+(1-\beta)\bar{r}_{jit}^{r}=1\\ \beta\in[0,1], \varepsilon_{ijt}\geqslant 0, i<j, i,j\in N, t\in M \end{cases}$$

Theorem 2. *The linear equivalent model of UCCPM $(M-2)$ is $(M-3)$.*

3.3. Consistency Analysis of UPRs

3.3.1. Individual additive consistency analysis

Let $\overline{R}'^{t} = (\bar{r}'_{ijt})_{n\times n}$ be the associated additively consistent UPR of $\overline{R}^{t}$. $\bar{r}'_{ijt}$ is an uncertain variable and its UD is Φ'_{ijt}, IUD is Φ'^{-1}_{ijt}. When the completed UPR $\overline{R}^{t}$ is equal to $\overline{R}'^{t}$, then $\overline{R}^{t}$ is additively consistent. When $i<j$, the deviations of the upper triangular matrix and lower triangular matrix between $\overline{R}^{t}$ and $\overline{R}'^{t}$ are respectively defined as follows:

$$d\left(\bar{r}_{ijt}, \bar{r}'_{ijt}\right) = \left|\Phi_{ijt}^{-1}(\beta) - \Phi'^{-1}_{ijt}(1-\beta)\right| \tag{11}$$

$$d\left(\bar{r}_{jit}, \bar{r}'_{jit}\right) = \left|\Phi'^{-1}_{jit}(\beta) - \Phi_{jit}^{-1}(1-\beta)\right| \tag{12}$$

The smaller the value of d, the closer is $\overline{R}^{t}$ to $\overline{R}'^{t}$. Additionally, the consistency level of $\overline{R}^{t}$ is higher. Considering that

$$\begin{aligned} d\left(\bar{r}_{jit}, \bar{r}'_{jit}\right) &= \left|\Phi'^{-1}_{jit}(\beta) - \Phi_{jit}^{-1}(1-\beta)\right| \\ &= \left|1-\Phi'^{-1}_{ijt}(1-\beta) - \left[1-\Phi_{ijt}^{-1}(\beta)\right]\right| \\ &= \left|-\Phi'^{-1}_{ijt}(1-\beta) + \Phi_{ijt}^{-1}(\beta)\right| \\ &= d\left(\bar{r}_{ijt}, \bar{r}'_{ijt}\right) \end{aligned}$$

the deviations of the upper triangular matrix and lower triangular matrix can be expressed in the same formula; thus, we will no longer calculate the deviation of the lower triangular matrix separately.

Definition 16. [16] Let $\overline{R}'^{t} = \left(\bar{r}'_{ijt}\right)_{n\times n}$ be the associated additively consistent UPR of $\overline{R}^{t}$. The additive consistency index (ACI) of $\overline{R}^{t}$ is defined as follows:

$$ACI\left(\overline{R}^{t}\right) = 1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n} d\left(\bar{r}_{ijt}, \bar{r}'_{ijt}\right) \tag{13}$$

The larger the $ACI\left(\overline{R}^{t}\right)$, the higher the consistency level of $\overline{R}^{t}$. If and only if $d\left(\bar{r}_{ijt}, \bar{r}'_{ijt}\right) = 0$, $ACI(\overline{R}^{t}) = 1$ and $\overline{R}^{t}$ is fully additively consistent.

3.3.2. Adjustment of inconsistent UPRs

Definition 17. Let $\overline{R}'^{t} = \left(\bar{r}'_{ijt}\right)_{n\times n}$ be the associated additively consistent UPR of $\overline{R}^{t}$. Suppose the UD of $\tilde{\overline{R}}^{t} = (\tilde{r}_{ijt})_{n\times n}$ is $\tilde{\Phi}_{ijt}$, $\theta\in(0,1)$. $\tilde{\overline{R}}^{t}$ is an improved UPR, if it satisfies

$$\tilde{\Phi}_{ijt}^{-1}(\beta) = \theta\Phi_{ijt}^{-1}(\beta) + (1-\theta)\Phi'^{-1}_{ijt}(1-\beta) \tag{14}$$

Theorem 3. *Let $\tilde{\overline{R}}^{t}$ be an improved UPR and $\overline{R}^{t}$ be a complete UPR. Therefore, we can derive that $ACI(\tilde{\overline{R}}^{t}) > ACI(\overline{R}^{t})$.*

Proof.

$$
\begin{aligned}
& d\left(\tilde{\bar{r}}_{ijt}, \bar{r}'_{ijt}\right) \\
& = \left|\tilde{\Phi}^{-1}_{ijt}(\beta) - \Phi'^{-1}_{ijt}(1-\beta)\right| \\
& = \left|\theta\Phi^{-1}_{ijt}(\beta) + (1-\theta)\Phi'^{-1}_{ijt}(1-\beta) - \Phi'^{-1}_{ijt}(1-\beta)\right| \\
& = \left|\theta\left[\Phi^{-1}_{ijt}(\beta) - \Phi'^{-1}_{ijt}(1-\beta)\right]\right| \\
& = \theta\left|\left[\Phi^{-1}_{ijt}(\beta) - \Phi'^{-1}_{ijt}(1-\beta)\right]\right| \\
& = \theta d\left(\bar{r}_{ijt}, \bar{r}'_{ijt}\right) \\
& < d\left(\bar{r}_{ijt}, \bar{r}'_{ijt}\right)
\end{aligned}
$$

Thus, $ACI\left(\tilde{\bar{R}}^{t}\right) > ACI\left(\bar{R}^{t}\right)$.

Corollary 1. Let $\tilde{\bar{R}}^{t(h)} = \left(\tilde{\bar{r}}^{(h)}_{ijt}\right)_{n\times n}$ $(h \in Z)$ be the hth improved UPR. After an adjustment, the consistency index of the UPR is better than that of the previous one.

Proof. Let $\tilde{\Phi}^{(h)}_{ijt}$ be the UD of $\tilde{\bar{R}}^{t(h)}$ and its IUD is $\tilde{\Phi}^{(h)-1}_{ijt}$. Let $\bar{R}'^{t(h-1)} = \left(\bar{r}'^{(h-1)}_{ijt}\right)_{n\times n}$ be the additively consistent UPR with UD $\Phi'^{(h-1)}_{ijt}$ and IUD $\Phi'^{(h-1)-1}_{ijt}$.

$$
\begin{aligned}
& d\left(\tilde{\bar{r}}^{(h)}_{ijt}, \bar{r}'^{(h-1)}_{ijt}\right) \\
& = \left|\tilde{\Phi}^{(h)-1}_{ijt}(\beta) - \Phi'^{(h-1)-1}_{ijt}(1-\beta)\right| \\
& = \theta\left|\left[\tilde{\Phi}^{(h-1)-1}_{ijt}(\beta) - \Phi'^{(h-1)-1}_{ijt}(1-\beta)\right]\right| \\
& = \theta d\left(\tilde{\bar{r}}^{(h-1)}_{ijt}, \bar{r}'^{(h-1)}_{ijt}\right)
\end{aligned}
$$

Since $\theta \in (0,1)$, $d\left(\tilde{\bar{r}}^{(h-1)}_{ijt}, \bar{r}'^{(h-1)}_{ijt}\right)$ are known constants, the deviation after each adjustment is smaller than that of the previous one.

In Corollary 1, after h_0 number of iterations, a certain threshold Z_0 is obtained. When $d\left(\tilde{\bar{r}}^{(h_0)}_{ijt}, \bar{r}'^{(h_0-1)}_{ijt}\right) < Z_0$, we consider the additive consistency of $\tilde{\bar{R}}^{t(h_0)}$ to be acceptable.

3.3.3. Consensus analysis

To improve the level of consensus in GDM, all individual FPRs are typically aggregated to obtain a collective FPR. The latter is used to obtain the individual preference relation which deviates significantly from the consensus [4,52,53]. Combining the uncertainty theory and the aggregation operator, a consensus index which measures the deviation between an individual UPR and a collective UPR is introduced.

Let $\bar{R}^{c} = (\bar{r}_{ijc})_{n\times n}$ be the collective UPR of all individual DMs. $\bar{r}_{ijc}$ is an uncertain variable with an UD Φ_{ijc}. For any β $(\beta \in [0,1])$, we have $\Phi^{-1}_{ijc}(\beta) + \Phi^{-1}_{jic}(1-\beta) = 1$, $\Phi^{-1}_{iic}(\beta) = 0.5$.

Definition 18. [16] Let $\bar{R}^{t} = \left(\bar{r}_{ijt}\right)_{n\times n}$ be the UPR of d_t, and its IUD is Φ^{-1}_{ijt}. The consensus index of $\bar{R}^{t}$ is defined as follows:

$$
COI\left(\bar{R}^{t}, \bar{R}^{c}\right) = 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} d\left(\bar{r}_{ijt}, \bar{r}_{ijc}\right) \tag{15}
$$

The smaller the $d\left(\bar{r}_{ijt}, \bar{r}_{ijc}\right)$, the larger the consensus index of $\bar{R}^{t}$. If and only if $d\left(\bar{r}_{ijt}, \bar{r}_{ijc}\right) = 0$, $COI\left(\bar{R}^{t}, \bar{R}^{c}\right) = 1$, and the individuals reach full consensus.

Since $d(\bar{r}_{ijt}, \bar{r}_{ijc}) = |\Phi^{-1}_{ijt}(\beta) - \Phi^{-1}_{ijc}(1-\beta)|$, we have

$$
\begin{aligned}
d\left(\bar{r}_{jit}, \bar{r}_{jic}\right) & = \left|\Phi^{-1}_{jit}(\beta) - \Phi^{-1}_{jic}(1-\beta)\right| \\
& = \left|1 - \Phi^{-1}_{ijt}(1-\beta) - \left[1 - \Phi^{-1}_{ijc}(\beta)\right]\right| \\
& = \left|-\Phi^{-1}_{ijt}(1-\beta) + \Phi^{-1}_{ijc}(\beta)\right| \\
& = \left|\Phi^{-1}_{ijt}(\beta) - \Phi^{-1}_{ijc}(1-\beta)\right| \\
& = d\left(\bar{r}_{ijt}, \bar{r}_{ijc}\right)
\end{aligned}
$$

with $i < j$. The deviations of the upper triangular matrix and lower triangular matrix between $\bar{R}^{t}$ and $\bar{R}^{c}$ can be expressed in the same formula.

- IHWA Operator

Meng and Chen [16] propose an IHWA operator to calculate the elements of a collective FPR based on the importance of DMs (or criteria) and ordered positions. Extending the operator to interval UPRs, we propose the following definition of the collective UPR.

Definition 19. [16] An IHWA operator with dimension n is a mapping IHWA: Let $R^n \rightarrow R$ be defined on the set of the second arguments of two tuples $\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle$ with a set of order-inducing variables $\{u_i\}$ $(i \in N)$, denoted by

$$
\begin{aligned}
& IHWA_{\lambda,v}\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle\right) \\
& = \sum_{j=1}^{n} \frac{v_j \lambda_{(j)}}{\sum_{j=1}^{n} v_j \lambda_{(j)}} a_{(j)}
\end{aligned} \tag{16}
$$

where (.) is a permutation on u_i such that $u_{(j)}$ is the jth smallest value of u_i, $v = (v_1\ v_2\ \dots\ v_n)^T$ is the weight vector on the ordered set O = $\{1, 2, \dots, n\}$, and $\lambda = (\lambda_1\ \lambda_2\ \dots\ \lambda_n)^T$ is the weight vector on object set A = $\{a_1, a_2, \dots, a_n\}$.

Definition 20. Let $\bar{R}^{t} = \left(\bar{r}_{ijt}\right)_{n\times n}$ be the UPR of DM d_t, $v = (v_1\ v_2\ \dots\ v_n)^T$ and $\lambda = (\lambda_1\ \lambda_2\ \dots\ \lambda_m)^T$ be the respective weight vectors on the ordered set O and on the DM set D = $\{d_1, d_2, \dots, d_m\}$. Then, the collective UPR $\bar{R}^{c} = \left(\bar{r}_{ijc}\right)_{n\times n}$ is defined by

$$
\begin{aligned}
& IHWA\left(\left\langle ACI\left(\bar{R}^{1}\right), \Phi^{-1}_{ij1}\right\rangle, \left\langle ACI\left(\bar{R}^{2}\right), \Phi^{-1}_{ij2}\right\rangle, \dots, \right. \\
& \left. \left\langle ACI\left(\bar{R}^{m}\right), \Phi^{-1}_{ijm}\right\rangle\right) \\
& = \sum_{k=1}^{m} \frac{v_k \lambda_{(k)}}{\sum_{k=1}^{m} v_k \lambda_{(k)}} \Phi^{-1}_{ij(k)} = \Phi^{-1}_{ijc}
\end{aligned} \tag{17}
$$

Φ_{ijt}^{-1} represents the IUD of $\overline{R}^t$, and Φ_{ijc}^{-1} represents the IUD of $\overline{R}^c$. u_i is represented as the ACI. (.) is a permutation on $ACI\left(\overline{R}^k\right)$ $(k \in M)$ such that $ACI\left(\overline{R}^{(k)}\right)$ is the kth smallest value of $ACI\left(\overline{R}^k\right)$.

Theorem 4. *Let $\overline{R}^t = \left(\bar{r}_{ijt}\right)_{n\times n}$ be the UPR of DM d_t, and $\overline{R}^c = (\bar{r}_{ijc})_{n\times n}$ be the collective UPR; then, the consistency index of the collective UPR is no less than the minimum value of any individual consistency index. That is, $ACI\left(\overline{R}^c\right) \geqslant \min_{1\leqslant t\leqslant m} ACI\left(\overline{R}^t\right)$.*

Proof. From Eq. (13), we have

$$
\begin{aligned}
&ACI\left(\overline{R}^c\right)\\
&= 1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n} d\left(\bar{r}_{ijc}, \bar{r}'_{ijc}\right)\\
&= 1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n} \left|\Phi_{ijc}^{-1}(\beta) - \Phi'^{-1}_{ijc}(1-\beta)\right|\\
&= 1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n} \left|\Phi_{ijc}^{-1}(\beta) - \sum_{p=1,p\neq i,j}^{n}\left[\frac{\Phi_{ipc}^{-1}(1-\beta)+\Phi_{pjc}^{-1}(1-\beta)}{n-2}\right] + 0.5\right|\\
&= 1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n} \left|\sum_{k=1}^{m}\frac{v_k\lambda_{(k)}}{\sum_{k=1}^{m} v_k\lambda_{(k)}}\Phi_{ij(k)}^{-1}(\beta) - \frac{1}{n-2}\sum_{k=1}^{m}\frac{v_k\lambda_{(k)}}{\sum_{k=1}^{m} v_k\lambda_{(k)}}\sum_{p=1,p\neq i,j}^{n}\left[\Phi_{ip(k)}^{-1}(1-\beta)+\Phi_{pj(k)}^{-1}(1-\beta)\right] + 0.5\right|\\
&= 1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n}\sum_{k=1}^{m}\frac{v_k\lambda_{(k)}}{\sum_{k=1}^{m} v_k\lambda_{(k)}}\left|\Phi_{ij(k)}^{-1}(\beta) - \frac{1}{n-2}\sum_{p=1,p\neq i,j}^{n}\left[\Phi_{ip(k)}^{-1}(1-\beta)+\Phi_{pj(k)}^{-1}(1-\beta)\right] + 0.5\right|\\
&= \sum_{k=1}^{m}\frac{v_k\lambda_{(k)}}{\sum_{k=1}^{m} v_k\lambda_{(k)}}\left\{1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n}\left|\Phi_{ij(k)}^{-1}(\beta) - \frac{1}{n-2}\sum_{p=1,p\neq i,j}^{n}\left[\Phi_{ip(k)}^{-1}(1-\beta)+\Phi_{pj(k)}^{-1}(1-\beta)\right] + 0.5\right|\right\}\\
&\geqslant \min_{1\leqslant k\leqslant m}\left\{1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n}\left|\Phi_{ij(k)}^{-1}(\beta) - \frac{\sum_{p=1,p\neq i,j}^{n}\left[\Phi_{ip(k)}^{-1}(1-\beta)+\Phi_{pj(k)}^{-1}(1-\beta)\right]}{n-2} + 0.5\right|\right\}\\
&= \min_{1\leqslant k\leqslant m}\left\{1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n}\left|\Phi_{ijk}^{-1}(\beta) - \frac{\sum_{p=1,p\neq i,j}^{n}\left[\Phi_{ipk}^{-1}(1-\beta)+\Phi_{pjk}^{-1}(1-\beta)\right]}{n-2} + 0.5\right|\right\}\\
&= \min_{1\leqslant k\leqslant m}\left\{1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n}\left|\Phi_{ijk}^{-1}(\beta) - \Phi'^{-1}_{ijk}(1-\beta)\right|\right\}\\
&= \min_{1\leqslant k\leqslant m}\left\{1 - \frac{1}{n(n-1)}\sum_{i,j=1}^{n} d\left(\bar{r}_{ijk}, \bar{r}'_{ijk}\right)\right\}\\
&= \min_{1\leqslant k\leqslant m} ACI\left(\overline{R}^k\right)
\end{aligned}
$$

- Weighted Averaging Consensus Index $\overline{COI}\left(\overline{R}^c\right)$

Definition 21. [16] Let $\overline{R}^t = \left(\bar{r}_{ijt}\right)_{n\times n}$ be the UPR of DM d_t, $\lambda = (\lambda_1\ \lambda_2\ \ldots\ \lambda_m)^T$ be the weight vector of DM set $D = \{d_1, d_2, \ldots, d_m\}$, $\overline{R}^c = \left(\bar{r}_{ijc}\right)_{n\times n}$ be the collective UPR. Then, the weighted averaging consensus index is defined by

$$\overline{COI}\left(\overline{R}^c\right) = \sum_{t=1}^{m}\lambda_t COI\left(\overline{R}^t, \overline{R}^c\right) \tag{18}$$

Theorem 5. *Let ω be the consensus threshold value, $\tilde{\overline{R}}^{t(h,g+1)} = \left(\tilde{\bar{r}}_{ijt}^{(h,g+1)}\right)_{n\times n}$ $(g \in Z)$ be the $(g+1)$th improved individual UPR for $COI\left(\tilde{\overline{R}}^{t(h,g)}, \overline{R}^{c(h,g)}\right) < \omega$, $\tilde{\Phi}_{ijt}^{(h,g+1)-1}$ be the IUD of $\tilde{\overline{R}}^{t(h,g+1)}$, and $\Phi_{ijc}^{(h,g)-1}$ be the IUD of $\overline{R}^{c(h,g)}$. When $\tilde{\Phi}_{ijt}^{(h,g+1)-1}(\beta) = \theta\tilde{\Phi}_{ijt}^{(h,g)-1}(\beta) + (1-\theta)\Phi_{ijc}^{(h,g)-1}(1-\beta)$, $\theta \in (0,1)$, we have $COI\left(\tilde{\overline{R}}^{t(h,g+1)}, \overline{R}^{c(h,g)}\right) > COI\left(\tilde{\overline{R}}^{t(h,g)}, \overline{R}^{c(h,g)}\right)$.*

Proof. Based on Eq. (15), we have

$$
\begin{aligned}
& d\left(\tilde{\bar{r}}_{ijt}^{(h,g+1)}, \bar{r}_{ijc}^{(h,g)}\right) \\
&= \left|\tilde{\Phi}_{ijt}^{(h,g+1)-1}(\beta) - \Phi_{ijc}^{(h,g)-1}(1-\beta)\right| \\
&= \left|\theta\tilde{\Phi}_{ijt}^{(h,g)-1}(\beta) + (1-\theta)\,\Phi_{ijc}^{(h,g)-1}(1-\beta) - \Phi_{ijc}^{(h,g)-1}(1-\beta)\right| \\
&= \theta\left|\tilde{\Phi}_{ijt}^{(h,g)-1}(\beta) - \Phi_{ijc}^{(h,g)-1}(1-\beta)\right| \\
&= \theta d\left(\tilde{\bar{r}}_{ijt}^{(h,g)}, \bar{r}_{ijc}^{(h,g)}\right)
\end{aligned}
$$

Thus, the deviation after each adjustment is smaller than that of the previous one, namely, the consensus index of the UPR is better than that of the previous one.

- An Algorithm for GDM

Step 1. Determine the preference relations and weight vectors.

Let $\overline{R}^{t} = (\bar{r}_{ijt})_{n\times n}$ be an independent UPR of the DM d_t, $\nu = (\nu_1\ \nu_2\ \dots\ \nu_n)^T$ and $\lambda = (\lambda_1\ \lambda_2\ \dots\ \lambda_m)^T$ be the weight vectors on the ordered set O and on the DM set $\mathrm{D} = \{d_1, d_2, \dots, d_m\}$, respectively.

Step 2. Complete the incomplete UPRs.

Use UCCPM $(M-3)$ to compute the incomplete values in individual preference relation $\overline{R}^{t}$. Additionally, use $\overline{R}^{t}$ to represent the completed individual preference relation.

Step 3. Calculate the individual consistency index.

Let $\overset{\simeq}{R}^{t(h)} = \left(\tilde{\bar{r}}_{ijt}^{(h)}\right)_{n\times n}$ be the hth improved individual preference relation, $h = 0$, $\overset{\simeq}{R}^{t(h)} = \overline{R}^{t}$. The individual ACI threshold value is λ $(\lambda \in [0,1])$. If $ACI\left(\overset{\simeq}{R}^{t(h)}\right) \geqslant \lambda$, then proceed to Step 5; otherwise, proceed to the next step.

Step 4. Adjust inconsistent individual UPRs.

Let $\tilde{\Phi}_{ijt}^{(h)-1}$ be the IUD of $\overset{\simeq}{R}^{t(h)}$. If $ACI\left(\overset{\simeq}{R}^{t(h)}\right) < \lambda$, let $\tilde{\Phi}_{ijt}^{(h+1)-1}(\beta) = \theta\tilde{\Phi}_{ijt}^{(h)-1}(\beta) + (1-\theta)\,\Phi_{ijt}^{\prime(h)-1}(1-\beta)$, where $\Phi_{ijt}^{\prime -1}(\beta) = \sum_{k=1,k\neq i,j}^{n}\left[\frac{\Phi_{ikt}^{-1}(\beta) + \Phi_{kjt}^{-1}(\beta)}{n-2}\right] - 0.5$, $\theta \in (0,1)$. Let $h = h+1$ and return to Step 3.

Step 5. Calculate the individual consensus index.

Let ω be the weighted averaging consensus index threshold value, g be the number of iterations, and $\overset{\simeq}{R}^{t(h,g)} = \left(\tilde{\bar{r}}_{ijt}^{(h,g)}\right)_{n\times n}$ be the gth improved individual UPR for $COI\left(\overset{\simeq}{R}^{t(h,g-1)}, \overline{R}^{c(h,g-1)}\right) < \omega$. When $g = 1$, we have $\overset{\simeq}{R}^{t(h,g-1)} = \overset{\simeq}{R}^{t(h)}$. Use the IHWA operator to calculate the collective UPR $\overline{R}^{c(h,g)}$. Use Eq. (15) to calculate the individual consensus index $COI\left(\overset{\simeq}{R}^{t(h,g)}, \overline{R}^{c(h,g)}\right)$.

Step 6. Calculate the weighted averaging consensus index.

Use Eq. (18) to calculate the weighted averaging consensus index. If $\overline{COI}\left(\overline{R}^{c(h,g)}\right) \geqslant \omega$, proceed to Step 8; otherwise, proceed to Step 7.

Step 7. Adjust the individual UPRs.

Let $\tilde{\Phi}_{ijt}^{(h,g)-1}$ be the IUD of $\overset{\simeq}{R}^{t(h,g)}$ and $\Phi_{ijc}^{(h,g)-1}$ be the IUD of $\overline{R}^{c(h,g)}$. If $COI\left(\overset{\simeq}{R}^{t(h,g)}, \overline{R}^{c(h,g)}\right) < \omega$, let $\tilde{\Phi}_{ijt}^{(h,g+1)-1}(\beta) = \theta\tilde{\Phi}_{ijt}^{(h,g)-1}(\beta) + (1-\theta)\,\Phi_{ijc}^{(h,g)-1}(1-\beta)$. Let $g = g+1$ and return to Step 5.

Step 8. Calculate the consistency index of the collective UPR.

Calculate the ultimate collective UPR $\overline{R}^{c(h,g)}$, still denoted as $\overline{R}^{c}$. Let ϑ be the consistency index threshold value of $\overline{R}^{c}$, s be the number of iterations, and $\overline{R}^{c(s)} = \left(\bar{r}_{ijc}^{(s)}\right)_{n\times n}$ be the sth collective UPR improved for $ACI\left(\overline{R}^{c(s-1)}\right) < \vartheta$. When $s = 0$, $\overline{R}^{c(s)} = \overline{R}^{c}$. If $ACI(\overline{R}^{c(s)}) \geqslant \vartheta$, then $\overline{R}^{c(s)}$ is additively consistent; otherwise, proceed to Step 9.

Step 9. Adjust inconsistent collective preference relation.

Let $\Phi_{ijc}^{(s)-1}$ be the IUD of $\overline{R}^{c(s)}$. If $ACI(\overline{R}^{c(s)}) < \vartheta$, let $\Phi_{ijc}^{(s+1)-1}(\beta) = \theta\Phi_{ijc}^{(s)-1}(\beta) + (1-\theta)\,\Phi_{ijc}^{\prime(s)-1}(1-\beta)$, where $\Phi_{ijc}^{\prime -1}(\beta) = \sum_{k=1,k\neq i,j}^{n}\left[\frac{\Phi_{ikc}^{-1}(\beta) + \Phi_{kjc}^{-1}(\beta)}{n-2}\right] - 0.5$, $\theta \in (0,1)$. Let $s = s+1$ and return to Step 8. In addition, we have $ACI\left(\overline{R}^{c}\right) \geqslant \min_{1\leqslant t\leqslant m} ACI\left(\overline{R}^{t}\right) \geqslant \lambda$ based on Theorem 4. That is, when $\lambda = \vartheta$, Step 9 can be omitted.

3.4. Illustrative Example

Let $\mathrm{X} = \{x_1, x_2, x_3, x_4\}$ be the set of alternatives, $\mathrm{D} = \{d_1, d_2, d_3\}$ be the set of DMs, $\overline{R}^{t} = (\bar{r}_{ijt})_{4\times 4}$ $(i,j \in N = \{1,2,3,4\})$ be the independent UPR of d_t $(t \in M = \{1,2,3\})$. $\bar{r}_{ijt}$ is a LUV, i.e., $\bar{r}_{ijt} \sim \mathcal{L}(\bar{r}_{ijt}^{l}, \bar{r}_{ijt}^{r})$, with an UD Φ_{ijt}. Let $\overline{R}^{c} = (\bar{r}_{ijc})_{4\times 4}$ be the collective UPR, $\bar{r}_{ijc} \sim \mathcal{L}(\bar{r}_{ijc}^{l}, \bar{r}_{ijc}^{r})$ be a LUV, and Φ_{ijc} be its UD function. For any β $(\beta \in [0,1])$, we have $\Phi_{ijc}^{-1}(\beta) + \Phi_{jic}^{-1}(1-\beta) = 1$, and $\Phi_{iic}^{-1}(\beta) = 0.5$.

The incomplete UPRs $\overline{R}^{1}, \overline{R}^{2}, \overline{R}^{3}$ provided separately by the three DMs are as follows:

$$
\begin{pmatrix}
\mathcal{L}(0.5,0.5) & \mathcal{L}(0.6,0.8) & \mathcal{L}\left(x_{13}^{l},0.75\right) & \mathcal{L}(0.4,0.7) \\
\mathcal{L}(0.2,0.4) & \mathcal{L}(0.5,0.5) & \mathcal{L}\left(x_{23}^{l},x_{23}^{r}\right) & \mathcal{L}(0.35,0.55) \\
\mathcal{L}\left(0.25,x_{31}^{r}\right) & \mathcal{L}\left(x_{32}^{l},x_{32}^{r}\right) & \mathcal{L}(0.5,0.5) & \mathcal{L}\left(0.3,x_{34}^{r}\right) \\
\mathcal{L}(0.3,0.6) & \mathcal{L}(0.45,0.65) & \mathcal{L}\left(x_{43}^{l},0.7\right) & \mathcal{L}(0.5,0.5)
\end{pmatrix}
$$

$$
\begin{pmatrix}
\mathcal{L}(0.5,0.5) & \mathcal{L}\left(x_{12}^{l},0.7\right) & \mathcal{L}(0.5,0.75) & \mathcal{L}\left(x_{14}^{l},x_{14}^{r}\right) \\
\mathcal{L}\left(0.3,x_{21}^{r}\right) & \mathcal{L}(0.5,0.5) & \mathcal{L}\left(0.3,x_{23}^{r}\right) & \mathcal{L}(0.55,0.8) \\
\mathcal{L}(0.25,0.5) & \mathcal{L}\left(x_{32}^{l},0.7\right) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.7,0.8) \\
\mathcal{L}\left(x_{41}^{l},x_{41}^{r}\right) & \mathcal{L}(0.2,0.45) & \mathcal{L}(0.2,0.3) & \mathcal{L}(0.5,0.5)
\end{pmatrix}
$$

$$\begin{pmatrix} \mathcal{L}(0.5,0.5) & \mathcal{L}(0.1,0.3) & \mathcal{L}(x_{13}^l,x_{13}^r) & \mathcal{L}(0.3,x_{14}^r) \\ \mathcal{L}(0.7,0.9) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.2,0.4) & \mathcal{L}(x_{24}^l,x_{24}^r) \\ \mathcal{L}(x_{31}^l,x_{31}^r) & \mathcal{L}(0.6,0.8) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.7,0.8) \\ \mathcal{L}(x_{41}^l,0.7) & \mathcal{L}(x_{42}^l,x_{42}^r) & \mathcal{L}(0.2,0.3) & \mathcal{L}(0.5,0.5) \end{pmatrix}$$

Step 1. Let the belief degree $\alpha = 0.8$. Based on $(M-3)$, the incomplete values are estimated. In $\overline{R}^1$, $x_{13}^l = 0.75, x_{23}^l = 0.58, x_{23}^r = 0.58, x_{34}^r = 0.37$. In $\overline{R}^2$, $x_{12}^l = 0.7, x_{14}^l = 0.875, x_{14}^r = 0.875, x_{23}^r = 0.34$. In $\overline{R}^3$, $x_{13}^l = 0.025, x_{13}^r = 0.025, x_{14}^r = 0.3, x_{24}^l = 0.575, x_{24}^r = 0.575$.

Step 2. Let the individual ACI threshold value λ be 0.95. Since the actual deviation is an interval with the same end points, it is treated as a real number. Based on Eq. (13), the individual ACI is calculated. $ACI\left(\overline{R}^1\right) = 0.952$, $ACI\left(\overline{R}^2\right) = 0.932$, and $ACI\left(\overline{R}^3\right) = 0.953$. Let $\theta = 0.8$ and the adjusted $ACI\left(\overline{R}^{2(1)}\right) = 0.955$, $\overline{R}^{2(1)}$ is defined as follows:

$$\begin{pmatrix} \mathcal{L}(0.5,0.5) & \mathcal{L}(0.7,0.7) & \mathcal{L}(0.52,0.71) & \mathcal{L}(0.88,0.88) \\ \mathcal{L}(0.3,0.3) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.34,0.34) & \mathcal{L}(0.57,0.76) \\ \mathcal{L}(0.29,0.48) & \mathcal{L}(0.66,0.66) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.75,0.77) \\ \mathcal{L}(0.12,0.12) & \mathcal{L}(0.24,0.43) & \mathcal{L}(0.23,0.25) & \mathcal{L}(0.5,0.5) \end{pmatrix}$$

Step 3. Let the consensus threshold value ω be 0.85. The weight vector on the set of DMs $D = \{d_1, d_2, d_3\}$ is $\lambda = (\lambda_1 \quad \lambda_2 \quad \lambda_3)^T = \left(\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}\right)^T$, and the weight vector on the ordered set $O = \{1,2,3\}$ is $v = (v_1 \quad v_2 \quad v_3)^T = \left(\frac{1}{9} \quad \frac{3}{9} \quad \frac{5}{9}\right)^T$. As $ACI\left(\overline{R}^1\right) < ACI\left(\overline{R}^3\right) < ACI\left(\overline{R}^{2(1)}\right)$, the weights of $\overline{R}^1$, $\overline{R}^{2(1)}$, and $\overline{R}^3$ are $\frac{1}{9}$, $\frac{5}{9}$, and $\frac{3}{9}$, respectively. The collective UPR $\overline{R}^c = \left(\bar{r}_{ijc}\right)_{4\times4}$ is defined as follows:

$$\begin{pmatrix} \mathcal{L}(0.5,0.5) & \mathcal{L}(0.49,0.58) & \mathcal{L}(0.38,0.49) & \mathcal{L}(0.63,0.67) \\ \mathcal{L}(0.42,0.51) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.32,0.39) & \mathcal{L}(0.55,0.68) \\ \mathcal{L}(0.51,0.62) & \mathcal{L}(0.61,0.68) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.68,0.74) \\ \mathcal{L}(0.33,0.37) & \mathcal{L}(0.33,0.45) & \mathcal{L}(0.26,0.32) & \mathcal{L}(0.5,0.5) \end{pmatrix}$$

e.g.,

$$\begin{aligned} &\Phi_{12c}^{-1}(\beta) \\ &= \sum_{k=1}^{3} \frac{v_k \lambda_{(k)}}{\sum_{k=1}^{3} v_k \lambda_{(k)}} \Phi_{ij(k)}^{-1}(\beta) \\ &= (1-\beta)\left(\frac{1}{9}\bar{r}_{121}^l + \frac{3}{9}\bar{r}_{123}^l + \frac{5}{9}\bar{r}_{122}^{(1)l}\right) \\ &\quad + \beta\left(\frac{1}{9}\bar{r}_{121}^r + \frac{3}{9}\bar{r}_{123}^r + \frac{5}{9}\bar{r}_{122}^{(1)r}\right) \\ &= 0.49(1-\beta) + 0.58\beta \end{aligned}$$

Step 4. From Eq. (15), $COI\left(\overline{R}^1, \overline{R}^c\right) = 0.767$, $COI\left(\overline{R}^{2(1)}, \overline{R}^c\right) = 0.904$, $COI\left(\overline{R}^3, \overline{R}^c\right) = 0.778$. From Eq. (18), $\overline{COI}\left(\overline{R}^c\right) = \frac{1}{3} \times 0.767 + \frac{1}{3} \times 0.904 + \frac{1}{3} \times 0.778 = 0.816 < \omega$.

Step 5. As $COI\left(\overline{R}^1, \overline{R}^c\right) < \omega$, $COI\left(\overline{R}^3, \overline{R}^c\right) < \omega$, $\overline{R}^1$, and $\overline{R}^3$ must be adjusted. The ultimate individual UPRs are as follows:

$$\overline{R}^{1(0,2)} = \begin{pmatrix} \mathcal{L}(0.5,0.5) & \mathcal{L}(0.58,0.67) & \mathcal{L}(0.61,0.61) & \mathcal{L}(0.47,0.64) \\ \mathcal{L}(0.33,0.42) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.45,0.55) \\ \mathcal{L}(0.39,0.39) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.44,0.47) \\ \mathcal{L}(0.36,0.53) & \mathcal{L}(0.45,0.55) & \mathcal{L}(0.53,0.56) & \mathcal{L}(0.5,0.5) \end{pmatrix}$$

$$\overline{R}^{2(1,1)} = \begin{pmatrix} \mathcal{L}(0.5,0.5) & \mathcal{L}(0.65,0.65) & \mathcal{L}(0.49,0.64) & \mathcal{L}(0.8,0.8) \\ \mathcal{L}(0.35,0.35) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.35,0.35) & \mathcal{L}(0.57,0.71) \\ \mathcal{L}(0.36,0.51) & \mathcal{L}(0.65,0.65) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.73,0.74) \\ \mathcal{L}(0.2,0.2) & \mathcal{L}(0.29,0.43) & \mathcal{L}(0.26,0.27) & \mathcal{L}(0.5,0.5) \end{pmatrix}$$

$$\overline{R}^{3(0,2)} = \begin{pmatrix} \mathcal{L}(0.5,0.5) & \mathcal{L}(0.2,0.34) & \mathcal{L}(0.1,0.1) & \mathcal{L}(0.37,0.37) \\ \mathcal{L}(0.66,0.8) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.24,0.38) & \mathcal{L}(0.58,0.58) \\ \mathcal{L}(0.9,0.9) & \mathcal{L}(0.62,0.76) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.71,0.78) \\ \mathcal{L}(0.63,0.63) & \mathcal{L}(0.42,0.42) & \mathcal{L}(0.22,0.29) & \mathcal{L}(0.5,0.5) \end{pmatrix}$$

$COI\left(\overline{R}^{1(0,2)}, \overline{R}^{c(0,2)}\right) = 0.909$, $COI\left(\overline{R}^{2(1,1)}, \overline{R}^{c(0,2)}\right) = 0.87$, $COI\left(\overline{R}^{3(0,1)}, \overline{R}^{c(0,2)}\right) = 0.819$, $\overline{COI}\left(\overline{R}^{c(0,2)}\right) = 0.866 > \omega$. The ultimate collective UPR $\overline{R}^c$ is

$$\begin{pmatrix} \mathcal{L}(0.5,0.5) & \mathcal{L}(0.46,0.56) & \mathcal{L}(0.43,0.44) & \mathcal{L}(0.47,0.57) \\ \mathcal{L}(0.44,0.54) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.4,0.44) & \mathcal{L}(0.51,0.58) \\ \mathcal{L}(0.56,0.57) & \mathcal{L}(0.56,0.6) & \mathcal{L}(0.5,0.5) & \mathcal{L}(0.56,0.6) \\ \mathcal{L}(0.43,0.53) & \mathcal{L}(0.42,0.49) & \mathcal{L}(0.4,0.44) & \mathcal{L}(0.5,0.5) \end{pmatrix}$$

Let the ACI threshold value of the collective UPR be $\vartheta = \lambda = 0.95$; then, $ACI(\overline{R}^c) = 0.982 > 0.95$, the priority problem is written separately, and this indicates that the individuals have reached the optimal consensus.

To verify the feasibility and efficiency of the proposed method, comparative analysis is conducted using existing methods to estimate the missing values and calculate ACI.

Let $\overline{R} = \left(\bar{r}_{ij}\right)_{4\times4}$ is defined as follows:

$$\begin{pmatrix} [0.5,0.5] & [0.6,0.8] & [x_{13}^l,0.75] & [0.4,0.7] \\ [0.2,0.4] & [0.5,0.5] & [x_{23}^l,x_{23}^r] & [x_{24}^l,0.55] \\ [0.25,x_{31}^r] & [x_{32}^l,x_{32}^r] & [0.5,0.5] & [0.3,x_{34}^r] \\ [0.3,0.6] & [0.45,x_{42}^r] & [x_{43}^l,0.7] & [0.5,0.5] \end{pmatrix}$$

Meng *et al.* [54] propose the concepts of quasi intervals with its additive consistency presentation. It is independent of the permutation of object labels and considers the additive consistency of lower and upper endpoints of IPRs simultaneously. Further, Meng *et al.* [55] discover that the additive consistency of quasi intervals is included in Krejčí's [56] which is more flexible.

Using model (26) in [54] whose solutions of missing values has the highest additive consistency level with respect to known values and model (M-3) in [55] separately, the results are shown in Table 1. Although the proposed method is dependent on the labels of objects, the estimated values of proposed method are all included in the results of [54,55] when the belief degree $\alpha = 0.8$. And it lessens uncertainty and has higher consistency level. Meanwhile, [54,55] are based on the interval with end points transformation

Table 1 Determined missing values with different methods.

Methods	Consistency	Missing Values					ACI
		r_{13}^l	r_{23}^l	r_{23}^r	r_{24}^l	r_{34}^r	
Method of Meng *et al.* [54]	Quasi intervals additive consistency	0.6	0.3	0.63	0.1	0.43	0.893
Method of Meng *et al.* [55]	Krejčí's additive consistency	0.65	0.6	0.6	0.25	0.4	0.918
The proposed method with $\alpha = 0.8$	UPRs additive consistency	0.75	0.62	0.62	0.55	0.43	0.919
The proposed method with $\alpha = 0.9$	UPRs additive consistency	0.75	0.65	0.65	0.55	0.3	0.909
The proposed method with $\alpha = 0.95$	UPRs additive consistency	0.75	0.65	0.65	0.55	0.3	0.909

which ignores internal values. The proposed method treats subjective preference as certain UD which handles the interval preference collectively. Furthermore, when the belief degree is higher, the stricter the requirement for deviation between estimated values and ideal values is, the greater the influence on ACI is. When end points of missing values are equal, the estimated values will not change whatever the belief degree changes.

4. CONCLUSION

Based on the LUD and its consistency condition, the algorithm to fill in the incomplete values and the optimisation of group consistency of completed UPRs are investigated in this study.

The main contributions of this study are as follows:

- An UCCPM is introduced to calculate the missing values in incomplete UPRs, which allows DMs to measure the confidence level of deviation between the supplementary values and the original incomplete information and guarantees the effectiveness of estimated values via a belief degree. It also proves that the operation of incomplete UPRs is an extension of that of traditional IPRs under a certain belief degree.
- A novel distance measure and the ACI of incomplete UPRs are proposed to calculate the consistency and consensus degree of preference relations based on LUD. They are also used to improve the consistency and consensus index of UPRs iteratively.
- The interval preference is treated collectively by obeying the LUD, which avoids the decision distortion and discretisation operation of intervals in the traditional interval operation.

Our future research will focus on two aspects. We discuss the situation of independent DMs in current work. If social relationships of individuals are considered, then GDM can be more scientific. Besides the interval preference information of DMs obeys a LUD, it may obey a NUD, zigzag uncertainty distribution, or lognormal uncertainty distribution, etc. Nonlinear distributions of other types of preference relations with their multiplicative consistency indices will be further investigated.

CONFLICT OF INTEREST

Authors have no conflict of interest to declare.

AUTHORS' CONTRIBUTIONS

The study is guided by Zaiwu Gong and written by all authors.

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